J. Math. Anal. Appl. 379 (2011) 150–170

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Fast reaction limit of a three-component reaction–diffusion system

H. Murakawa [a,*], H. Ninomiya [b]

[a] Graduate School of Science and Engineering for Research, University of Toyama, 3190 Gofuku, Toyama 930-8555, Japan
[b] Department of Mathematics, Meiji University, 1-1-1 Higashimita, Tamaku, Kawasaki 214-8571, Japan

ARTICLE INFO

Article history:
Received 24 July 2010
Available online 23 December 2010
Submitted by Goong Chen

Keywords:
Reaction–diffusion system
Fast reaction limit
Reaction limit set
Free boundary problem
Nonlinear diffusion problem

ABSTRACT

We consider a three-component reaction–diffusion system with a reaction rate parameter, and investigate its singular limit as the reaction rate tends to infinity. The limit problem is given by a free boundary problem which possesses three regions separated by the free boundaries. One component vanishes and the other two components remain positive in each region. Therefore, the dynamics is governed by a system of two equations.

1. Introduction

The study of the evolution of species in natural environments is one of the most exciting problems. According to the Gause's principle of competitive exclusion, two competing species cannot coexist under strong competition. The migration or the spatial distribution changes the situation and then species can coexist due to the segregation of their habitats.

Recently, many researchers have studied these problems from a mathematical viewpoint. One of the mathematical tools to deal with them is so-called *fast reaction limit* or *reaction–diffusion system approximation*. Many systems are considered in this context (see [9] and references therein). Dancer, Hilhorst, Mimura and Peletier [3] considered the Lotka–Volterra competition–diffusion system including a large parameter k:

$$(\mathrm{LV})^k \quad \begin{cases} u_t = d_1 \Delta u + \lambda u(1-u) - kuv & \text{in } Q := \Omega \times (0,T), \\ v_t = d_2 \Delta v + \mu v(1-v) - kuv & \text{in } Q \end{cases}$$

with Neumann boundary conditions and non-negative initial conditions, where Ω is a smooth domain of $\mathbb{R}^N$ ($N \in \mathbb{N}$). Here, λ, μ, d_1, d_2 and T are positive constants. The solution pair (u, v) represents densities of two competing species and k is an interspecific competition rate. They showed that the two species are spatially segregated as k tends to infinity and that the interface between two habitats is governed by a Stefan-type free boundary problem.

We encounter the following natural question: *How can we extend this result to the case of the three or more component systems?* A possible extension is the Lotka–Volterra competition–diffusion system with three species:

$$(\mathrm{LV3})^k \quad \begin{cases} u_t = d_1 \Delta u + \mu_1 u(1-u) - k(v+w)u & \text{in } Q, \\ w_t = d_2 \Delta v + \mu_2 v(1-v) - k(w+u)v & \text{in } Q, \\ w_t = d_2 \Delta w + \mu_3 w(1-w) - k(u+v)w & \text{in } Q, \end{cases}$$

* Corresponding author.
E-mail addresses: murakawa@sci.u-toyama.ac.jp (H. Murakawa), ninomiya@math.meiji.ac.jp (H. Ninomiya).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.12.040

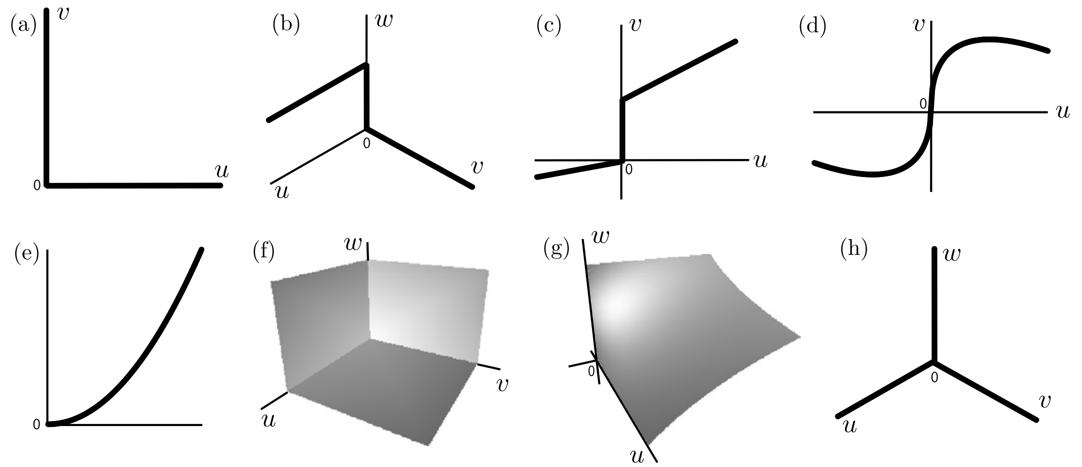

Fig. 1. The reaction limit sets. The limit problems are (a) the two-phase Stefan problem without latent heat or the one-phase Stefan problem, (b) the two-phase Stefan problem, (c) the two-phase Stefan problem, (d) the porous medium equation, (e) a nonlinear-diffusion equation, (f) our problem, (g) Shigesada–Kawasaki–Teramoto cross-diffusion system, (h) open problem.

where $\mu_i \geqslant 0$ for $i = 1, 2, 3$. Hilhorst, Iida, Mimura and Ninomiya [7] have treated $2m$-component systems including $(\mathrm{LV3})^k$ and showed that the species segregate as k tends to infinity. However, they did not derive any explicit limit problem.

In this paper we consider a different type of extension of $(\mathrm{LV})^k$, namely the three-component system as follows:

$$(\mathrm{RD})^k \quad \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + f_1(u, v, w) - kuvw & \text{in } Q, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + f_2(u, v, w) - kuvw & \text{in } Q, \\ \dfrac{\partial w}{\partial t} = d_3 \Delta w + f_3(u, v, w) - kuvw & \text{in } Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0^k, \quad v(\cdot, 0) = v_0^k, \quad w(\cdot, 0) = w_0^k & \text{in } \Omega, \end{cases}$$

where f_i are given functions, ν is the unit outward normal vector to the boundary $\partial\Omega$, and u_0^k, v_0^k and w_0^k are non-negative initial functions.

In order to derive the limit problem, we begin by examining $(\mathrm{LV})^k$ and its limiting equation as k tends to infinity. Letting $k \to \infty$ in

$$\frac{u_t}{k} = \frac{d_1}{k} \Delta u + \frac{1}{k} \lambda u(1 - u) - uv,$$

we can expect that

$$0 = uv$$

if u, u_t and Δu are bounded with respect to k. Hence, the dynamics is restricted to the following one-dimensional set:

$$\mathcal{A}_{\mathrm{LV}} = \{(u, 0) \mid u \geqslant 0\} \cup \{(0, v) \mid v \geqslant 0\}.$$

This set $\mathcal{A}_{\mathrm{LV}}$ consists of equilibria of the fast reaction system:

$$\begin{cases} u_t = -kuv, \\ v_t = -kuv. \end{cases}$$

We call the equilibria of the fast reaction system a *reaction limit set*. The reaction limit set $\mathcal{A}_{\mathrm{LV}}$ of $(\mathrm{LV})^k$ is shown in Fig. 1(a). The set consists of two axis. The solution diffuses with the diffusion coefficient d_1 on $\{(u, 0) \mid u \geqslant 0\}$, while it does with the coefficient d_2 on $\{(0, v) \mid v \geqslant 0\}$. The flux is discontinuous across the corner in $\mathcal{A}_{\mathrm{LV}}$. This may indicate the presence of a free boundary in the limit problem. Indeed, it was proved that the limiting system as the reaction rate k tends to infinity is represented by the one-phase Stefan problem for the case $d_1 > 0$, $d_2 = 0$ in [5] and that the limit equation can be described by the two-phase Stefan problem without latent heat for the case $d_1, d_2 > 0$ in [3]. Similar reaction limit sets are also observed in [8] and [2] and the corresponding limit problems are given by the two-phase Stefan problem without latent heat and the one-phase Stefan problem. Hilhorst, Iida, Mimura and Ninomiya [6] proposed the following three-component reaction–diffusion system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + f_1(u) - ku(1-w) & \text{in } Q, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + f_2(v) - kvw & \text{in } Q, \\ \dfrac{\partial w}{\partial t} = ku(1-w) - kvw & \text{in } Q \end{cases} \tag{1.1}$$

with initial data satisfying $0 \leqslant u_0, v_0, w_0 \leqslant 1$. The reaction limit set of (1.1) is shown in Fig. 1(b). We can observe that it consists of two $\mathcal{A}_{\mathrm{LV}}$s. This suggests us that the limit problem is the two-phase Stefan problem with positive latent heat. In fact, this was proved in [6]. Murakawa [13] also proved that the solution of the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + f_1(u) - k\big(u - \beta(u+v)\big) & \text{in } Q, \\ \dfrac{\partial v}{\partial t} = k\big(u - \beta(u+v)\big) & \text{in } Q \end{cases} \tag{1.2}$$

converges to that of the two-phase Stefan problem when $\beta(r) = d_1 \max\{r-1, 0\} + d_2 \min\{r, 0\}$ $(r \in \mathbb{R})$. The corresponding reaction limit set of (1.2) is illustrated in Fig. 1(c). The shapes of the reaction limit sets of (b) and (c) are based on a combination of two sets of Fig. 1(a). Although the number of components of the original system (1.1) is different from that of (1.2), the limits are represented by the same problem. Thus, the reaction limit sets must play an important role in singular limit analysis. It is shown in [13] that the porous medium equation is also approximated by (1.2) when $\beta(r) = |r|^{m-1} r$ $(r \in \mathbb{R})$ for some $m > 1$. The reaction limit set is shown in Fig. 1(d). Bothe and Hilhorst [1] considered a reversible chemical reaction between two mobile species, and studied the limit to an instantaneous reaction:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u - k\big(r_A(u) - r_B(v)\big) & \text{in } Q, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v - k\big(r_B(u) - r_A(v)\big) & \text{in } Q \end{cases} \tag{1.3}$$

(see [1] for the detailed assumptions of r_A and r_B). The reaction limit set of (1.3) is given in Fig. 1(e) for a usual choice of r_A and r_B. They proved that the limit problem becomes a single nonlinear-diffusion equation and that the nonlinear diffusivities in the limit problems are determined by the reaction limit sets. We note that the reaction limit sets in both cases (d) and (e) are smooth curves contrary to (a)–(c), so the diffusivities are given by smooth functions.

All of the above examples illustrate the importance of the shapes of reaction limit sets in presuming the limit problems. Since these reaction limit sets are one-dimensional, the limit problems in all these examples are represented by single nonlinear-diffusion equations. The existence of corners or non-smooth points in the reaction limit set indicates the appearance of interfaces, because they create the discontinuity of the flux, which exhibits the interfaces.

Now we go back to our problem. The reaction limit set of $(\mathrm{RD})^k$ is

$$\mathcal{A}_{\mathrm{RD}} = \big\{(0, v, w) \mid v \geqslant 0, w \geqslant 0\big\} \cup \big\{(u, 0, w) \mid u \geqslant 0, w \geqslant 0\big\} \cup \big\{(u, v, 0) \mid u \geqslant 0, v \geqslant 0\big\}$$

(see Fig. 1(f) for its shape). From the above observations, we can imagine the limit problem of the system $(\mathrm{RD})^k$ as k tends to infinity. Since the reaction limit set $\mathcal{A}_{\mathrm{RD}}$ is a two-dimensional surface, the limit problem will consist of two equations, which will be proved in Theorem 3.2. Moreover, $\mathcal{A}_{\mathrm{RD}}$ has corners, which may imply the appearance of interfaces in the limit problem. Actually, we will prove this in Theorems 1.1 and 4.1.

There are few results dealing with two- or multi-dimensional reaction limit sets. Iida et al. [10] studied Shigesada–Kawasaki–Teramoto cross-diffusion system [15]. For a deeper understanding of the cross-diffusion mechanism, they replaced cross-diffusion by a different way of avoiding the congestion of the other species. Then, they proposed a three-component reaction–diffusion system and showed that its solution approximates the solution of the cross-diffusion system. (For more general cases, see [11,14].) In their study, the reaction limit set $\{(u, v, w) \mid (1-w)v = uw\}$ is a two-dimensional smooth set as in Fig. 1(g).

We note that the reaction limit set of $(\mathrm{LV3})^k$ consists of three lines as in Fig. 1(h) and is neither associated with a one-dimensional curve nor with a two-dimensional surface by continuous map. This prevents us from being able to derive the explicit expression of the limit problem. Thus we consider the limit problem of $(\mathrm{RD})^k$ instead of $(\mathrm{LV3})^k$.

