Cleavability, pseudo-radial and R-monolithic spaces

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Abstract


The problem whether a compact space cleavable over the class of pseudo-radial spaces is itself pseudo-radial is considered and a consistent positive solution is given. It is also shown (in ZFC) that a compact space which is cleavable over the class of R-monolithic spaces is R-monolithic.

Keywords: Cleavability; R-monolithic; Pseudo-radial.

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Let $\mathcal{P}$ be a class of topological spaces. A space $X$ is said to be cleavable over $\mathcal{P}$ if for any subset $A$ of $X$ there exist a space $Y \in \mathcal{P}$ and a continuous function $f : X \to Y$ such that $f^{-1}(f(A)) = A$. A general problem is to see whether $X$ cleavable over $\mathcal{P}$ implies $X \in \mathcal{P}$. If $X$ is compact then very often (but not always) this is the case. Notice that $X$ cleavable over $\mathcal{P}$ is a substantial weakening of the requirement that there exists a one to one continuous mapping from $X$ into some $Y \in \mathcal{P}$.

The notion of cleavability has recently received increasing attention in particular thanks to the work of its creator Arhangel’skii. For further details the reader is referred to [2] and the bibliography listed there.

The aim of this paper is to investigate the following:

**Question (∗).** Let $\mathcal{P}$ be the class of pseudo-radial spaces and $X$ be a compact space cleavable over $\mathcal{P}$. Does $X$ belong to $\mathcal{P}$?

This question was first asked by Arhangel’skii, who gave a positive answer for the special case when $\mathcal{P}$ is the class of sequential spaces (see Theorem 14 for the precise statement).

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We begin by presenting a consistent positive answer to Question (\(\ast\)). Then we provide a complete solution to the analogous problem obtained by considering the class of R-monolithic spaces (see definitions below).

Henceforth all spaces are assumed to be Hausdorff. \(\lambda\) and \(\kappa\) denote infinite cardinal numbers and \(\alpha, \beta, \gamma, \delta\) and \(\xi\) ordinal numbers. \(c\) denotes the cardinality of the continuum and \(\mathfrak{c}_\alpha\) and \(\alpha\th\) successor of \(c\) defined inductively by letting \(c_0 = c, \mathfrak{c}_{\beta+1} = (\mathfrak{c}_\beta)^+ \) and \(\mathfrak{c}_\alpha = \sup(\mathfrak{c}_\beta : \beta \in \alpha)\) when \(\alpha\) is limit.

The tightness of the topological space \(X\), denoted by \(t(X)\), is the smallest cardinal number \(\kappa\) with the property that for any set \(A \subset X\) and any point \(x \in A\) there exists a set \(B \subset A\) such that \(|B| \leq \kappa\) and \(x \in B\).

The density of the topological space \(X\), denoted by \(d(X)\), is the smallest cardinality of a dense subset of \(X\).

A space \(X\) is said to be pseudo-radial provided that for any nonclosed set \(A \subset X\) there exists a (transfinite) sequence \(\{x_\alpha : \alpha \in \kappa\} \subset A\) which converges to a point \(x \in A \setminus A\).

The chain character of a pseudo-radial space \(X\), denoted by \(\sigma_c(X)\), is the smallest cardinal \(\lambda\) such that the above definition can hold for \(\kappa \leq \lambda\).

Clearly the inequality \(t(X) \leq \sigma_c(X)\) holds for every pseudo-radial space \(X\).

A pseudo-radial space \(X\) is said to be R-monolithic provided that \(\sigma_c(A) \leq |A|\) for any \(A \subset X\).

Every compact monolithic (or even quasi-monolithic) space is R-monolithic (see [3]).

R-monolithic spaces were first considered in [5] without any name.

A subset \(A\) of a topological space \(X\) is said to be \(\lambda\)-closed (respectively \(< \lambda\)-closed) provided that if \(B \subset A\) and \(|B| \leq \lambda\) (respectively \(|B| < \lambda\)) then \(\overline{B} \subset A\).

