Bernstein-Durrmeyer Operators*

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Abstract—In this paper, we consider the Durrmeyer-type modifications of the classical Bernstein, Szász and Baskakov operators, and deal with two different kind of problems. Firstly, we obtain new results concerning preservation of shape properties, Lipschitz constants and global smoothness, as well as monotonic convergence under convexity. To do this, we use a probabilistic approach based on representations of these operators in terms of stochastic processes having a.s. nondecreasing paths and satisfying a suitable martingale-type condition. Secondly, we show that the Szász-Durrmeyer operator is the limit, in an appropriate sense, of both the Bernstein-Durrmeyer and the Baskakov-Durrmeyer operators. We provide rates of convergence which are derived from the bounds for the total variation distance between the probability measures involved.

Keywords—Bernstein-Durrmeyer operator, Szász-Durrmeyer operator, Baskakov-Durrmeyer operator, Poisson process, Gamma process.

1. INTRODUCTION

The one-dimensional Bernstein-Durrmeyer operator $D_n$ is defined by

$$D_n(f, x) := \int_0^1 f(u)K_n(x, u)\, du, \quad x \in [0, 1], \quad n = 1, 2, \ldots,$$

where $f$ is any real-valued function on $[0, 1]$ which is integrable with respect to the kernel

$$K_n(x, u) := (n + 1) \sum_{k=0}^n p_{n,k}(x)p_{n,k}(u),$$

where

$$p_{n,k}(x) := \binom{n}{k}x^k(1 - x)^{n-k}, \quad k = 0, 1, \ldots, n.$$

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This operator is a modification of the classical Bernstein operator

\[ B_n(f, x) := \sum_{k=0}^{n} f \left( \frac{k}{n} \right) p_{n,k}(x), \]

and was introduced by Durrmeyer [1] and, independently, by Lupaș [2]. It has been extensively studied by Derronnic [3], Ditzian and Ivanov [4], Gonska and Zhou [5] and several other authors.

In the present paper, we establish new properties for \( D_n \). First, we are interested in the following topics:

(a) Preservation of shape properties, such as monotonicity and convexity,
(b) preservation of Lipschitz constants, and
(c) the so-called (by Anastassiou, Cottin and Gonska [6]) preservation of global smoothness, that is, estimates of the moduli of continuity of first and second order of \( D_n f \) in terms of the moduli of \( f \).

To deal with these problems, we use a probabilistic approach based on the following representation

\[ D_n(f, x) = Ef(Y^n_x), \]

where \( E \) denotes mathematical expectation and \( Y^n_x \) is a random variable having the probability density \( K_n(x, \cdot) \). The key point in formula (4) is that all the random variables \( Y^n_x \) are defined on the same probability space in such a way that the stochastic process \( \{Y^n_x : x \in [0, 1]\} \) has a.s. nondecreasing paths and satisfies a suitable martingale-type condition for every \( n = 1, 2, \ldots \) (see Lemma 1 below). This enables us to apply to the case at hand the theory and techniques developed by the authors in [7] and [8], thus achieving a unified treatment of the aforementioned problems. The construction of the process and the consequent results are contained in the next section, which also includes a brief discussion of the problem of monotonic convergence under convexity in connection with a modification of \( D_n \) introduced by Goodman and Sharma [9].

Durrmeyer-type modifications of other classical operators have also been considered during the last decade. We can mention, for instance, the Szász-Durrmeyer operator introduced by Mazhar and Totik [10]. In Section 3, we construct an appropriate probabilistic representation for this operator and sketch some of its most significant consequences. Proofs are omitted, for they are similar to those given in Section 2. Nevertheless, the property of monotonic convergence under convexity is considered in some detail. Finally, the Baskakov-Durrmeyer operator proposed by Sahai and Prasad [11] is briefly discussed in Section 4.