In the following, we present our main result, that is, the convergence of the solution (u^k, v^k, w^k) of $(\mathrm{RD})^k$ and the limit problem. Define

$$\begin{aligned} \Omega_1(t) &:= \big\{x \in \Omega \mid v(x,t) > 0, w(x,t) > 0\big\}, \\ \Omega_2(t) &:= \big\{x \in \Omega \mid w(x,t) > 0, u(x,t) > 0\big\}, \\ \Omega_3(t) &:= \big\{x \in \Omega \mid u(x,t) > 0, v(x,t) > 0\big\}, \end{aligned}$$

$$Q_i := \bigcup_{t\in(0,T)} \Omega_i(t) \quad (i=1,2,3).$$

Then

$$\Omega_i(t)\cap\Omega_j(t)=\emptyset \quad (i\neq j).$$

We also denote the interfaces by

$$\Gamma_1(t) := \partial\Omega_2(t)\cap\partial\Omega_3(t)\cap\Omega,$$
$$\Gamma_2(t) := \partial\Omega_3(t)\cap\partial\Omega_1(t)\cap\Omega,$$
$$\Gamma_3(t) := \partial\Omega_1(t)\cap\partial\Omega_2(t)\cap\Omega.$$

Throughout this paper, the following assumptions are imposed on the initial data and on the given functions f_i:

(H1) The initial data $u_0^k, v_0^k, w_0^k \in C(\overline{\Omega})$ satisfy

$$0\leqslant u_0^k, v_0^k, w_0^k \leqslant M,$$
$$u_0^k \rightharpoonup u_0,\quad v_0^k \rightharpoonup v_0,\quad w_0^k \rightharpoonup w_0 \quad \text{weakly in } L^2(\Omega) \text{ as } k\to\infty$$

for some positive constant M and for some functions $u_0, v_0, w_0 \in L^\infty(\Omega)$.

(H2) There exist C^1-functions $\tilde{f}_i$ $(i=1,2,3)$ such that for all $\boldsymbol{s}=(s_1,s_2,s_3)\in\mathbb{R}^3_+$,

$$f_i(\boldsymbol{s}) = \tilde{f}_i(\boldsymbol{s})s_i,$$
$$\tilde{f}_i(\boldsymbol{s}) \leqslant 0 \quad \text{if } s_i \geqslant M.$$

Under these assumptions, there exists a unique solution of $(\mathrm{RD})^k$ (see [12]).

We now state our main result.

Theorem 1.1. *Assume that* (H1) *and* (H2) *hold. Let* (u^k, v^k, w^k) *be the solution of* $(\mathrm{RD})^k$. *Then, there are subsequences* $\{u^{k_n}\}$, $\{v^{k_n}\}$ *and* $\{w^{k_n}\}$ *of* $\{u^k\}$, $\{v^k\}$ *and* $\{w^k\}$, *respectively, and functions* u, v, w *such that*

$$u^{k_n}\to u,\quad v^{k_n}\to v,\quad w^{k_n}\to w$$

strongly in $L^2(Q)$, *a.e. in* Q, *and weakly in* $L^2(0,T;H^1(\Omega))$ *as* k_n *tends to infinity. Moreover, assume that each of* $\Gamma_i(t)$ $(i=1,2,3)$ *defined as above is an* $(N-1)$*-dimensional smooth hypersurface or the empty set, and* $\Gamma_i(t)$ $(i=1,2,3)$ *and* $\partial\Omega$ *do not intersect each other for* $0\leqslant t\leqslant T$, *and* Q_i *are (piecewise) smooth, and* $u_0v_0w_0=0$. *If the functions* u, v *and* w *are smooth on* $\overline{Q_1}$, $\overline{Q_2}$ *and* $\overline{Q_3}$, *then* $(\Gamma_1,\Gamma_2,\Gamma_3,u,v,w)$ *is the solution of the following free boundary problem*:

$$\begin{cases} \dfrac{\partial v}{\partial t} = d_2\Delta v + f_2(0,v,w), \\ \dfrac{\partial w}{\partial t} = d_3\Delta w + f_3(0,v,w), \end{cases} \quad \text{in } Q_1, \tag{1.4}$$

$$\begin{cases} \dfrac{\partial w}{\partial t} = d_3\Delta w + f_3(u,0,w), \\ \dfrac{\partial u}{\partial t} = d_1\Delta u + f_1(u,0,w), \end{cases} \quad \text{in } Q_2, \tag{1.5}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + f_1(u,v,0), \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + f_2(u,v,0), \end{cases} \quad \text{in } Q_3, \tag{1.6}$$

$$\begin{cases} v=w=0 & \text{on } \Gamma_1, \\ w=u=0 & \text{on } \Gamma_2, \\ u=v=0 & \text{on } \Gamma_3, \end{cases} \tag{1.7}$$

$$d_2\frac{\partial v|_{Q_3}}{\partial n_1} + d_3\frac{\partial w|_{Q_2}}{\partial n_1} = 0, \qquad d_1\left(\frac{\partial u|_{Q_3}}{\partial n_1} - \frac{\partial u|_{Q_2}}{\partial n_1}\right) = d_2\frac{\partial v|_{Q_3}}{\partial n_1} \quad \text{on } \Gamma_1, \tag{1.8}$$

$$d_3\frac{\partial w|_{Q_1}}{\partial n_2} + d_1\frac{\partial u|_{Q_3}}{\partial n_2} = 0, \qquad d_2\left(\frac{\partial v|_{Q_1}}{\partial n_2} - \frac{\partial v|_{Q_3}}{\partial n_2}\right) = d_3\frac{\partial w|_{Q_1}}{\partial n_2} \quad \text{on } \Gamma_2, \tag{1.9}$$

$$d_1\frac{\partial u|_{Q_2}}{\partial n_3} + d_2\frac{\partial v|_{Q_1}}{\partial n_3} = 0, \qquad d_3\left(\frac{\partial w|_{Q_2}}{\partial n_3} - \frac{\partial w|_{Q_1}}{\partial n_3}\right) = d_1\frac{\partial u|_{Q_2}}{\partial n_3} \quad \text{on } \Gamma_3, \tag{1.10}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.11}$$

$$u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0, \quad w(\cdot,0) = w_0 \quad \text{in } \Omega, \tag{1.12}$$

where n_i are unit normal vectors on $\Gamma_i(t)$ oriented from $\Omega_j(t)$ to $\Omega_k(t)$ for $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$.

Three types of free boundaries appear in the limit problem. Furthermore, the dynamics is governed by a system of equations in each region separated by the free boundaries. The intersection of the three axes $\{(0,0,w) \mid w \geqslant 0\}$, $\{(u,0,0) \mid u \geqslant 0\}$ and $\{(0,v,0) \mid v \geqslant 0\}$ on $\mathcal{A}_{\mathrm{RD}}$ might imply the existence of triple (or multiple) junctions. This theorem excludes the multiple junction points by assumption, but these points are included in the limit problem in a weak sense. We will present numerical simulations later on.

Remark 1.2. If the diffusion coefficients satisfy the additional conditions (see (H3) or (H4) in Section 3.2), we can show the uniqueness of the weak solution of the limiting equation. Therefore this theorem holds for the full sequence (u^k, v^k, w^k) without taking subsequences.

Remark 1.3. Assume that $f_3(u,v,1) = 0$ and that $w_0(x) = 1$ for $x \in \Omega$. Then $w(x,t) = 1$ for $t \geqslant 0$, $x \in \Omega$. In this case, the problem $(\mathrm{RD})^k$ coincides with $(\mathrm{LV})^k$. Therefore, Theorem 1.1 is an extension of the result by Dancer et al. [3].

This paper is organized as follows. Several a priori estimates are provided in Section 2. We show that (u^k, v^k, w^k) converges to the solution of a weak form of the limiting system (1.4)–(1.12) in Section 3. We can prove the uniqueness of the weak solution under the additional conditions. Moreover, the rate of convergence is obtained. In Section 4, we derive the strong form of the limit problem. Section 5 gives the simulations of two examples.

2. Some basic properties

Before proving the convergence results, we first show several basic inequalities for the solutions (u^k, v^k, w^k).

Lemma 2.1. *Let (u^k, v^k, w^k) be a solution of $(\mathrm{RD})^k$. Then, there exists a positive constant C_1 independent of k such that*

$$0 \leqslant u^k, v^k, w^k \leqslant M \quad \text{in } Q, \tag{2.1}$$

$$\iint_Q u^k v^k w^k \, dx\, dt \leqslant \frac{C_1}{k}. \tag{2.2}$$

Proof. Let us define

$$\mathscr{L}_1(u) := u_t - d_1 \Delta u - f_1(u,v,w) + kuvw,$$
$$\mathscr{L}_2(v) := v_t - d_2 \Delta v - f_2(u,v,w) + kuvw,$$
$$\mathscr{L}_3(w) := w_t - d_3 \Delta w - f_3(u,v,w) + kuvw.$$

Since $\mathscr{L}_i(0) = 0$ and $\mathscr{L}_i(M) \geqslant 0$ for $i = 1,2,3$, the assertion (2.1) follows from the maximum principle. Integration of the equation for u^k in Q yields

$$k \iint_Q u^k v^k w^k \, dx\, dt = \int_\Omega \left(u_0^k - u^k(\cdot,T)\right) dx + \iint_Q f_1\left(u^k, v^k, w^k\right) dx\, dt.$$

The boundedness of f_1 on $[0,M]^3$ and (2.1) imply the desired estimate. □

The following lemma follows from a similar argument to that of [3].

Lemma 2.2. *The functions u^k, v^k and w^k are uniformly bounded with respect to k in $L^2(0,T;H^1(\Omega))$.*

Proof. Multiplying the equation for u^k by u^k and integrating by parts on Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(u^k\right)^2 dx + d_1 \int_\Omega \left|\nabla u^k\right|^2 dx = \int_\Omega f_1\left(u^k, v^k, w^k\right) u^k \, dx - k \int_\Omega \left(u^k\right)^2 v^k w^k \, dx.$$

Integrating on $(0,T)$ and using Lemma 2.1 yield

$$\left\|u^k\right\|_{L^2(0,T;H^1(\Omega))} \leqslant C_2$$

for some positive constant C_2 independent of k. Thus, we have verified the result for u^k. The estimates for v^k and w^k can be obtained similarly. □

Next, we consider a function $\boldsymbol{z}^k = (z_1^k, z_2^k) = (v^k - u^k, w^k - u^k)$ which appears when we eliminate the terms involving k from $(\mathrm{RD})^k$. The functions satisfy

$$\frac{\partial z_1^k}{\partial t} = d_2 \Delta v^k - d_1 \Delta u^k + f_2(u^k, v^k, w^k) - f_1(u^k, v^k, w^k) \quad \text{in } Q, \tag{2.3}$$

$$\frac{\partial z_2^k}{\partial t} = d_3 \Delta w^k - d_1 \Delta u^k + f_3(u^k, v^k, w^k) - f_1(u^k, v^k, w^k) \quad \text{in } Q, \tag{2.4}$$

$$\frac{\partial}{\partial \nu}(d_2 v^k - d_1 u^k) = \frac{\partial}{\partial \nu}(d_3 w^k - d_1 u^k) = 0 \quad \text{on } \partial\Omega \times (0,T),$$

$$z_1^k(\cdot,0) = v_0^k - u_0^k, \qquad z_2^k(\cdot,0) = w_0^k - u_0^k \quad \text{in } \Omega.$$

Lemma 2.3. *The functions z_1^k, z_2^k are uniformly bounded with respect to k in $H^1(0,T; H^1(\Omega)^*)$.*

Proof. Multiplying (2.3) by $\zeta \in L^2(0,T; H^1(\Omega))$ and integrating it over Q by parts yield

$$\int_0^T \left\langle \frac{\partial z_1^k}{\partial t}, \zeta \right\rangle dt = -d_2 \int_0^T \langle \nabla v^k, \nabla \zeta \rangle dt + d_1 \int_0^T \langle \nabla u^k, \nabla \zeta \rangle dt + \int_0^T \langle f_2(u^k, v^k, w^k) - f_1(u^k, v^k, w^k), \zeta \rangle dt. \tag{2.5}$$

Here, $\langle \cdot,\cdot \rangle$ denotes both the inner product in $L^2(\Omega)$ and the duality pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$. Applying the Cauchy–Schwarz inequality to (2.5) and using Lemma 2.2, we see that there exists a positive constant C_3 independent of k such that

$$\left| \int_0^T \left\langle \frac{\partial z_1^k}{\partial t}, \zeta \right\rangle dt \right| \leqslant C_3 \|\zeta\|_{L^2(0,T;H^1(\Omega))}.$$

Namely, we have

$$\|z_1^k\|_{H^1(0,T;H^1(\Omega)^*)} \leqslant C_3,$$

which concludes the statement for z_1^k. The same argument gives the estimate for z_2^k. □

We introduce several auxiliary functions to state the limiting equation. Set

$$I := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 > 0,\ z_2 \geqslant 0\},$$
$$II := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \leqslant 0,\ z_1 < z_2\},$$
$$III := \{(z_1, z_2) \in \mathbb{R}^2 \mid z_2 < 0,\ z_1 \geqslant z_2\},$$

and define for $\boldsymbol{z} = (z_1, z_2) \in \mathbb{R}^2$

$$\varphi(\boldsymbol{z}) := \begin{cases} 0 & \text{if } \boldsymbol{z} \in I, \\ z_1 & \text{if } \boldsymbol{z} \in II, \\ z_2 & \text{if } \boldsymbol{z} \in III, \end{cases}$$

$$\gamma_1(\boldsymbol{z}) := -\varphi(\boldsymbol{z}), \qquad \gamma_2(\boldsymbol{z}) := z_1 - \varphi(\boldsymbol{z}), \qquad \gamma_3(\boldsymbol{z}) := z_2 - \varphi(\boldsymbol{z}),$$

$$\phi_1(\boldsymbol{z}) := d_2\gamma_2(\boldsymbol{z}) - d_1\gamma_1(\boldsymbol{z}) = d_2 z_1 + (d_1 - d_2)\varphi(\boldsymbol{z}),$$

$$\phi_2(\boldsymbol{z}) := d_3\gamma_3(\boldsymbol{z}) - d_1\gamma_1(\boldsymbol{z}) = d_3 z_2 + (d_1 - d_3)\varphi(\boldsymbol{z}).$$

The functions γ_i satisfy the following useful relations.