Furthermore \(A\) is said to be \(\lambda\)-chain-closed if it contains the limit points of all its converging subsequences of length not exceeding \(\lambda\). Clearly every \(\lambda\)-closed set is \(\lambda\)-chain-closed and if \(t(X) \leq \lambda\) then every \(\lambda\)-closed subset of \(X\) is closed.

Spaces in which any \(\lambda\)-chain-closed set is \(\lambda\)-closed for any \(\lambda\) were called semiradial in [7]. There it was also pointed out that every R-monolithic space is semiradial.

A topological space \(X\) is called \(\omega\)-scattered if every nonempty closed subset \(F\) of \(X\) has a point of countable character (in \(F\)).

**Theorem 1.** Let \(\mathcal{P}\) be a class of spaces, \(X\) a countably compact regular space and \(f : X \to Y\) a perfect mapping having \(\omega\)-scattered fibers. If \(X\) is cleavable over \(\mathcal{P}\) and the product of \(Y\) with any member of \(\mathcal{P}\) is pseudo-radial then \(X\) is pseudo-radial.

**Proof.** Fix a nonclosed subset \(A\) of \(X\) and choose a space \(Y' \in \mathcal{P}\) and a continuous mapping \(g : X \to Y'\) satisfying \(g^{-1}(g(A)) = A\). Denote by \(h : X \to Y \times Y'\) the diagonal mapping of \(f\) and \(g\). Since \(h\) is a perfect mapping it follows that the set \(Z = h(X)\) is closed in \(Y \times Y'\) and therefore it is pseudo-radial. The fact that \(h^{-1}(h(A)) = A\) implies that the set \(h(A)\) is not closed in \(Z\) and thus there exists a
sequence \( \{z_\alpha; \alpha \in \kappa\} \subset h(A) \) which converges to a point \( z \in h(A) \setminus h(A) \). Clearly \( \kappa \) can be assumed regular. For every \( \alpha \in \kappa \) pick a point \( x_\alpha \in h^{-1}(z_\alpha) \) and denote by \( B \) the sequence \( \{x_\alpha; \alpha \in \kappa\} \subset A \). Since \( z \in h(B) \), it follows that the set \( h^{-1}(z) \cap B \) is not empty. Observe also that the set \( h^{-1}(z) \cap B \), being a subset of \( f^{-1}(\pi_1(z)) \) (here \( \pi_1: Z \to Y \) is the canonical projection) is \( \omega \)-scattered. We distinguish now two cases.

Case 1: \( \kappa = \omega \). The set \( \{z\} \cup h(B) \) is compact and therefore \( C = h^{-1}(\{z\} \cup h(B)) \) is compact as well. Select a point \( x \in h^{-1}(z) \cap B \) which has countable character in \( h^{-1}(z) \cap B \). Since \( B \) is countable, we see that \( x \) is a \( G_\delta \)-point in \( C \). As \( C \) is compact, \( x \) is actually of countable character in \( C \) and consequently we can extract from \( B \) a subsequence converging to \( x \).

Case 2: \( \kappa > \omega \). Denote by \( G \) the set of all complete accumulation points of \( B \). It is not difficult to see that \( G \) must be a subset of \( h^{-1}(z) \). We claim that \( G \) is not empty. On the contrary, for each \( y \in h^{-1}(z) \) we could select an open neighbourhood \( U_y \) such that \( |U_y \cap B| < \kappa \). By compactness there exist points \( y_1, \ldots, y_n \) such that \( h^{-1}(z) \subset U_{y_1} \cup \cdots \cup U_{y_n} = U \). Since \( \kappa \) is regular each \( U_{y_i} \) misses a final segment of \( B \) and consequently also \( U \) does. As \( h \) is closed there exists an open neighbourhood \( V \) of \( z \) such that \( h^{-1}(V) \subset U \) and this is a contradiction with the fact that \( U \) misses a final segment of \( B \). Therefore the set \( G \) is not empty.

Now select a point \( x \in G \) which has countable character in \( G \) and let \( \{U_n; n \in \omega\} \) be a decreasing family of closed neighbourhoods of \( x \) in \( X \) such that \( \{U_n \cap G; n \in \omega\} \) is a local base of \( x \) in \( G \). For each \( n \in \omega \) the set \( U_n \cap B \) has cardinality \( \kappa \) and so it is a cofinal subsequence of \( B \), namely \( U_n \cap B = \{x_{\phi_n(\alpha)}; \alpha \in \kappa\} \) for some increasing mapping \( \phi_n: \kappa \to \kappa \). If for some \( \alpha \) the set \( \{x_{\phi_n(\alpha)}; n \in \omega\} \) is infinite then, by the countable compactness of \( X \), it has some accumulation point. If none of these points is in \( A \) then the set \( h((x_{\phi_n(\alpha)}; n \in \omega)) \) is a not closed set without accumulation points in \( h(A) \) and therefore it contains a subsequence converging to a point in \( h(A) \setminus h(A) \). In this situation we may again proceed as in Case 1. So let us assume that for every \( \alpha \) the set \( \{x_{\phi_n(\alpha)}; n \in \omega\} \) is finite or it has an accumulation point \( y_\alpha \) in \( A \). When \( \{x_{\phi_n(\alpha)}; n \in \omega\} \) is finite there is a point \( y_\alpha \) equal to \( x_{\phi_n(\alpha)} \) for infinitely many \( n \). Since \( x \notin A \), the proof will be complete if we show that \( \{y_\alpha; \alpha \in \kappa\} \) converges to \( x \). To this end, fix an open neighbourhood \( W \) of \( x \) and choose \( m \in \omega \) in such a way that \( U_0 \cap G \subset W \). The set \( U_0 \setminus W \) is compact and contains no complete accumulation point of \( B \). Then each point of \( U_0 \setminus W \) has a neighbourhood missing a final segment of \( B \) and consequently all the \( U_m \setminus W \) can be included in an open set \( W_1 \) which misses a final segment of \( B \). Denote by \( x_{\alpha}^* \) the minimum of such a segment. Furthermore let \( W_2 \) be an open set satisfying \( G \setminus W \subset W_2 \) and \( W_2 \cap U_0 = \emptyset \). Let \( V \) be an open neighbourhood of \( z \) such that \( h^{-1}(V) \subset W \cup W_1 \cup W_2 \). Since \( \{z_\alpha; \alpha \in \kappa\} \) converges to \( z \), there exists \( \alpha^{**} \) such that \( z_\alpha \in V \) whenever \( \alpha \geq \alpha^{**} \). To finish, observe that if \( B = \text{max}\{\alpha^*, \alpha^{**}\} \) and \( \alpha \geq B \) then \( x_{\phi_n(\alpha)} \in W \) for each \( n \geq m \). It follows that \( y_\alpha \in W \) whenever \( \alpha \geq \beta \) and this shows that the sequence \( \{y_\alpha; \alpha \in \kappa\} \) converges to \( x \).\( \square \)
Lemma 2. If \( X \) is a separable regular space then there exists a subset \( S \) of \( X \) such that every closed set contained either in \( S \) or in \( X \setminus S \) has cardinality less than \( 2^{\alpha} \).

Proof. By the well-known formula \( w(X) \leq 2^{\delta(X)} \), the weight of \( X \) does not exceed \( c \) and hence the topology of \( X \) has cardinality at most \( 2^{\alpha} \). Let \( \mathcal{F} \) be the family of all subsets of \( X \) having cardinality \( 2^{\alpha} \). If \( \mathcal{F} = \emptyset \) then put \( S = X \). If \( \mathcal{F} \neq \emptyset \) then choose an onto mapping \( \phi : 2^{\alpha} \rightarrow \mathcal{F} \) and proceed as in the classical construction of a Bernstein's subset of the reals. That is for any \( \alpha \in 2^{\alpha} \) select distinct points \( x_\alpha, y_\alpha, z_\alpha, w_\alpha, \beta, \gamma \in \alpha \). Finally put \( S = \{ x_\alpha; \alpha \in 2^{\alpha} \} \).