Lemma 2.4. *The following properties hold for all non-negative real numbers* u, v *and* w:

$$\left|u-\gamma_1(v-u,w-u)\right|^3 \leqslant uvw, \tag{2.6}$$

$$\left|v-\gamma_2(v-u,w-u)\right|^3 \leqslant uvw, \tag{2.7}$$

$$\left|w-\gamma_3(v-u,w-u)\right|^3 \leqslant uvw. \tag{2.8}$$

Proof. Set

$$\boldsymbol{z}=(z_1,z_2)=(v-u,w-u).$$

If $\boldsymbol{z}\in I$, i.e., $z_1=v-u>0$ and $z_2=w-u\geqslant 0$, then $0\leqslant u\leqslant v$, $0\leqslant u\leqslant w$. Hence,

$$\begin{aligned}
&\left|u-\gamma_1(\boldsymbol{z})\right|^3=|u-0|^3\leqslant uvw,\\
&\left|v-\gamma_2(\boldsymbol{z})\right|^3=\left|v-(v-u)\right|^3\leqslant uvw,\\
&\left|w-\gamma_3(\boldsymbol{z})\right|^3=\left|w-(w-u)\right|^3\leqslant uvw.
\end{aligned}$$

If $\boldsymbol{z}\in II$, that is, $z_1=v-u\leqslant 0$ and $z_2-z_1=w-v>0$, then $0\leqslant v\leqslant u$, $0\leqslant v\leqslant w$. Therefore, we have

$$\begin{aligned}
&\left|u-\gamma_1(\boldsymbol{z})\right|^3=\left|u+(v-u)\right|^3\leqslant uvw,\\
&\left|v-\gamma_2(\boldsymbol{z})\right|^3=|v-0|^3\leqslant uvw,\\
&\left|w-\gamma_3(\boldsymbol{z})\right|^3=\left|w-(w-u)+(v-u)\right|^3\leqslant uvw.
\end{aligned}$$

Similarly, if $\boldsymbol{z}\in III$, then $0\leqslant w\leqslant u$, $0\leqslant u\leqslant v$ and consequently,

$$\begin{aligned}
&\left|u-\gamma_1(\boldsymbol{z})\right|^3=\left|u+(w-u)\right|^3\leqslant uvw,\\
&\left|v-\gamma_2(\boldsymbol{z})\right|^3=\left|v-(v-u)+(w-u)\right|^3\leqslant uvw,\\
&\left|w-\gamma_3(\boldsymbol{z})\right|^3=|w-0|^3\leqslant uvw.
\end{aligned}$$

Thus, the proof is complete. □

This lemma and Lemma 2.1 give the estimates on the solutions.

Lemma 2.5. *There exists a positive constant* C_4 *independent of* k *such that*

$$\begin{aligned}
&\left\|u^k-\gamma_1(\boldsymbol{z}^k)\right\|_{L^3(Q)}+\left\|v^k-\gamma_2(\boldsymbol{z}^k)\right\|_{L^3(Q)}+\left\|w^k-\gamma_3(\boldsymbol{z}^k)\right\|_{L^3(Q)}\\
&\quad+\left\|\left(d_2v^k-d_1u^k\right)-\phi_1(\boldsymbol{z}^k)\right\|_{L^3(Q)}+\left\|\left(d_3w^k-d_1u^k\right)-\phi_2(\boldsymbol{z}^k)\right\|_{L^3(Q)}\leqslant C_4k^{-1/3}.
\end{aligned}$$

Proof. From Lemma 2.4, we get for all $(x,t)\in Q$

$$\begin{aligned}
&\left|u^k(x,t)-\gamma_1\left(\boldsymbol{z}^k(x,t)\right)\right|^3\leqslant u^k(x,t)v^k(x,t)w^k(x,t),\\
&\left|v^k(x,t)-\gamma_2\left(\boldsymbol{z}^k(x,t)\right)\right|^3\leqslant u^k(x,t)v^k(x,t)w^k(x,t),\\
&\left|w^k(x,t)-\gamma_3\left(\boldsymbol{z}^k(x,t)\right)\right|^3\leqslant u^k(x,t)v^k(x,t)w^k(x,t),
\end{aligned}$$

which together with Lemma 2.1 implies that

$$\begin{cases}
\left\|u^k-\gamma_1(\boldsymbol{z}^k)\right\|_{L^3(Q)}\leqslant C_1^{1/3}k^{-1/3},\\
\left\|v^k-\gamma_2(\boldsymbol{z}^k)\right\|_{L^3(Q)}\leqslant C_1^{1/3}k^{-1/3},\\
\left\|w^k-\gamma_3(\boldsymbol{z}^k)\right\|_{L^3(Q)}\leqslant C_1^{1/3}k^{-1/3}.
\end{cases} \tag{2.9}$$

By (2.9) we have

$$\begin{aligned}
\left\|\left(d_2v^k-d_1u^k\right)-\phi_1(\boldsymbol{z}^k)\right\|_{L^3(Q)}&=\left\|\left(d_2v^k-d_1u^k\right)-\left(d_2\gamma_2(\boldsymbol{z}^k)-d_1\gamma_1(\boldsymbol{z}^k)\right)\right\|_{L^3(Q)}\\
&\leqslant d_2\left\|v^k-\gamma_2(\boldsymbol{z}^k)\right\|_{L^3(Q)}+d_1\left\|u^k-\gamma_1(\boldsymbol{z}^k)\right\|_{L^3(Q)}\\
&\leqslant (d_1+d_2)C_1^{1/3}k^{-1/3}.
\end{aligned}$$

Similarly, we obtain

$$\left\| \left(d_3 w^k - d_1 u^k\right) - \phi_2\left(\boldsymbol{z}^k\right) \right\|_{L^3(Q)} \leqslant (d_1 + d_3) C_1^{1/3} k^{-1/3},$$

which completes the proof. □

3. Convergence to a nonlinear-diffusion system

The arguments in the previous section suggest that $(u^k, v^k, w^k) - (\gamma_1(z^k), \gamma_2(z^k), \gamma_3(z^k))$ converges to zero as k tends to infinity. Thus we can expect from (2.3) and (2.4) that the limit functions (z_1, z_2) of (z_1^k, z_2^k) satisfy the following nonlinear-diffusion system:

$$\text{(ND)} \quad \begin{cases} \dfrac{\partial z_1}{\partial t} = \Delta\phi_1(\boldsymbol{z}) + F_1(\boldsymbol{z}) & \text{in } Q, \\ \dfrac{\partial z_2}{\partial t} = \Delta\phi_2(\boldsymbol{z}) + F_2(\boldsymbol{z}) & \text{in } Q, \\ \dfrac{\partial \phi_1(\boldsymbol{z})}{\partial \nu} = \dfrac{\partial \phi_2(\boldsymbol{z})}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{z}(\cdot,0) = \boldsymbol{z}_0 & \text{in } \Omega, \end{cases}$$

where $\boldsymbol{z}_0 := (v_0 - u_0, w_0 - u_0)$ and

$$F_1(\boldsymbol{z}) := f_2\big(\gamma_1(\boldsymbol{z}), \gamma_2(\boldsymbol{z}), \gamma_3(\boldsymbol{z})\big) - f_1\big(\gamma_1(\boldsymbol{z}), \gamma_2(\boldsymbol{z}), \gamma_3(\boldsymbol{z})\big),$$
$$F_2(\boldsymbol{z}) := f_3\big(\gamma_1(\boldsymbol{z}), \gamma_2(\boldsymbol{z}), \gamma_3(\boldsymbol{z})\big) - f_1\big(\gamma_1(\boldsymbol{z}), \gamma_2(\boldsymbol{z}), \gamma_3(\boldsymbol{z})\big).$$

In this section, we show this expectation in a weak sense. To this end, we define a weak solution of (ND).

Definition 3.1. A function $\boldsymbol{z} \in L^\infty(Q)^2$ is a weak solution of (ND) with an initial datum $\boldsymbol{z}_0 \in L^\infty(\Omega)^2$ if it satisfies $\phi_i(\boldsymbol{z}) \in L^2(0,T; H^1(\Omega))^2$ and

$$\int_0^T \left\langle z_i, \frac{\partial \zeta_i}{\partial t} \right\rangle dt + \left\langle z_{0i}, \zeta_i(\cdot,0) \right\rangle = \int_0^T \left\langle \nabla\phi_i(\boldsymbol{z}), \nabla\zeta_i \right\rangle dt - \int_0^T \left\langle F_i(\boldsymbol{z}), \zeta_i \right\rangle dt \tag{3.1}$$

for all functions $\boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in H^1(Q)^2$ with $\zeta_i(\cdot, T) = 0$ and for $i = 1, 2$.

In Section 3.1, we establish the existence of the weak solution of (ND). The uniqueness of the weak solution is discussed in Section 3.2.

3.1. Convergence

We present our result on the convergence.

Theorem 3.2. *Assume that* (H1) *and* (H2) *hold. Let* (u^k, v^k, w^k) *be the solution of* (RD)^k. *Then, there exist a weak solution* $\boldsymbol{z} = (z_1, z_2) \in (L^\infty(Q) \cap L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^1(\Omega)^*))^2$ *of* (ND) *and subsequences* $\{u^{k_n}\}$, $\{v^{k_n}\}$ *and* $\{w^{k_n}\}$ *of* $\{u^k\}$, $\{v^k\}$ *and* $\{w^k\}$, *respectively, such that*

$$u^{k_n} \to \gamma_1(\boldsymbol{z}), \quad v^{k_n} \to \gamma_2(\boldsymbol{z}), \quad w^{k_n} \to \gamma_3(\boldsymbol{z})$$

strongly in $L^2(Q)$, *a.e. in* Q, *and weakly in* $L^2(0,T; H^1(\Omega))$,

$$z_1^{k_n} := v^{k_n} - u^{k_n} \to z_1, \qquad z_2^{k_n} := w^{k_n} - u^{k_n} \to z_2$$

strongly in $L^2(Q)$, *a.e. in* Q, *and weakly in* $L^2(0,T; H^1(\Omega))$ *and* $H^1(0,T; H^1(\Omega)^*)$ *as* k_n *tends to infinity.*

Proof. By virtue of Lemmas 2.1–2.3 and the compactness of the embedding $L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^1(\Omega)^*) \subset L^2(Q)$ [16, Theorem 2.1], there exist subsequences $\{u^{k_n}\}$, $\{v^{k_n}\}$, $\{w^{k_n}\}$ and $\{\boldsymbol{z}^{k_n}\}$ and functions $u^*, v^*, w^* \in L^\infty(Q) \cap L^2(0,T; H^1(\Omega))$ and $\boldsymbol{z}^* \in (L^\infty(Q) \cap H^1(0,T; H^1(\Omega)^*) \cap L^2(0,T; H^1(\Omega)))^2$ such that

$$0 \leqslant u^*, v^*, w^* \leqslant M \quad \text{a.e. in } Q,$$
$$u^* v^* w^* = 0 \tag{3.2}$$

and

$$u^{k_n} \rightharpoonup u^*, \quad v^{k_n} \rightharpoonup v^*, \quad w^{k_n} \rightharpoonup w^* \quad \text{weakly in } L^2\big(0,T;H^1(\Omega)\big),$$
$$\boldsymbol{z}^{k_n} \to \boldsymbol{z}^* \quad \text{strongly in } L^2(Q)^2, \text{ a.e. in } Q,$$
$$\text{and weakly in } L^2\big(0,T;H^1(\Omega)\big)^2 \text{ and } H^1\big(0,T;H^1(\Omega)^*\big)^2$$

as k_n tends to infinity. It follows from the Lipschitz continuities of γ_i $(i=1,2,3)$ and ϕ_i $(i=1,2)$ that $\gamma_i(\boldsymbol{z}^{k_n})$ and $\phi_i(\boldsymbol{z}^{k_n})$ also converge to $\gamma_i(\boldsymbol{z}^*)$ and $\phi_i(\boldsymbol{z}^*)$ strongly in $L^2(Q)$ and a.e. in Q, respectively. Lemma 2.5 implies that u^{k_n}, v^{k_n} and w^{k_n} converge to $\gamma_1(\boldsymbol{z}^*)$, $\gamma_2(\boldsymbol{z}^*)$ and $\gamma_3(\boldsymbol{z}^*)$ strongly in $L^2(Q)$ and a.e. in Q, respectively, also $u^*=\gamma_1(\boldsymbol{z}^*)$, $v^*=\gamma_2(\boldsymbol{z}^*)$, $w^*=\gamma_3(\boldsymbol{z}^*)$, $d_2v^*-d_1u^*=\phi_1(\boldsymbol{z}^*)$ and $d_3w^*-d_1u^*=\phi_2(\boldsymbol{z}^*)$ a.e. Since f_i $(i=1,2,3)$ are Lipschitz continuous, we see that $f_i(u^{k_n},v^{k_n},w^{k_n})$ converge to $f_i(\gamma_1(\boldsymbol{z}^*),\gamma_2(\boldsymbol{z}^*),\gamma_3(\boldsymbol{z}^*))$ strongly in $L^2(Q)$ and a.e. in Q as k_n tends to infinity.