Assuming CH and applying the Čech–Pospíšil’s Theorem, we see also that the set \( S \) in the above lemma has the property that every compact set contained either in \( S \) or in \( X \setminus S \) is \( \omega \)-scattered.

Lemma 3 (see [8, Corollary 8.7]). If \( X \) is a Hausdorff space and \( \alpha \leq |X| \leq \omega_\lambda \) for some \( \lambda < \kappa \) then there exists a set \( S \subset X \) such that every compact set contained either in \( S \) or in \( X \setminus S \) is scattered.

Observe that, by the Čech–Pospíšil’s Theorem, the above assertion trivially holds when \( |X| < \kappa \).

Lemma 4 (see [10]). (a) If \( \kappa \leq \omega_2 \) then every compact sequentially compact space is pseudo-radial. In particular the product of countably many compact pseudo-radial spaces is pseudo-radial.

(b) If MA(\( \sigma \)-centered) holds then every compact sequentially compact space of countable tightness is pseudo-radial.

Theorem 5. If either CH or \( \kappa = \omega_2 \) and \( 2^{\alpha} \leq \omega_\omega \) holds then a compact space \( X \) which is cleavable over the class of pseudo-radial spaces is itself pseudo-radial.

Proof. According to Lemma 4(a), it is enough to show that \( X \) is sequentially compact. This will be achieved by checking that each closed separable subspace of \( X \) is pseudo-radial. Hence we are reduced to the case that \( X \) itself is separable. By the well-known inequality \( |X| \leq 2^{\delta(X)} \), we have \( |X| \leq 2^{\alpha} \) and therefore, by applying either Lemma 2 or Lemma 3, there exists a set \( S \subset X \) such that every compact set which is contained either in \( S \) or in \( X \setminus S \) is \( \omega \)-scattered. Now choose a compact pseudo-radial space \( Y \) and a continuous mapping \( f : X \rightarrow Y \) satisfying \( f^{-1}(f(S)) = S \). The mapping \( f \) has clearly \( \omega \)-scattered fibres. Taking as \( \mathcal{P} \) the class of pseudo-radial compact spaces and using again Lemma 4(a), we see that all the hypotheses of Theorem 1 are fulfilled and we can conclude that \( X \) is pseudo-radial. \( \square \)
As another application of Lemma 2 we have the following:

**Theorem 6.** If $\text{MA}(\sigma\text{-centered})$ holds then a compact space which is cleavable over the class of sequentially compact spaces is sequentially compact.

**Proof.** Let $X$ be a compact space cleavable over the class of sequentially compact spaces and $A = \{a_n; n \in \omega \setminus \{0\}\}$ a countable subset of $X$. Without any loss of generality, we can assume that $X = \bar{A}$. Let $S$ be a subset of $X$ satisfying the conclusion of Lemma 2 and choose sequentially compact spaces $\{Y_n; n \in \omega\}$ and continuous mappings $f_n : X \to Y_n$ in such a way that $f_0^{-1}(f_0(S)) = S$ and $f_n^{-1}(f_0(a_n)) = \{a_n\}$ for every $n \in \omega \setminus \{0\}$. Let $f : X \to \prod_{n \in \omega} Y_n = Y$ be the diagonal product of $f_0, f_1, \ldots$. Since $Y$ is sequentially compact, the set $f(A)$ has a subsequence $B$ converging to some $y \in Y$. Put $C = f^{-1}(B)$. The set $f^{-1}(y) \cap C$, being contained either in $S$ or in $X \setminus S$, has cardinality less than $2^\omega$. By applying the Čech–Pospisil’s Theorem, we can find a point $x \in f^{-1}(y) \cap C$ which has character less than $c$ in $f^{-1}(y) \cap C$. Since $f^{-1}(B)$ is countable, it is clear that $x$ has also character less than $c$ in $C$. Now, by a well-known consequence of $\text{MA}(\sigma\text{-centered})$ (see [12]), the set $f^{-1}(B) \subset A$ has a subsequence converging to $x$ and the proof is complete. $\blacksquare$

**Corollary 7.** If $\text{MA}(\sigma\text{-centered})$ holds then a compact space $X$ which is cleavable over the class of pseudo-radial spaces of countable tightness is pseudo-radial.