Next, we show that $\boldsymbol{z}^*$ is a weak solution of (ND). Taking $\zeta\in H^1(Q)$ with $\zeta(\cdot,T)=0$ in (2.5) and integrating by parts yield

$$\int_0^T \left\langle z_1^k, \frac{\partial\zeta}{\partial t}\right\rangle dt + \left\langle v_0^k-u_0^k, \zeta(\cdot,0)\right\rangle$$
$$= \int_0^T \left\langle \nabla\big(d_2v^k-d_1u^k\big), \nabla\zeta\right\rangle dt - \int_0^T \left\langle f_2\big(u^k,v^k,w^k\big)-f_1\big(u^k,v^k,w^k\big), \zeta\right\rangle dt.$$

Passing to the limit along the subsequences, we have

$$\int_0^T \left\langle z_1^*, \frac{\partial\zeta}{\partial t}\right\rangle dt + \left\langle z_{01}, \zeta(\cdot,0)\right\rangle = \int_0^T \left\langle \nabla\phi_1\big(\boldsymbol{z}^*\big), \nabla\zeta\right\rangle dt - \int_0^T \left\langle F_1\big(\boldsymbol{z}^*\big), \zeta\right\rangle dt.$$

Analogously, we deduce from (2.4) that

$$\int_0^T \left\langle z_2^*, \frac{\partial\zeta}{\partial t}\right\rangle dt + \left\langle z_{02}, \zeta(\cdot,0)\right\rangle = \int_0^T \left\langle \nabla\phi_2\big(\boldsymbol{z}^*\big), \nabla\zeta\right\rangle dt - \int_0^T \left\langle F_2\big(\boldsymbol{z}^*\big), \zeta\right\rangle dt$$

for all functions $\zeta\in H^1(Q)$ with $\zeta(\cdot,T)=0$. Thus, we observe that $\boldsymbol{z}^*$ is a weak solution of (ND). □

If the weak solution of (ND) is unique, then we do not need to take subsequences in Theorem 3.2. Indeed, we will prove the uniqueness of the weak solutions under some assumptions on the diffusion coefficients in the following subsection.

3.2. Uniqueness of the weak solutions of the limiting system (ND)

Here and in the next subsection, we impose the following conditions on the diffusion coefficients.

(H3) d_1 is the largest number among d_j $(j=1,2,3)$ and

$$\begin{cases} (d_1-d_2)^2 < 4d_2d_3, \qquad (d_1-d_3)^2 < 4d_2d_3, \\ \sqrt{d_2d_3}(d_1-d_2) < d_2d_3 + \dfrac{(d_1-d_2)^2}{4} - \dfrac{(d_1-d_3)^2}{4}; \end{cases} \tag{3.3}$$

or

(H4) $d_1=d_2$.

Assumption (H3) looks rather complicated, but it is satisfied if

$$\frac{1}{2}\max_{j=1,2,3} d_j < d_i \leqslant \max_{j=1,2,3} d_j \quad (i=1,2,3).$$

We deduce from (3.3) that there exist positive constants α_i $(i = 1, \dots, 4)$ satisfying

$$\begin{cases} d_2 - \dfrac{d_1 - d_3}{2\alpha_1} > 0, \qquad d_3 - \dfrac{(d_1 - d_3)\alpha_1}{2} > 0, \\ d_2 - \dfrac{d_1 - d_2}{2\alpha_2} > 0, \qquad d_3 - \dfrac{(d_1 - d_2)\alpha_2}{2} > 0, \\ d_2 - \dfrac{d_1 - d_2}{2\alpha_3} - \dfrac{d_1 - d_3}{2\alpha_4} > 0, \\ d_3 - \dfrac{(d_1 - d_2)\alpha_3}{2} - \dfrac{(d_1 - d_3)\alpha_4}{2} > 0. \end{cases} \tag{3.4}$$

Indeed the existence of α_1 and α_2 immediately follows from the first two inequalities of (3.3). The last two inequalities of (3.4) imply that

$$\frac{(d_1 - d_3)}{2d_2 - \frac{(d_1 - d_2)}{\alpha_3}} < \alpha_4 < \frac{2d_3 - (d_1 - d_2)\alpha_3}{d_1 - d_3}.$$

The inequalities (3.3) guarantee the existence of α_3 and α_4.

Assumption (H4) implies that the system (ND) is weakly coupled. These assumptions are necessary to show the uniqueness in our proof. We postpone to future work the problem of relaxing these assumptions.

Theorem 3.3. *Suppose that* (H1) *and* (H2) *hold. Moreover, we assume that* (H3) *or* (H4) *is satisfied. Let $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$ be weak solutions of* (ND) *with initial data $\boldsymbol{z}_0$ and $\bar{\boldsymbol{z}}_0$, respectively. Then there exists a positive constant C_5 independent of the data such that*

$$\|\boldsymbol{z} - \bar{\boldsymbol{z}}\|_{L^2(Q)^2} + \left\| \int_0^t \big(\boldsymbol{\phi}(\boldsymbol{z}) - \boldsymbol{\phi}(\bar{\boldsymbol{z}})\big)\, d\tau \right\|_{L^\infty(0,T;H^1(\Omega))^2} \leqslant C_5 \|\boldsymbol{z}_0 - \bar{\boldsymbol{z}}_0\|_{L^2(\Omega)^2}.$$

Theorem 3.3 guarantees the uniqueness of the weak solutions of (ND) and that $\{u^k\}$, $\{v^k\}$ and $\{w^k\}$ converge to $\gamma_1(\boldsymbol{z})$, $\gamma_2(\boldsymbol{z})$ and $\gamma_3(\boldsymbol{z})$ respectively as k tends to infinity if (H3) or (H4) holds.

To show Theorem 3.3, we need to estimate

$$A = A(\boldsymbol{z}, \bar{\boldsymbol{z}}) := (z_1 - \bar{z}_1)\big(\phi_1(\boldsymbol{z}) - \phi_1(\bar{\boldsymbol{z}})\big) + (z_2 - \bar{z}_2)\big(\phi_2(\boldsymbol{z}) - \phi_2(\bar{\boldsymbol{z}})\big),$$
$$\widetilde{A} = \widetilde{A}(\boldsymbol{z}, \bar{\boldsymbol{z}}) := (z_2 - \bar{z}_2)\big(\phi_2(\boldsymbol{z}) - \phi_2(\bar{\boldsymbol{z}})\big).$$

We present the following lemma.

Lemma 3.4. *If* (H3) *holds, then there exists a positive constant C_6 such that*

$$A(\boldsymbol{z}, \bar{\boldsymbol{z}}) \geqslant C_6 |\boldsymbol{z} - \bar{\boldsymbol{z}}|^2$$

for all $\boldsymbol{z}, \bar{\boldsymbol{z}} \in \mathbb{R}^2$.

If (H4) *holds, then there exist positive constants C_7 and C_8 such that*

$$\widetilde{A}(\boldsymbol{z}, \bar{\boldsymbol{z}}) \geqslant C_7 |z_2 - \bar{z}_2|^2 - C_8 |z_1 - \bar{z}_1|^2$$

for all $\boldsymbol{z}, \bar{\boldsymbol{z}} \in \mathbb{R}^2$.

Proof. Using φ defined in Section 2, we can rewrite A as

$$\begin{aligned} A &= d_2 |z_1 - \bar{z}_1|^2 + (d_1 - d_2)(z_1 - \bar{z}_1)\big(\varphi(\boldsymbol{z}) - \varphi(\bar{\boldsymbol{z}})\big) \\ &\quad + d_3 |z_2 - \bar{z}_2|^2 + (d_1 - d_3)(z_2 - \bar{z}_2)\big(\varphi(\boldsymbol{z}) - \varphi(\bar{\boldsymbol{z}})\big) \\ &= d_2 |z_1 - \bar{z}_1|^2 + d_3 |z_2 - \bar{z}_2|^2 + B, \end{aligned}$$

where

$$B := (d_1 - d_2)(z_1 - \bar{z}_1)\big(\varphi(\boldsymbol{z}) - \varphi(\bar{\boldsymbol{z}})\big) + (d_1 - d_3)(z_2 - \bar{z}_2)\big(\varphi(\boldsymbol{z}) - \varphi(\bar{\boldsymbol{z}})\big).$$

First we assume (H3). If $\boldsymbol{z}, \bar{\boldsymbol{z}} \in I$, then we have $B = 0$ and

$$A = d_2 |z_1 - \bar{z}_1|^2 + d_3 |z_2 - \bar{z}_2|^2.$$

If $\boldsymbol{z} \in I$ and $\bar{\boldsymbol{z}} \in II$, that is, $z_1 > 0$, $z_2 \geqslant 0$, $\bar{z}_1 \leqslant 0$, $\bar{z}_2 > \bar{z}_1$, then

$$\begin{aligned} B &= (d_1-d_2)(z_1-\bar{z}_1)(-\bar{z}_1)+(d_1-d_3)(z_2-\bar{z}_2)(-\bar{z}_1) \\ &\geqslant -(d_1-d_3)|z_2-\bar{z}_2||z_1-\bar{z}_1|, \end{aligned}$$

which implies

$$\begin{aligned} A &\geqslant d_2|z_1-\bar{z}_1|^2+d_3|z_2-\bar{z}_2|^2-(d_1-d_3)|z_1-\bar{z}_1||z_2-\bar{z}_2| \\ &\geqslant \big(d_2-(d_1-d_3)/(2\alpha_1)\big)|z_1-\bar{z}_1|^2+\big(d_3-(d_1-d_3)\alpha_1/2\big)|z_2-\bar{z}_2|^2, \end{aligned} \tag{3.5}$$

where α_1 is a positive constant satisfying (3.4).

Similarly we can treat the case where $\boldsymbol{z} \in I$ and $\bar{\boldsymbol{z}} \in III$, i.e., $z_1>0$, $z_2 \geqslant 0$, $\bar{z}_2<0$, $\bar{z}_2<\bar{z}_1$. Since

$$\begin{aligned} B &= (d_1-d_2)(z_1-\bar{z}_1)(-\bar{z}_2)+(d_1-d_3)(z_2-\bar{z}_2)(-\bar{z}_2) \\ &\geqslant -(d_1-d_2)|z_1-\bar{z}_1||z_2-\bar{z}_2|, \end{aligned}$$

we obtain

$$A \geqslant \big(d_2-(d_1-d_2)/(2\alpha_2)\big)|z_1-\bar{z}_1|^2+\big(d_3-(d_1-d_2)\alpha_2/2\big)|z_2-\bar{z}_2|^2 \tag{3.6}$$

with a positive constant α_2 as in (3.4).

Since

$$\begin{aligned} B &= (d_1-d_2)(z_1-\bar{z}_1)(z_1-\bar{z}_1)+(d_1-d_3)(z_2-\bar{z}_2)(z_1-\bar{z}_1) \\ &\geqslant (d_1-d_2)|z_1-\bar{z}_1|^2-(d_1-d_3)|z_1-\bar{z}_1||z_2-\bar{z}_2| \end{aligned}$$

for the case where $\boldsymbol{z}, \bar{\boldsymbol{z}} \in II$, we have (3.5).