**Proof.** By Theorem 6 $X$ is sequentially compact and by Lemma 11 below it has countable tightness. To finish, apply Lemma 4(b). $\blacksquare$

Now we turn to the case of $R$-monolithic spaces.

The first two lemmas below may be found somewhere in the literature, but for the reader’s convenience we outline a proof of them.

**Lemma 8.** If $X$ is a pseudo-radial space then $|X| \leq d(X)^{\sigma_\alpha(X)}$.

**Proof.** Let $A$ be a dense subset of $X$ such that $|A| \leq d(X)$. Denote by $[A]_R$ the set obtained by adjoining to $A$ the limits of all its converging subsequences and for any $\alpha \in \sigma_\alpha(X)^+$ put $A_\alpha = \{\bigcup_{\beta \in \alpha} A_\beta\}_R$ (we begin with $A_\emptyset = A$). It is easy to check that $\bigcup_{\alpha \in \sigma_\alpha(X)} A_\alpha = X$ and $|A_\alpha| \leq d(X)^{\sigma_\alpha(X)}$ for each $\alpha$. Therefore we have $|X| \leq d(X)^{\sigma_\alpha(X)}$. $\blacksquare$

**Lemma 9.** Let $\mathcal{P}$ be a $\kappa$-productive class of spaces. If the space $X$ is cleavable over $\mathcal{P}$ and $|X| \leq 2^\kappa$ then there exist $Y \in \mathcal{P}$ and a one to one continuous mapping $f : X \to Y$.

**Proof.** Embed (not continuously) $X$ into the space $D^\kappa$, $D$ is the two-point discrete space. $D^\kappa$ has a base of cardinality $\kappa$ and so the trace of it on $X$ gives a separating
family, say \( \{A_\alpha; \alpha \in \kappa\} \). For every \( \alpha \), fix a space \( Y_\alpha \in \mathcal{P} \) and a continuous mapping \( f_\alpha : X \to Y_\alpha \) which cleaves \( X \) along \( A_\alpha \). The desired mapping is just the diagonal of the family \( \{f_\alpha; \alpha \in \kappa\} \). \( \Box \)

The next lemma is a version of a theorem in [9]. It was formulated assuming that every decomposition consists of compact sets. The proof, however, works replacing compact sets with closed countably compact sets. This observation will be useful in Theorem 14.

**Lemma 10** If every subset of the space \( X \) is the union of \( \kappa \) closed countably compact sets then \( |X| \leq \kappa \).

**Lemma 11** (see [4]). If a space \( X \) is cleavable with respect to closed mappings over the class of spaces having tightness not exceeding \( \kappa \) then \( t(X) \leq \kappa \).

The key point to get the second result announced at the beginning is a theorem which strengthens the countable productivity of the class of R-monolithic spaces established in [6].

**Theorem 12.** Let \( \{X_\alpha; \alpha \in \kappa\} \) be a family of R-monolithic compact spaces. If \( \lambda \geq \kappa \) then every \( <\kappa \)-closed \( \lambda \)-chain-closed subset of \( \prod_{\alpha \in \kappa} X_\alpha \) is \( \lambda \)-closed.

**Proof.** We proceed by transfinite induction. Since in [6] it was shown that the theorem holds for \( \kappa \leq \omega \), we take \( \kappa > \omega \) and assume that the theorem is true for each cardinal below \( \kappa \). Obviously it is enough to verify that if \( A \) is a \( <\kappa \)-closed non \( \lambda \)-closed subset of \( \prod_{\alpha \in \kappa} X_\alpha \) then \( A \) is not \( \lambda \)-chain-closed. Evidently, by replacing \( X \) with a smaller cardinal, we can assume that \( A \) is actually \( <\lambda \)-closed. Let us choose a set \( B \subset A \) satisfying \( |B| = \lambda \) and \( \overline{B} \neq \emptyset \). Denote by \( \pi_\beta : \prod_{\alpha \in \kappa} X_\alpha \to \prod_{\alpha \in \beta} X_\alpha \) the canonical projection and select a point \( x \in \overline{B} \setminus A \). If \( \pi_\beta(x) \in \pi_\beta(A) \) for each \( \beta \in \kappa \) then pick points \( x_\beta \in A \) in such a way that \( \pi_\beta(x) = \pi_\beta(x_\beta) \). It is clear that the sequence \( \{x_\beta; \beta \in \kappa\} \) converges to \( x \) and therefore \( A \) is not \( \lambda \)-chain-closed. Next we assume that for some \( \beta \in \kappa \), \( \pi_\beta(x) \in \overline{\pi_\beta(B)} \setminus \pi_\beta(A) \). This means that \( \pi_\beta(A) \) is a non \( \lambda \)-closed subset of \( \prod_{\alpha \in \beta} X_\alpha \) which moreover is \( <\lambda \)-closed. By the inductive assumption there exists a sequence \( \{x_\alpha^\beta; \alpha \in \lambda\} \subset A \) and a point \( x^\beta \in \prod_{\alpha \in \beta} X_\alpha \setminus \pi_\beta(A) \) such that the sequence \( \{\pi_\gamma(x_\alpha^\beta); \alpha \in \lambda\} \) converges to \( x^\beta \). Clearly \( \lambda \) must be regular. Let \( C_\gamma = \{x_\alpha^\gamma; \alpha \in \gamma\} \) and \( C = \bigcup_{\gamma \in \lambda} C_\gamma \). Observe that \( \pi_{\beta+1}(C) \) is a \( <\lambda \)-closed non \( \lambda \)-closed subset of \( \prod_{\alpha \in \beta+1} X_\alpha \). Again by the inductive assumption, there exists a sequence \( \{x_\alpha^{\beta+1}; \alpha \in \kappa\} \subset C \) and a point \( x^{\beta+1} \in \prod_{\alpha \in \beta+1} X_\alpha \) such that \( \{\pi_{\beta+1}(x_\alpha^{\beta+1}); \alpha \in \kappa\} \) converges to \( x^{\beta+1} \). Furthermore, the sequence \( \{x_\alpha^{\beta+1}; \alpha \in \kappa\} \) can be chosen in such a way that \( x^{\beta+1} \notin C_\gamma \) for every \( \gamma \in \lambda \) and consequently \( x^{\beta+1} \notin \{x_\alpha^{\gamma}; \gamma \in \alpha \in \lambda\} \).

It is easy to see that the sequence \( \{\pi_\beta(x_\alpha^{\beta+1}); \alpha \in \lambda\} \) must converge to \( x^\beta \) and hence \( x^{\beta+1} \) is an extension of \( x_\beta \), i.e., \( x^{\beta+1}(\alpha) = x^\beta(\alpha) \) for every \( \alpha \in \beta \).
Iterating this procedure we construct sequences \( \{x_{\alpha}^{\beta+n}: \alpha \in \lambda \} \subset A \) in such a way that \( \{\pi_{\beta+n}(x_{\alpha}^{\beta+n}): \alpha \in \lambda \} \) converges to \( x_{\alpha}^{\beta+n} \in \prod_{\alpha \in \beta+n} X_\alpha \),

\[
x_{\gamma}^{\beta+n+1} = \left\{x_{\alpha}^{\beta+n}: \gamma \in \alpha \in \lambda \right\}
\]