If $\boldsymbol{z} \in II$ and $\bar{\boldsymbol{z}} \in III$, that is, $z_1 \leqslant 0$, $z_1<z_2$, $\bar{z}_2<0$, $\bar{z}_1 \geqslant \bar{z}_2$, then

$$z_2-z_1 \leqslant \bar{z}_1-z_1+z_2-\bar{z}_2, \qquad \bar{z}_1-\bar{z}_2 \leqslant \bar{z}_1-z_1+z_2-\bar{z}_2.$$

Therefore, we have

$$\begin{aligned} B &= (d_1-d_2)(z_1-\bar{z}_1)(z_1-\bar{z}_2)+(d_1-d_3)(z_2-\bar{z}_2)(z_1-\bar{z}_2) \\ &= (d_1-d_2)|z_1-\bar{z}_1|^2+(d_1-d_2)(z_1-\bar{z}_1)(\bar{z}_1-\bar{z}_2) \\ &\quad +(d_1-d_3)|z_2-\bar{z}_2|^2+(d_1-d_3)(z_2-\bar{z}_2)(z_1-z_2) \\ &\geqslant (d_1-d_2)|z_1-\bar{z}_1|^2-(d_1-d_2)|z_1-\bar{z}_1|\big(|z_1-\bar{z}_1|+|z_2-\bar{z}_2|\big) \\ &\quad +(d_1-d_3)|z_2-\bar{z}_2|^2-(d_1-d_3)|z_2-\bar{z}_2|\big(|z_1-\bar{z}_1|+|z_2-\bar{z}_2|\big) \\ &= -(d_1-d_2+d_1-d_3)|z_1-\bar{z}_1||z_2-\bar{z}_2|. \end{aligned}$$

Then taking α_3 and α_4 as in (3.4) implies

$$\begin{aligned} A &\geqslant \big(d_2-(d_1-d_2)/(2\alpha_3)-(d_1-d_3)/(2\alpha_4)\big)|z_1-\bar{z}_1|^2 \\ &\quad +\big(d_3-(d_1-d_2)\alpha_3/2-(d_1-d_3)\alpha_4/2\big)|z_2-\bar{z}_2|^2. \end{aligned}$$

If $\boldsymbol{z}, \bar{\boldsymbol{z}} \in III$, then

$$B \geqslant (d_1-d_3)|z_2-\bar{z}_2|^2-(d_1-d_2)|z_1-\bar{z}_1||z_2-\bar{z}_2|,$$

which implies (3.6).

Since $A(\boldsymbol{z}, \bar{\boldsymbol{z}})=A(\bar{\boldsymbol{z}}, \boldsymbol{z})$, we have shown the first statement of the lemma.

Next, we show the last statement under the assumption (H4). We note that

$$\widetilde{A}(\boldsymbol{z}, \bar{\boldsymbol{z}})=d_3|z_2-\bar{z}_2|^2+(d_1-d_3)(z_2-\bar{z}_2)\big(\varphi(\boldsymbol{z})-\varphi(\bar{\boldsymbol{z}})\big).$$

If $\boldsymbol{z}, \bar{\boldsymbol{z}} \in I$ or $\boldsymbol{z} \in I$, $\bar{\boldsymbol{z}} \in III$ or $\boldsymbol{z} \in III$, $\bar{\boldsymbol{z}} \in I$ or $\boldsymbol{z}, \bar{\boldsymbol{z}} \in III$, then $\widetilde{A} \geqslant \min\{d_1, d_3\}|z_2-\bar{z}_2|^2$. In the other cases, we can obtain

$$\widetilde{A} \geqslant d_3|z_2-\bar{z}_2|^2-|d_1-d_3||z_1-\bar{z}_1||z_2-\bar{z}_2| \geqslant \frac{d_3}{2}|z_2-\bar{z}_2|^2-\frac{(d_1-d_3)^2}{2d_3}|z_1-\bar{z}_1|^2,$$

which completes the proof. □

Proof of Theorem 3.3. From Definition 3.1, the weak solutions $\boldsymbol{z}$ and $\bar{\boldsymbol{z}}$ satisfy

$$-\int_0^T \left\langle z_i - \bar{z}_i, \frac{\partial \zeta_i}{\partial t} \right\rangle dt + \int_0^T \left\langle \nabla\big(\phi_i(\boldsymbol{z}) - \phi_i(\bar{\boldsymbol{z}})\big), \nabla\zeta_i \right\rangle dt = \left\langle z_{0i} - \bar{z}_{0i}, \zeta_i(\cdot,0)\right\rangle + \int_0^T \left\langle F_i(\boldsymbol{z}) - F_i(\bar{\boldsymbol{z}}), \zeta_i\right\rangle dt \tag{3.7}$$

for all functions $\boldsymbol{\zeta} = (\zeta_1, \zeta_2) \in H^1(Q)^2$ with $\zeta_i(\cdot, T) = 0$ and for $i = 1, 2$. Take

$$\zeta_i(x,t) = \begin{cases} \displaystyle\int_t^{t_0} \Big(\phi_i\big(\boldsymbol{z}(x,\tau)\big) - \phi_i\big(\bar{\boldsymbol{z}}(x,\tau)\big)\Big)\, d\tau & \text{for } 0 \leqslant t < t_0, \\ 0 & \text{for } t_0 \leqslant t \leqslant T, \end{cases}$$

where t_0 is an arbitrary point in $(0, T)$. The first term of the left-hand side of (3.7) is

$$-\int_0^T \left\langle z_i - \bar{z}_i, \frac{\partial \zeta_i}{\partial t}\right\rangle dt = \int_0^{t_0} \left\langle z_i - \bar{z}_i, \phi_i(\boldsymbol{z}) - \phi_i(\bar{\boldsymbol{z}})\right\rangle dt. \tag{3.8}$$

The second term of the left-hand side of (3.7) can be estimated easily as follows:

$$\int_0^T \left\langle \nabla\big(\phi_i(\boldsymbol{z}) - \phi_i(\bar{\boldsymbol{z}})\big), \nabla\zeta_i\right\rangle dt = \frac{1}{2}\left\| \nabla \int_0^{t_0} \big(\phi_i(\boldsymbol{z}) - \phi_i(\bar{\boldsymbol{z}})\big)\, dt \right\|^2_{L^2(\Omega)}. \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7), we have

$$\begin{aligned}
&\int_0^{t_0} \left\langle z_i - \bar{z}_i, \phi_i(\boldsymbol{z}) - \phi_i(\bar{\boldsymbol{z}})\right\rangle dt + \frac{1}{2}\left\| \nabla \int_0^{t_0} \big(\phi_i(\boldsymbol{z}) - \phi_i(\bar{\boldsymbol{z}})\big)\, dt \right\|^2_{L^2(\Omega)} \\
&\quad = \left\langle z_{0i} - \bar{z}_{0i}, \zeta_i(\cdot,0)\right\rangle + \int_0^T \left\langle F_i(\boldsymbol{z}) - F_i(\bar{\boldsymbol{z}}), \zeta_i\right\rangle dt \\
&\quad \leqslant L_\phi \|z_{0i} - \bar{z}_{0i}\|_{L^2(\Omega)} \int_0^{t_0} \big\|\boldsymbol{z}(\tau) - \bar{\boldsymbol{z}}(\tau)\big\|_{L^2(\Omega)^2}\, d\tau \\
&\qquad + L_F L_\phi \int_0^{t_0}\int_0^t \big\|\boldsymbol{z}(\tau) - \bar{\boldsymbol{z}}(\tau)\big\|_{L^2(\Omega)^2} \big\|\boldsymbol{z}(t) - \bar{\boldsymbol{z}}(t)\big\|_{L^2(\Omega)^2}\, d\tau\, dt,
\end{aligned}$$

where L_ϕ and L_F are the Lipschitz constants of $\boldsymbol{\phi}$ and $\boldsymbol{F}$, respectively. Using the Schwarz inequality yields

$$\begin{aligned}
&\int_0^{t_0} \left\langle z_i - \bar{z}_i, \phi_i(\boldsymbol{z}) - \phi_i(\bar{\boldsymbol{z}})\right\rangle dt + \frac{1}{2}\left\| \nabla \int_0^{t_0} \big(\phi_i(\boldsymbol{z}) - \phi_i(\bar{\boldsymbol{z}})\big)\, dt \right\|^2_{L^2(\Omega)} \\
&\quad \leqslant \frac{L_\phi^2 T}{2\beta_1}\|z_{0i} - \bar{z}_{0i}\|^2_{L^2(\Omega)} + \frac{(L_F L_\phi)^2 T}{2\beta_2}\int_0^{t_0} \|\boldsymbol{z} - \bar{\boldsymbol{z}}\|^2_{L^2(0,t;L^2(\Omega))^2}\, dt \\
&\qquad + \frac{\beta_1 + \beta_2}{2}\|\boldsymbol{z} - \bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2},
\end{aligned} \tag{3.10}$$

where β_i ($i = 1, 2$) are positive constants specified later.

Now, we prove the theorem in the case where (H3) holds. Lemma 3.4 implies

$$\sum_{i=1}^2 \int_0^{t_0} \left\langle z_i - \bar{z}_i, \phi_i(\boldsymbol{z}) - \phi_i(\bar{\boldsymbol{z}})\right\rangle dt = \int_0^{t_0} A(\boldsymbol{z}, \bar{\boldsymbol{z}})\, dt \geqslant C_6 \|\boldsymbol{z} - \bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2}. \tag{3.11}$$

Choose positive constants β_i satisfying $\beta_1+\beta_2=C_6/2$. Summing (3.10) over $i=1,2$ and using (3.11), we have

$$\frac{C_6}{2}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2}+\frac{1}{2}\sum_{i=1}^{2}\left\|\nabla\int_0^{t_0}\big(\phi_i(\boldsymbol{z})-\phi_i(\bar{\boldsymbol{z}})\big)\,dt\right\|^2_{L^2(\Omega)}$$
$$\leqslant\frac{L_\phi^2T}{2\beta_1}\|\boldsymbol{z}_0-\bar{\boldsymbol{z}}_0\|^2_{L^2(\Omega)}+\frac{(L_FL_\phi)^2T}{\beta_2}\int_0^{t_0}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t;L^2(\Omega))^2}\,dt.$$

The desired estimate follows from the above inequality and the Gronwall lemma when (H3) holds.

Next, we consider the case where (H4) is satisfied instead of (H3). Since $d_1=d_2$, $\phi_1(\boldsymbol{s})=d_1s_1$ for all $\boldsymbol{s}\in\mathbb{R}^2$, and so

$$\int_0^{t_0}\big\langle z_1-\bar{z}_1,\phi_1(\boldsymbol{z})-\phi_1(\bar{\boldsymbol{z}})\big\rangle\,dt=d_1\|z_1-\bar{z}_1\|^2_{L^2(0,t_0;L^2(\Omega))}.$$

By (3.10) with $i=1$, we have

$$d_1\|z_1-\bar{z}_1\|^2_{L^2(0,t_0;L^2(\Omega))}+\frac{1}{2}\left\|\nabla\int_0^{t_0}\big(\phi_1(\boldsymbol{z})-\phi_1(\bar{\boldsymbol{z}})\big)\,dt\right\|^2_{L^2(\Omega)}$$
$$\leqslant\frac{L_\phi^2T}{2\beta_1}\|z_{01}-\bar{z}_{01}\|^2_{L^2(\Omega)}+\frac{(L_FL_\phi)^2T}{2\beta_2}\int_0^{t_0}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t;L^2(\Omega))^2}\,dt$$
$$+\frac{\beta_1+\beta_2}{2}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2}. \tag{3.12}$$

For $i=2$, we deduce from Lemma 3.4 and (3.10) with β_1,β_2 replaced by β_3,β_4 that

$$C_7\|z_2-\bar{z}_2\|^2_{L^2(0,t_0;L^2(\Omega))}+\frac{1}{2}\left\|\nabla\int_0^{t_0}\big(\phi_2(\boldsymbol{z})-\phi_2(\bar{\boldsymbol{z}})\big)\,dt\right\|^2_{L^2(\Omega)}$$
$$\leqslant C_8\|z_1-\bar{z}_1\|^2_{L^2(0,t_0;L^2(\Omega))}+\frac{L_\phi^2T}{2\beta_3}\|z_{02}-\bar{z}_{02}\|^2_{L^2(\Omega)}$$
$$+\frac{(L_FL_\phi)^2T}{2\beta_4}\int_0^{t_0}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t;L^2(\Omega))^2}\,dt+\frac{\beta_3+\beta_4}{2}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2}.$$