and \( x_{\gamma}^{\beta+n+1} \) is an extention of \( x_{\alpha}^{\beta+n} \). At the limit step \( \beta + \omega \) define \( x_{\alpha}^{\beta+\omega} = \bigcup_{n \in \omega} x_{\alpha}^{\beta+n} \) and for any \( \alpha \in \lambda \) pick a point \( x_{\alpha}^{\beta+\omega} \in \left\{x_{\alpha}^{\beta+n}: n \in \omega \right\} \subset A \). It is easy to check that the sequence \( \{\pi_{\beta+\omega}(x_{\alpha}^{\beta+\omega}): \alpha \in \lambda \} \) converges to \( x_{\alpha}^{\beta+\omega} \). Mimicking the same pattern, we can construct for any \( \gamma \in \kappa \) sequences \( \{x_{\alpha}^{\beta+\gamma}: \alpha \in \lambda \} \subset A \) and points \( x_{\alpha}^{\beta+\gamma} \in \prod_{\alpha \in \beta+\gamma} X_\alpha \) in such a way that \( \{\pi_{\beta+\gamma}(x_{\alpha}^{\beta+\gamma}): \alpha \in \lambda \} \) converges to \( x_{\alpha}^{\beta+\gamma} \) and \( x_{\alpha}^{\beta+\delta} \) is an extention of \( x_{\alpha}^{\beta+\gamma} \) whenever \( \gamma \in \delta \). Furthermore we require that a condition analogous to \((\ast)\) holds for every successor ordinal and for \( \gamma \) limit \( x_{\alpha}^{\beta+\gamma} \) is a complete accumulation point of the set \( \{x_{\alpha}^{\beta+\delta}: \delta \in \gamma \} \). To finish put \( z = \bigcup_{\alpha \in \kappa} x_{\alpha}^{\beta+\gamma} \in \prod_{\alpha \in \beta+\gamma} X_\alpha \setminus A \). We distinguish two cases. If \( \lambda = \kappa \) then put \( z_{\alpha} = x_{\alpha}^{\beta+\alpha} \) for each \( \alpha \in \kappa \). We claim that the sequence \( \{z_{\alpha}: \alpha \in \kappa \} \) converges to \( z \). To this end fix a neighbourhood \( W \) of \( z \). Without any loss of generality, we can take \( W = \pi_{\beta+\gamma}^{-1}(V) \) where \( V \) is a closed neighbourhood of \( x_{\alpha}^{\beta+\gamma} \) in \( \prod_{\alpha \in \beta+\gamma} X_\alpha \). Clearly there exists \( \hat{\alpha} \in \kappa \) such that \( x_{\alpha}^{\beta+\gamma} \in \pi_{\beta+\gamma}^{-1}(V) \) for any \( \alpha \geq \hat{\alpha} \). Since by construction for \( \alpha \geq \hat{\alpha} \) we have \( x_{\alpha}^{\beta+\gamma+1} \in \pi_{\beta+\gamma+1}^{-1}(V) \) we see that \( x_{\alpha}^{\beta+\gamma+1} \in \pi_{\beta+\gamma}^{-1}(V) \) for any \( \alpha \geq \hat{\alpha} \). With the same argument we deduce that \( x_{\alpha}^{\beta+\gamma+n} \in \pi_{\beta+\gamma}^{-1}(V) \) for any \( \alpha \geq \hat{\alpha} \) and any \( n \in \omega \). But then \( x_{\alpha}^{\beta+\gamma+n} \in \pi_{\beta+\gamma}^{-1}(V) \) and continuing in this manner we obtain that \( x_{\alpha}^{\beta+\delta} \in \pi_{\beta+\gamma}^{-1}(V) \) for any \( \alpha \geq \hat{\alpha} \) and any \( \gamma \leq \delta \in \kappa \). Now replacing \( \hat{\alpha} \) with \( \max(\hat{\alpha}, \gamma) \) we see that \( z_{\alpha} = x_{\alpha}^{\beta+\alpha} \in W \) for any \( \alpha \geq \hat{\alpha} \) and therefore \( \{z_{\alpha}: \alpha \in \kappa \} \) converges to \( z \). If \( \lambda > \kappa \) then for any \( \alpha \in \lambda \) pick a complete accumulation point \( z_{\alpha} \) of the set \( \{x_{\alpha}^{\beta+\gamma}: \gamma \in \kappa \} \). It is evident that \( z_{\alpha} \in A \) for every \( \alpha \in \lambda \). We claim that \( \{z_{\alpha}: \alpha \in \lambda \} \) converges to \( z \). Fix as before a closed neighbourhood of \( z \) of the form \( W = \pi_{\beta+\gamma}^{-1}(V) \). For every \( \delta \geq \gamma \) there exists \( \alpha_\delta \in \lambda \) such that \( \pi_{\beta+\delta}(x_{\alpha}^{\beta+\delta}) \in \pi_{\beta+\gamma}(W) \) whenever \( \alpha \geq \alpha_\delta \). As \( \lambda \) is regular we can select \( \hat{\alpha} \in \lambda \) such that \( \alpha_\delta \in \hat{\alpha} \) for every \( \delta \in \kappa \). We have \( x_{\hat{\alpha}}^{\beta+\delta} \in W \) for every \( \delta \in \kappa \) and every \( \hat{\alpha} \in \alpha \in \lambda \) and therefore \( z_{\alpha} \in W \) for every \( \hat{\alpha} \in \alpha \in \lambda \).