Substituting (3.12) into the above inequality yields

$$C_7\|z_2-\bar{z}_2\|^2_{L^2(0,t_0;L^2(\Omega))}+\frac{1}{2}\left\|\nabla\int_0^{t_0}\big(\phi_2(\boldsymbol{z})-\phi_2(\bar{\boldsymbol{z}})\big)\,dt\right\|^2_{L^2(\Omega)}$$
$$\leqslant\frac{C_8L_\phi^2T}{2d_1\beta_1}\|z_{01}-\bar{z}_{01}\|^2_{L^2(\Omega)}+\frac{C_8(L_FL_\phi)^2T}{2d_1\beta_2}\int_0^{t_0}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t;L^2(\Omega))^2}\,dt$$
$$+\frac{C_8(\beta_1+\beta_2)}{2d_1}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2}+\frac{L_\phi^2T}{2\beta_3}\|z_{02}-\bar{z}_{02}\|^2_{L^2(\Omega)}$$
$$+\frac{(L_FL_\phi)^2T}{2\beta_4}\int_0^{t_0}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t;L^2(\Omega))^2}\,dt+\frac{\beta_3+\beta_4}{2}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2}$$
$$\leqslant C_9\|\boldsymbol{z}_0-\bar{\boldsymbol{z}}_0\|^2_{L^2(\Omega)}+C_{10}\int_0^{t_0}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t;L^2(\Omega))^2}\,dt$$
$$+\left(\frac{C_8(\beta_1+\beta_2)}{2d_1}+\frac{\beta_3+\beta_4}{2}\right)\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2}, \tag{3.13}$$

where C_9 and C_{10} are positive constants independent of the solutions. By (3.12) and (3.13) we obtain

$$\begin{aligned}
&\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2}+\frac{1}{2d_1}\left\|\nabla\int_0^{t_0}\big(\phi_1(\boldsymbol{z})-\phi_1(\bar{\boldsymbol{z}})\big)\,dt\right\|^2_{L^2(\Omega)}+\frac{1}{2C_7}\left\|\nabla\int_0^{t_0}\big(\phi_2(\boldsymbol{z})-\phi_2(\bar{\boldsymbol{z}})\big)\,dt\right\|^2_{L^2(\Omega)}\\
&\quad\leqslant C_{11}\|\boldsymbol{z}_0-\bar{\boldsymbol{z}}_0\|^2_{L^2(\Omega)}+C_{12}\int_0^{t_0}\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t;L^2(\Omega))^2}\,dt\\
&\qquad+\left(\frac{\beta_1+\beta_2}{2d_1}+\frac{C_8(\beta_1+\beta_2)}{2d_1C_7}+\frac{\beta_3+\beta_4}{2C_7}\right)\|\boldsymbol{z}-\bar{\boldsymbol{z}}\|^2_{L^2(0,t_0;L^2(\Omega))^2}
\end{aligned}$$

for positive constants C_{11} and C_{12} independent of the solutions. If we choose positive constants β_j $(j=1,\dots,4)$ satisfying

$$\frac{\beta_1+\beta_2}{2d_1}+\frac{C_8(\beta_1+\beta_2)}{2d_1C_7}+\frac{\beta_3+\beta_4}{2C_7}<1,$$

then the last statement follows from the Gronwall lemma for the case where (H4) holds. □

3.3. Rate of convergence

Using similar arguments to those in the previous subsection, we can estimate the rate of convergence when (H3) or (H4) holds.

Theorem 3.5. *Suppose that* (H1) *and* (H2) *hold and that* (H3) *or* (H4) *is satisfied. Let* (u^k,v^k,w^k) *be a solution of* $(\mathrm{RD})^k$ *with an initial datum* (u^k_0,v^k_0,w^k_0) *and* $\boldsymbol{z}$ *a weak solution of* (ND) *with an initial datum* $\boldsymbol{z}_0$. *Put* $\boldsymbol{z}^k=(z^k_1,z^k_2):=(v^k-u^k,w^k-u^k)$ *and* $\boldsymbol{\phi}^k=(\phi^k_1,\phi^k_2):=(d_2v^k-d_1u^k,d_3w^k-d_1u^k)$. *Then there exists a positive constant* C_{13} *independent of k such that*

$$\begin{aligned}
&\big\|\gamma_1(\boldsymbol{z})-u^k\big\|_{L^2(Q)}+\big\|\gamma_2(\boldsymbol{z})-v^k\big\|_{L^2(Q)}+\big\|\gamma_3(\boldsymbol{z})-w^k\big\|_{L^2(Q)}+\big\|\boldsymbol{z}-\boldsymbol{z}^k\big\|_{L^2(Q)^2}\\
&\quad+\left\|\int_0^t\big(\boldsymbol{\phi}(\boldsymbol{z})-\boldsymbol{\phi}^k\big)\,d\tau\right\|_{L^\infty(0,T;H^1(\Omega))^2}\leqslant C_{13}\big(\big\|\boldsymbol{z}_0-\boldsymbol{z}^k_0\big\|_{L^2(\Omega)^2}+k^{-1/3}\big).
\end{aligned}$$

Proof. We prove the theorem in the case where (H3) is satisfied. We use the following notations: for $t\in(0,T)$

$$\begin{aligned}
&e_z(t):=\big\|\boldsymbol{z}-\boldsymbol{z}^k\big\|^2_{L^2(0,t;L^2(\Omega))^2},\\
&e_\gamma(t):=\big\|\gamma_1(\boldsymbol{z})-u^k\big\|^2_{L^2(0,t;L^2(\Omega))}+\big\|\gamma_2(\boldsymbol{z})-v^k\big\|^2_{L^2(0,t;L^2(\Omega))}+\big\|\gamma_3(\boldsymbol{z})-w^k\big\|^2_{L^2(0,t;L^2(\Omega))}.
\end{aligned}$$

Using the definition of γ_1 and Lemma 2.5, we have

$$\begin{aligned}
\big\|\gamma_1(\boldsymbol{z})-u^k\big\|_{L^2(0,t;L^2(\Omega))}&\leqslant\big\|\gamma_1(\boldsymbol{z})-\gamma_1\big(\boldsymbol{z}^k\big)\big\|_{L^2(0,t;L^2(\Omega))}+\big\|\gamma_1\big(\boldsymbol{z}^k\big)-u^k\big\|_{L^2(0,t;L^2(\Omega))}\\
&\leqslant\sqrt{5}\big\|\boldsymbol{z}-\boldsymbol{z}^k\big\|_{L^2(0,t;L^2(\Omega))^2}+C_4k^{-1/3},
\end{aligned}$$

which implies

$$\big\|\gamma_1(\boldsymbol{z})-u^k\big\|^2_{L^2(0,t;L^2(\Omega))}\leqslant10\big\|\boldsymbol{z}-\boldsymbol{z}^k\big\|^2_{L^2(0,t;L^2(\Omega))^2}+2C_4^2k^{-2/3}.$$

Similar calculations for $\gamma_2(\boldsymbol{z})-v^k$ and $\gamma_3(\boldsymbol{z})-w^k$ yield

$$e_\gamma(t)\leqslant30e_z(t)+6C_4^2k^{-2/3}.\tag{3.14}$$

Since

$$\begin{aligned}
&\big\|\phi_1(\boldsymbol{z})-\phi^k_1\big\|_{L^2(0,t;L^2(\Omega))}\leqslant d_1\big\|\gamma_1(\boldsymbol{z})-u^k\big\|_{L^2(0,t;L^2(\Omega))}+d_2\big\|\gamma_2(\boldsymbol{z})-v^k\big\|_{L^2(0,t;L^2(\Omega))},\\
&\big\|\phi_2(\boldsymbol{z})-\phi^k_2\big\|_{L^2(0,t;L^2(\Omega))}\leqslant d_1\big\|\gamma_1(\boldsymbol{z})-u^k\big\|_{L^2(0,t;L^2(\Omega))}+d_3\big\|\gamma_3(\boldsymbol{z})-w^k\big\|_{L^2(0,t;L^2(\Omega))},
\end{aligned}$$

we note that

$$\big\|\phi_i(\boldsymbol{z})-\phi^k_i\big\|^2_{L^2(0,t;L^2(\Omega))}\leqslant2d_1^2e_\gamma(t)$$

for $i=1,2$.

We see, from Definition 3.1 and (2.3) and (2.4), that $\boldsymbol{z}$ and $\boldsymbol{z}^k$ satisfy

$$-\int_0^T \left\langle z_i - z_i^k, \frac{\partial \zeta_i}{\partial t}\right\rangle dt + \int_0^T \left\langle \nabla\left(\phi_i(\boldsymbol{z}) - \phi_i^k\right), \nabla\zeta_i\right\rangle dt = \left\langle z_{0i} - z_{0i}^k, \zeta_i(\cdot,0)\right\rangle + \int_0^T \left\langle F_i(\boldsymbol{z}) - F_i^k, \zeta_i\right\rangle dt \tag{3.15}$$

for all functions $\boldsymbol{\zeta} \in H^1(Q)^2$ with $\zeta_i(\cdot,T) = 0$ and for $i = 1,2$. Here, we define $F_1^k := f_2(u^k, v^k, w^k) - f_1(u^k, v^k, w^k)$, $F_2^k := f_3(u^k, v^k, w^k) - f_1(u^k, v^k, w^k)$.

Set

$$\zeta_i(x,t) = \begin{cases} \displaystyle\int_t^{t_0} \left(\phi_i\left(\boldsymbol{z}(x,\tau)\right) - \phi_i^k(x,\tau)\right) d\tau & \text{for } 0 \leqslant t < t_0, \\ 0 & \text{for } t_0 \leqslant t \leqslant T, \end{cases}$$

where t_0 is an arbitrary point in $(0,T)$. The first term of the left-hand side of (3.15) is

$$-\int_0^T \left\langle z_i - z_i^k, \frac{\partial \zeta_i}{\partial t}\right\rangle dt = \int_0^{t_0} \left\langle z_i - z_i^k, \phi_i(\boldsymbol{z}) - \phi_i\left(\boldsymbol{z}^k\right)\right\rangle dt + \int_0^{t_0} \left\langle z_i - z_i^k, \phi_i\left(\boldsymbol{z}^k\right) - \phi_i^k\right\rangle dt.$$

Lemma 3.4 or (3.11) implies

$$\sum_{i=1}^2 \int_0^{t_0} \left\langle z_i - z_i^k, \phi_i(\boldsymbol{z}) - \phi_i\left(\boldsymbol{z}^k\right)\right\rangle dt \geqslant C_6 \left\|\boldsymbol{z} - \boldsymbol{z}^k\right\|^2_{L^2(0,t_0;L^2(\Omega))^2} = C_6 e_z(t_0).$$

Using (3.14), we have

$$\sum_{i=1}^2 \int_0^{t_0} \left\langle z_i - z_i^k, \phi_i(\boldsymbol{z}) - \phi_i\left(\boldsymbol{z}^k\right)\right\rangle dt \geqslant \frac{C_6}{60} e_\gamma(t_0) - \frac{C_4^2 C_6}{10} k^{-2/3} + \frac{C_6}{2} e_z(t_0).$$

By Lemma 2.5 we also have

$$\begin{aligned}\sum_{i=1}^2 \int_0^{t_0} \left\langle z_i - z_i^k, \phi_i\left(\boldsymbol{z}^k\right) - \phi_i^k\right\rangle dt &\leqslant \sum_{i=1}^2 \left\|z_i - z_i^k\right\|_{L^2(0,t_0;L^2(\Omega))} \left\|\phi\left(\boldsymbol{z}^k\right) - \phi_i^k\right\|_{L^2(0,t_0;L^2(\Omega))} \\ &\leqslant \frac{C_6}{4} e_z(t_0) + \frac{C_4^2 |\Omega|^{1/3} T^{1/3}}{C_6} k^{-2/3}.\end{aligned}$$

The second term of the left-hand side of (3.15) can be rewritten as

$$\int_0^T \left\langle \nabla\left(\phi_i(\boldsymbol{z}) - \phi_i^k\right), \nabla\zeta_i\right\rangle dt = \frac{1}{2}\left\|\nabla \int_0^{t_0} \left(\phi_i(\boldsymbol{z}) - \phi_i^k\right) dt\right\|^2_{L^2(\Omega)}.$$

Collecting these inequalities yields

$$\begin{aligned}&-\sum_{i=1}^2 \int_0^T \left\langle z_i - z_i^k, \frac{\partial \zeta_i}{\partial t}\right\rangle dt + \sum_{i=1}^2 \int_0^T \left\langle \nabla\left(\phi_i(\boldsymbol{z}) - \phi_i^k\right), \nabla\zeta_i\right\rangle dt \\ &\quad\geqslant \frac{C_6}{60} e_\gamma(t_0) + \frac{C_6}{4} e_z(t_0) + \frac{1}{2}\sum_{i=1}^2 \left\|\nabla \int_0^{t_0} \left(\phi_i(\boldsymbol{z}) - \phi_i^k\right) dt\right\|^2_{L^2(\Omega)} \\ &\quad\quad - \sum_{i=1}^2 \left(\frac{C_4^2 C_6}{10} + \frac{C_4^2 |\Omega|^{1/3} T^{1/3}}{C_6}\right) k^{-2/3}.\end{aligned} \tag{3.16}$$

Next, we consider the right-hand side of (3.15). By the definition of ϕ_i^k, we also get

$$\langle z_{0i} - z_{0i}^k, \zeta_i(\cdot,0)\rangle \leqslant \frac{60 d_1^2 T}{C_6}\|z_{0i} - z_{0i}^k\|_{L^2(\Omega)}^2 + \frac{C_6}{120} e_\gamma(t_0). \tag{3.17}$$

In view of the Lipschitz continuities of f_i, the last term of (3.15) can be estimated as follows:

$$\begin{aligned}\int_0^T \langle F_i(\boldsymbol{z}) - F_i^k, \zeta_i\rangle\,dt &= \int_0^{t_0}\left\langle \int_0^t \big(F_i(\boldsymbol{z}(\tau)) - F_i^k(\tau)\big)\,d\tau, \phi_i(\boldsymbol{z}(t)) - \phi_i^k(t)\right\rangle dt \\ &\leqslant C_{14}\int_0^{t_0} e_\gamma(t)\,dt + \frac{C_6}{240} e_\gamma(t_0),\end{aligned} \tag{3.18}$$

where C_{14} is a positive constant independent of k. Collecting (3.15)–(3.18) yields

$$\begin{aligned}&e_\gamma(t_0) + e_z(t_0) + \sum_{i=1}^2 \left\|\nabla \int_0^{t_0} \big(\phi_i(\boldsymbol{z}) - \phi_i^k\big)\,dt\right\|_{L^2(\Omega)}^2 \\ &\quad\leqslant C_{15}\left(\sum_{i=1}^2 \|z_{0i} - z_{0i}^k\|_{L^2(\Omega)}^2 + k^{-2/3} + \int_0^{t_0} e_\gamma(t)\,dt\right),\end{aligned}$$

where C_{15} is a constant independent of k. From the Gronwall inequality, the desired estimate is obtained.

Combining the above proof with the ideas in the proof of Theorem 3.3, we obtain the result in the case where (H4) holds.

4. Free boundaries

In the previous section, we have studied the weak solutions of (ND). By (3.2), free boundaries appear in the strong sense of the limit problem (ND) as k tends to infinity. In this section, we examine conditions on the free boundaries. Let $\boldsymbol{z}$ be a weak solution of (ND). Set three regions:

$$\begin{aligned}\Omega_1(t) &:= \{x \in \Omega \mid \gamma_2(\boldsymbol{z}(x,t)) > 0,\ \gamma_3(\boldsymbol{z}(x,t)) > 0\},\\ \Omega_2(t) &:= \{x \in \Omega \mid \gamma_3(\boldsymbol{z}(x,t)) > 0,\ \gamma_1(\boldsymbol{z}(x,t)) > 0\},\\ \Omega_3(t) &:= \{x \in \Omega \mid \gamma_1(\boldsymbol{z}(x,t)) > 0,\ \gamma_2(\boldsymbol{z}(x,t)) > 0\}.\end{aligned}$$

We also define Q_i, $\Gamma_i(t)$ and Γ_i $(i = 1,2,3)$ as in Section 1. To avoid some difficulties such as the appearance of multiple junctions among Γ_i, we introduce $\widetilde{\Gamma}_i$ and S as follows:

$$\widetilde{\Gamma}_1 := \left\{(x,t) \in \Gamma_1 \;\middle|\; \begin{array}{l}\text{there is a neighbourhood } D \text{ of } (x,t) \text{ such that}\\ \quad D = (Q_2 \cup Q_3 \cup \Gamma_1) \cap D\\ \quad \text{and } \Gamma_1 \text{ is an } (N-1)\text{-dimensional smooth hypersurface in } D\end{array}\right\}.$$

The interfaces $\widetilde{\Gamma}_2$ and $\widetilde{\Gamma}_3$ are similarly defined. Thus, $\widetilde{\Gamma}_i$ do not include multiple junction points. We recall the definition of n_i, which is the unit normal vector on $\Gamma_i(t)$ oriented from $\Omega_j(t)$ to $\Omega_k(t)$ for $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$. Let S be a set of all points $(x,t) \in \partial\Omega \times (0,T)$ so that there exists a normal segment at (x,t) which is located in $\overline{\Omega_i(t)}$ for some $i \in \{1,2,3\}$.

Now, we are ready to state our result.

Theorem 4.1. *Assume that* (H2) *holds. Let $\boldsymbol{z}$ be a weak solution of* (ND) *with an initial datum $\boldsymbol{z}_0 \in L^\infty(\Omega)^2$. Suppose that the functions $u = \gamma_1(\boldsymbol{z})$, $v = \gamma_2(\boldsymbol{z})$ and $w = \gamma_3(\boldsymbol{z})$ are smooth on $\overline{Q_1}$, $\overline{Q_2}$ and $\overline{Q_3}$. Also assume that Q_i are* (*piecewise*) *smooth. Then, u, v and w satisfy* (1.4)–(1.7) *and*

$$d_2\frac{\partial v|_{Q_3}}{\partial n_1} + d_3\frac{\partial w|_{Q_2}}{\partial n_1} = 0, \qquad d_1\left(\frac{\partial u|_{Q_3}}{\partial n_1} - \frac{\partial u|_{Q_2}}{\partial n_1}\right) = d_2\frac{\partial v|_{Q_3}}{\partial n_1} \quad \text{on } \widetilde{\Gamma}_1, \tag{4.1}$$

$$d_3\frac{\partial w|_{Q_1}}{\partial n_2} + d_1\frac{\partial u|_{Q_3}}{\partial n_2} = 0, \qquad d_2\left(\frac{\partial v|_{Q_1}}{\partial n_2} - \frac{\partial v|_{Q_3}}{\partial n_2}\right) = d_3\frac{\partial w|_{Q_1}}{\partial n_2} \quad \text{on } \widetilde{\Gamma}_2, \tag{4.2}$$

$$d_1\frac{\partial u|_{Q_2}}{\partial n_3}+d_2\frac{\partial v|_{Q_1}}{\partial n_3}=0, \qquad d_3\left(\frac{\partial w|_{Q_2}}{\partial n_3}-\frac{\partial w|_{Q_1}}{\partial n_3}\right)=d_1\frac{\partial u|_{Q_2}}{\partial n_3} \quad \text{on } \widetilde{\Gamma}_3, \tag{4.3}$$

$$\frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=\frac{\partial w}{\partial \nu}=0 \quad \text{on } S, \tag{4.4}$$

$$u(\cdot,0)=\gamma_1(\boldsymbol{z}_0), \qquad v(\cdot,0)=\gamma_2(\boldsymbol{z}_0), \qquad w(\cdot,0)=\gamma_3(\boldsymbol{z}_0) \quad \text{in } \Omega. \tag{4.5}$$

Proof. The conditions (1.7) and (4.5) follow immediately from the definitions of Γ_i and u, v, w. The boundary condition in (ND) and the definition of ϕ_i imply

$$d_1\frac{\partial u}{\partial \nu}=d_2\frac{\partial v}{\partial \nu}=d_3\frac{\partial w}{\partial \nu}$$

on $\partial\Omega\times(0,T)$. Noticing that $u=0$ in $\overline{Q_1}$, $v=0$ in $\overline{Q_2}$ and $w=0$ in $\overline{Q_3}$ yields (4.4).

Next, we show that v, w satisfy (1.4). Since $z_1=v$, $z_2=w$ and $u=0$ in Q_1, it follows from Definition 3.1 that for all $\zeta\in C_0^\infty(Q_1)$

$$\iint_{Q_1}\left(\frac{\partial v}{\partial t}-d_2\Delta v-f_2(0,v,w)\right)\zeta\,dx\,dt=0,$$

$$\iint_{Q_1}\left(\frac{\partial w}{\partial t}-d_3\Delta w-f_3(0,v,w)\right)\zeta\,dx\,dt=0,$$

which imply (1.4). The same arguments lead us to Eqs. (1.5) and (1.6) satisfied in Q_2 and Q_3.

Finally, we derive the free boundary conditions (4.1)–(4.3). For $(x_1,t_1)\in\widetilde{\Gamma}_1$, there is a cylinder $D=B(x_1,r)\times(t_1-r, t_1+r)\subset Q$ such that $D=(Q_2\cup Q_3\cup\Gamma_1)\cap D$ and Γ_1 is a smooth hypersurface in D. Noting that $z_1=-u$, $\phi_1(\boldsymbol{z})=-d_1u$ in Q_2 and $z_1=v-u$, $\phi_1(\boldsymbol{z})=d_2v-d_1u$ in Q_3, we deduce from Definition 3.1 for $i=1$ that

$$\begin{aligned}
&-\iint_{D\cap Q_2}u\frac{\partial\zeta}{\partial t}\,dx\,dt+\iint_{D\cap Q_3}(v-u)\frac{\partial\zeta}{\partial t}\,dx\,dt\\
&\quad=-d_1\iint_{D\cap Q_2}\nabla u\cdot\nabla\zeta\,dx\,dt+\iint_{D\cap Q_3}\nabla(d_2v-d_1u)\cdot\nabla\zeta\,dx\,dt\\
&\qquad+\iint_{D\cap Q_2}f_1(u,0,w)\zeta\,dx\,dt-\iint_{D\cap Q_3}\big(f_2(u,v,0)-f_1(u,v,0)\big)\zeta\,dx\,dt
\end{aligned}$$

for all functions $\zeta\in C_0^\infty(D)$. On the other hand, since $v=0$ on Γ_1, we have

$$\iint_{D\cap Q_3}\left(v\frac{\partial\zeta}{\partial t}+\frac{\partial v}{\partial t}\zeta\right)dx\,dt=\int_{t_1-r}^{t_1+r}\frac{d}{dt}\int_{B(x_1,r)\cap\Omega_3(t)}v\zeta\,dx\,dt=0.$$

Moreover,

$$\iint_{(D\cap Q_2)\cup(D\cap Q_3)}\left(u\frac{\partial\zeta}{\partial t}+\frac{\partial u}{\partial t}\zeta\right)dx\,dt=0.$$

Therefore, we find

$$\begin{aligned}
&\iint_{D\cap Q_2}\left(\frac{\partial u}{\partial t}-d_1\Delta u-f_1(u,0,w)\right)\zeta\,dx\,dt-\iint_{D\cap Q_3}\left(\frac{\partial v}{\partial t}-d_2\Delta v-f_2(u,v,0)\right)\zeta\,dx\,dt\\
&\qquad+\iint_{D\cap Q_3}\left(\frac{\partial u}{\partial t}-d_1\Delta u-f_1(u,v,0)\right)\zeta\,dx\,dt\\
&\quad=-d_1\iint_{D\cap\Gamma_1}\frac{\partial u|_{Q_2}}{\partial n_1}\zeta\,dx\,dt-\iint_{D\cap\Gamma_1}\left(d_2\frac{\partial v|_{Q_3}}{\partial n_1}-d_1\frac{\partial u|_{Q_3}}{\partial n_1}\right)\zeta\,dx\,dt
\end{aligned}$$

for all $\zeta \in C_0^\infty(D)$. It follows from (1.5) and (1.6) that

$$d_1\left(\frac{\partial u|_{Q_3}}{\partial n_1} - \frac{\partial u|_{Q_2}}{\partial n_1}\right) = d_2\frac{\partial v|_{Q_3}}{\partial n_1}$$

on $D \cap \Gamma_1$. Analogously, we deduce from Definition 3.1 for $i = 2$ that

$$d_1\left(\frac{\partial u|_{Q_2}}{\partial n_1} - \frac{\partial u|_{Q_3}}{\partial n_1}\right) = d_3\frac{\partial w|_{Q_2}}{\partial n_1}$$

on $D \cap \Gamma_1$. Hence, we get (4.1). We can obtain (4.2) and (4.3) in similar fashion. Thus, we conclude the proof. □

Proof of Theorem 1.1. Theorem 1.1 immediately follows from Theorems 3.2 and 4.1. □

5. Numerical experiments

In this section, we carry out numerical simulations. First, we deal with the competition system of Lotka–Volterra type:

$$\begin{cases} f_1(u,v,w) = u(1-u-2v-w), \\ f_2(u,v,w) = v(1-u-v-2w), \\ f_3(u,v,w) = w(1-2u-v-w). \end{cases} \tag{5.1}$$

The system of the ordinary differential equations

$$\begin{cases} u_t = f_1(u,v,w), \\ v_t = f_2(u,v,w), \\ w_t = f_3(u,v,w) \end{cases} \tag{5.2}$$

which corresponds to the diffusion-free system with $k = 0$ possesses the following equilibria:

$$\mathcal{E} = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1/4, 1/4, 1/4)\}.$$

Two equilibria $(0,0,0)$ and $(1/4,1/4,1/4)$ are unstable under the flow by (5.2) and the other three equilibria $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are stable. The numerical results are drawn in Fig. 2. We used $d_1 = d_2 = d_3 = 10^{-4}$ and $k = 10^5$. The figures in the first, second and third rows denote the profiles of solutions u^k, v^k and w^k, respectively. In order to make sure of our theoretical results, we painted the regions with different colors (Fig. 3) in the last row in Fig. 2. Three stable equilibria of (5.2) are on the reaction limit set $\mathcal{A}_{\mathrm{RD}}$ in Section 1. Moreover, the numerical solutions u^k, v^k and w^k in Fig. 2 look as if only one species survives at each point. However, Theorem 1.1 implies that only one component converges to zero at each point as k tends to infinity. We can observe that this is true from the numerical point of view. The domain Ω is divided into three regions after they diffuse enough, that is, one of the components goes to zero and others are positive at each point except for points on the interfaces.

Ei, Ikota and Mimura in [4] studied the following system:

$$\begin{cases} u_t = d_1\Delta u + kf_1(u,v,w), \\ v_t = d_2\Delta v + kf_2(u,v,w), \\ w_t = d_3\Delta w + kf_3(u,v,w). \end{cases}$$

It turns out that the reaction limit set of this system is $\mathcal{E}$. In this case, the solution converges to one of the three stable equilibria in each point. Thus, only one species can survive in each habitat.