Again we see that the sequence \( \{z_{\alpha}: \alpha \in \lambda \} \) converges to \( z \) and this completes the proof. \( \square \)

Comparing the above result with the notion of semiradial spaces we may reformulate Theorem 12 by saying that the product of \( \kappa \) many R-monolithic compact spaces is in some sense “semiradial from \( \kappa \) upwards”.

**Theorem 13.** If a compact space \( X \) is cleavable over the class of R-monolithic spaces then \( X \) is R-monolithic.

**Proof.** Since cleavability is a hereditary property it is enough to show that if \( X \) has density \( \lambda \) then \( X \) is pseudo-radial and \( \alpha_c(X) \leq \lambda \). Notice first that, as any
continuous image of $X$ is a compact space of density not exceeding $\lambda$, it follows that $X$ is actually cleavable over the class of $\mathcal{R}$-monolithic compact spaces having density not exceeding $\lambda$. These spaces have chain character and a fortiori tightness not exceeding $\lambda$. Applying Lemma 11 we then conclude that $t(X) \leq \lambda$. Now fix a nonclosed subset $A$ of $X$ and let $\kappa$ be the smallest cardinal such that $A$ is not $\kappa$-closed. Clearly $\kappa \leq \lambda$. Choose a set $B \subset A$ such that $|B| = \kappa$ and $\overline{B} \setminus A \neq \emptyset$. The subspace $Z = \overline{B}$ is cleavable over the class of $\mathcal{R}$-monolithic spaces having density not exceeding $\kappa$. By Lemma 8 every $\mathcal{R}$-monolithic space of this sort has cardinality not exceeding $2^\kappa$ and thus $Z$ is also cleavable over a class of $T_1$-spaces of cardinality not exceeding $2^\kappa$. In other words, every subset of $Z$ is the continuous preimage of some set of cardinality not exceeding $2^\kappa$ and consequently it is the union of at most $2^\kappa$ compact sets. By applying Lemma 10 we get $|Z| \leq 2^\kappa$. Next from Lemma 9 it follows that $Z$ can be homeomorphically embedded into the product of $\kappa$ many $\mathcal{R}$-monolithic spaces. The set $Z \cap A$ is evidently $<\kappa$-closed, but not $\kappa$-closed. By applying Theorem 12 we see that $Z \cap A$ is not $\kappa$-chain-closed and consequently there exists in $A$ a sequence of length not greater than $\lambda$ which converges to a point outside $A$ and the proof is complete. □

Sequential spaces are precisely the $\mathcal{R}$-monolithic spaces of countable tightness. Taking into account that any continuous mapping from a countably compact space into a sequential space is closed and that the class of countably compact sequential spaces is countably productive (see [11, Theorem 4.5]), the same reasoning applied in Theorem 13 yields a somewhat different proof of the following result, already established by Arhangel’skii:

**Theorem 14** (see [1]). *If a countably compact space $X$ is cleavable over the class of sequential spaces then $X$ is sequential.*

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**References**