Next, we consider the following example:

$$\begin{cases} f_1(u,v,w) = u(1-u-0.2v-0.6w), \\ f_2(u,v,w) = v(1-0.6u-v-0.2w), \\ f_3(u,v,w) = w(1-0.2u-0.6v-w). \end{cases} \tag{5.3}$$

The corresponding system (5.2) possesses the following equilibria:

$$\mathcal{E} = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), \\ (0, 10/11, 5/11), (5/11, 0, 10/11), (10/11, 5/11, 0), (5/9, 5/9, 5/9)\}.$$

The equilibrium $(5/9,5/9,5/9)$ is stable for the flow by (5.2) and the others are unstable. More precisely, since the eigenvalues of the linearized matrix of (5.2) near $(5/9,5/9,5/9)$ are -1 and $(-3 \pm \sqrt{3}i)/9$, most orbits of (5.2) rotate around the axis $(1,1,1)$ and converge to the stable equilibrium. Fig. 4 shows numerical solutions (u^k, v^k, w^k) with $k = 0$. On the

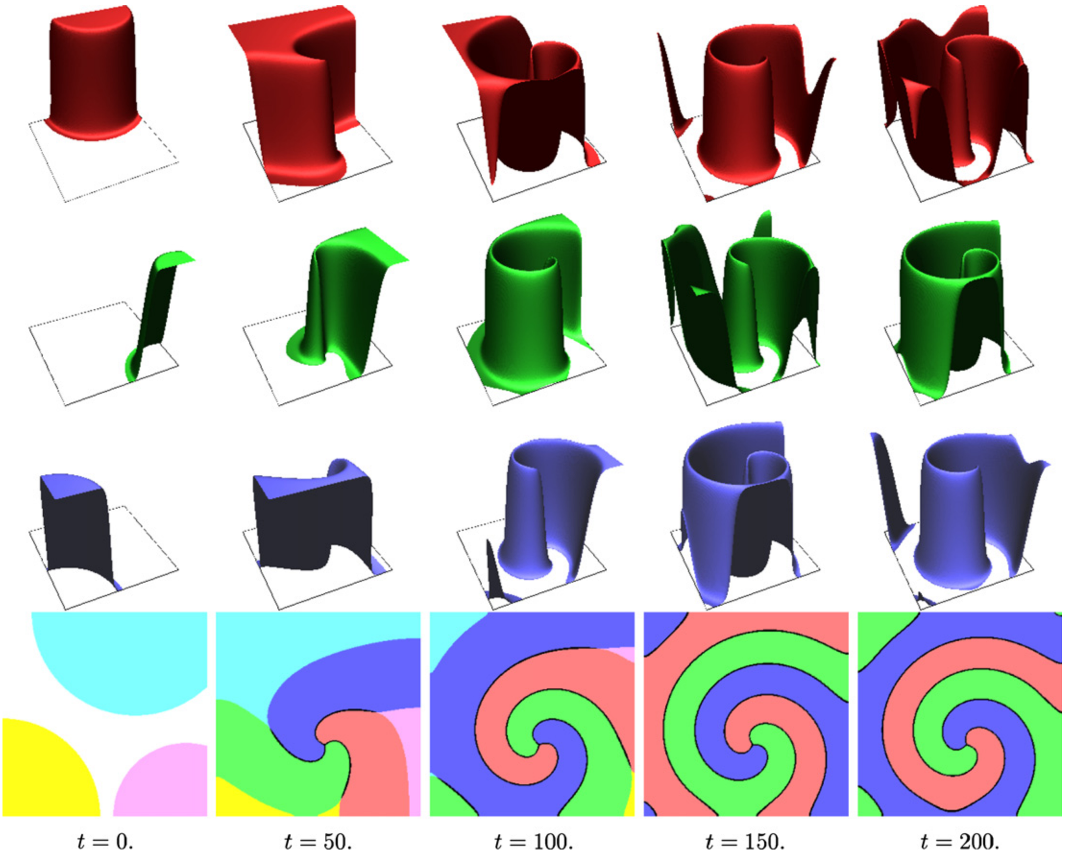

Fig. 2. Numerical solutions of $(\mathrm{RD})^k$ with (5.1).

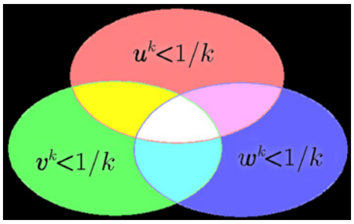

Fig. 3. Colors for approximated regions which correspond to $\Omega_1(t)$ (salmon), $\Omega_2(t)$ (yellow green) and $\Omega_3(t)$ (royal blue) respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

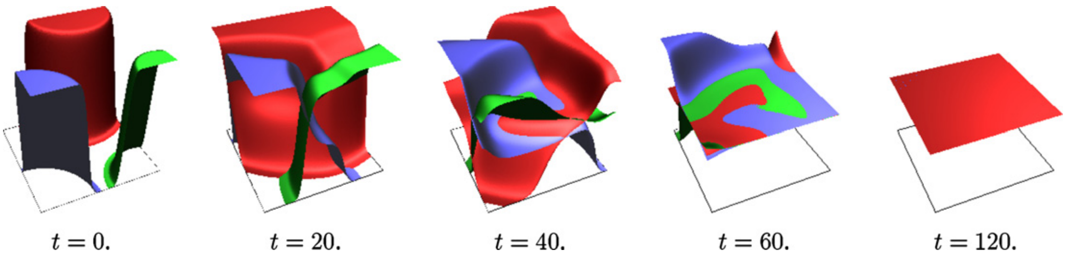

Fig. 4. Numerical solutions of $(\mathrm{RD})^k$ with (5.3) and $k = 0$.

other hand, the stable equilibrium $(5/9, 5/9, 5/9)$ does not belong to the reaction limit set $\mathcal{A}_{\mathrm{RD}}$. On the plane $\{w = 0\}$, for example, the system is reduced to

$$\begin{cases} u_t = u(1 - u - 0.2v), \\ v_t = v(1 - 0.6u - v), \end{cases}$$

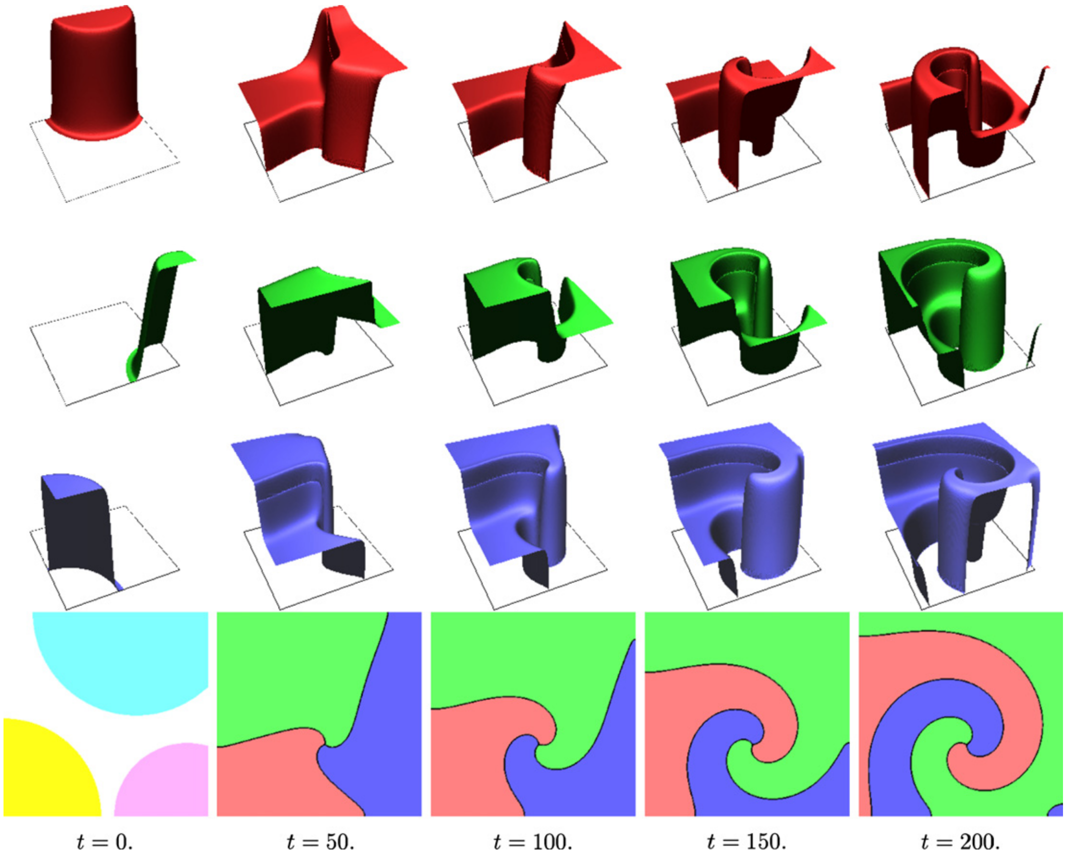

Fig. 5. Numerical solutions of $(\mathrm{RD})^k$ with (5.3).

which is a two-component system of mono-stable type. Three equilibria

$$(0, 10/11, 5/11), \quad (5/11, 0, 10/11), \quad (10/11, 5/11, 0)$$

are stable on the reaction limit set. Since these equilibria on the reaction limit set correspond to the coexistence of the species, we can expect the spiral waves with the coexistence. Actually by numerics we can observe the spiral waves as in Fig. 5. In this simulation, we can clearly see the discontinuities of the flux across the free boundaries.

Acknowledgments

The authors would like to thank Professor Danielle Hilhorst for her careful reading of the manuscript and for her numerous suggestions for improving the exposition of this paper. The first author is partially supported by Grant-in-Aid for Young Scientists (B) (No. 22740058), the Ministry of Education, Culture, Sports, Science and Technology of Japan. The last author is supported in part by Grant-in-Aid for Scientific Research (C) (No. 20540144), Japan Society for the Promotion of Science.

References

[1] D. Bothe, D. Hilhorst, A reaction–diffusion system with fast reversible reaction, J. Math. Anal. Appl. 268 (2003) 125–135.
[2] N. Bouillard, R. Eymard, M. Henry, R. Herbin, D. Hilhorst, A fast precipitation and dissolution reaction for a reaction–diffusion system arising in a porous medium, Nonlinear Anal. Real World Appl. 10 (2009) 629–638.
[3] E.N. Dancer, D. Hilhorst, M. Mimura, L.A. Peletier, Spatial segregation limit of a competition–diffusion system, European J. Appl. Math. 10 (1999) 97–115.
[4] S.-I. Ei, R. Ikota, M. Mimura, Segregating partition problem in competition–diffusion systems, Interfaces Free Bound. 1 (1999) 57–80.
[5] D. Hilhorst, R. van der Hout, L.A. Peletier, The fast reaction limit for a reaction–diffusion system, J. Math. Anal. Appl. 199 (1996) 349–373.
[6] D. Hilhorst, M. Iida, M. Mimura, H. Ninomiya, A competition–diffusion system approximation to the classical two-phase Stefan problem, Japan J. Indust. Appl. Math. 18 (2001) 161–180.
[7] D. Hilhorst, M. Iida, M. Mimura, H. Ninomiya, Relative compactness in L^p of solutions of some $2m$ components competition–diffusion systems, Discrete Contin. Dyn. Syst. 21 (2008) 233–244.
[8] D. Hilhorst, J.R. King, M. Röger, Mathematical analysis of a model describing the invasion of bacteria in burn wounds, Nonlinear Anal. 66 (2007) 1118–1140.
[9] D. Hilhorst, M. Mimura, H. Ninomiya, Fast reaction limit of competition–diffusion systems, in: C.M. Dafermos, Milan Pokorny (Eds.), Handbook of Differential Equations: Evolutionary Equations, vol. 5, North-Holland, Hungary, 2009, pp. 105–168.

[10] M. Iida, M. Mimura, H. Ninomiya, Diffusion, cross-diffusion and competitive interaction, J. Math. Biol. 53 (4) (2006) 617–641.

[11] M. Iida, H. Ninomiya, A reaction–diffusion approximation to a cross-diffusion system, in: M. Chipot, H. Ninomiya (Eds.), Recent Advances on Elliptic and Parabolic Issues, World Scientific, 2006, pp. 145–164.

[12] A. Lunardi, Analytic Semigroups and Regularity in Parabolic Problems, Progr. Nonlinear Differential Equations Appl., vol. 16, Birkhäuser, Basel–Boston–Berlin, 1995.

[13] H. Murakawa, Reaction–diffusion system approximation to degenerate parabolic systems, Nonlinearity 20 (2007) 2319–2332.

[14] H. Murakawa, A relation between cross-diffusion and reaction–diffusion, Discrete Contin. Dyn. Syst., in press.

[15] N. Shigesada, K. Kawasaki, E. Teramoto, Spatial segregation of interacting species, J. Theoret. Biol. 79 (1979) 83–99.

[16] R. Temam, Navier–Stokes Equation Theory and Numerical Analysis, AMS Chelsea Publishing, Providence, RI, 2001.