On the other hand, it is known that certain Bernstein-type operators can be approximated by other ones, whenever the parameters are conveniently chosen. By duality, this fact is closely related with the convergence of the probability measures appearing in the definition of such operators. In particular, the classical Szász operator is the limit, in an appropriate sense, of both the Bernstein and the Baskakov operators (cf. [12,13]). In Sections 5 and 6, we show that the analogous limiting properties hold for the corresponding Durrmeyer-type modifications. Furthermore, we provide rates of convergence which are derived from the bounds for the total variation distance between the probability measures involved.

Throughout the paper, we use the following notations. Given a real-valued continuous function \( f \) defined on an interval \( I \) of the real line, \( \omega(f; \cdot) \) will denote the usual first modulus of continuity, and \( \omega_2(f; \cdot) \) will stand for the second modulus of continuity defined by

\[ \omega_2(f; h) := \sup \left| f(x) + f(y) - 2f \left( \frac{x+y}{2} \right) \right|, \]

where the supremum is taken over all \( x, y \in I \) such that \( |x - y| \leq h \). The set of all functions \( f \) satisfying the Lipschitz condition \( \omega(f; h) \leq Ah^\mu \) (respectively, \( \omega_2(f; h) \leq Ah^\mu \), \( h \geq 0 \),
where $A > 0$ and $\mu \in (0, 1]$ (respectively, $\mu \in (0, 2]$) will be denoted by $\text{Lip}(A, \mu)$ (respectively, $\text{Lip}_2(A, \mu)$).

2. PROPERTIES OF $D_n$

We start by constructing a suitable probabilistic representation for $D_n$. Let $\{X_n : n = 1, 2, \ldots\}$ and $\{U_t : t \geq 0\}$ be two independent stochastic processes defined on the same probability space and such that: the former is a sequence of independent and on the interval $[0, 1]$ uniformly distributed random variables, and the later is a gamma process, i.e., a stochastic process starting at the origin, having stationary independent increments and such that, for each $t > 0$, $U_t$ has the gamma density $d_t$ given by

$$d_t(u) := \frac{u^{t-1}e^{-u}}{\Gamma(t)}, \quad u > 0.$$  

Without loss of generality (cf. [14]), it can be assumed that $\{U_t : t \geq 0\}$ has a.s. nondecreasing right-continuous paths. Set, for $x \in [0, 1]$ and $n = 1, 2, \ldots$,

$$S_n(x) := \sum_{k=1}^{n} I(X_k \leq x),$$

where (here and hereafter) $I(C)$ denotes the indicator function of the event $C$, and

$$Y_n^x := \frac{U_{S_n(x)+1}}{U_{n+2}}.$$  

The random variable $S_n(x)$ has the binomial distribution with parameters $n, x,$ and it is easy to check that $Y_n^x$ has the probability density $K_n(x, \cdot)$ described in the preceding section. Therefore, the relation (4) holds true.

In the next lemma, we show that the process $Y := \{Y_n^x : x \in [0, 1], n \geq 1\}$ satisfies some fundamental structural properties.

LEMMA 1. Let $n \geq 1$ be fixed. Then

(a) $Y_n^x \leq Y_n^u$ a.s., $0 \leq x < u \leq 1$.

(b) $E(Y_n^u - Y_n^x | \mathcal{G}_{n,x}) = \frac{u - x}{y - x} (Y_n^y - Y_n^x)$ a.s., $0 \leq x < u < y \leq 1$,

where $E(\cdot)$ denotes conditional expectation and $\mathcal{G}_{n,x}$ stands for the $\sigma$-algebra generated by the random variables $Y_n^u$, $Y_n^x$, $S_n(x)$, $S_n(y)$ and $U_{n+2}$.

(c) The process $\{Y_n^x : x \in [0, 1]\}$ has stationary increments.

PROOF. Properties (a) and (c) are immediate. To prove (b), let $0 \leq x < u < y \leq 1$ and let $\mathcal{H}_{n,x}^u$ the $\sigma$-algebra generated by $S_n(u)$ and the random variables which generate $\mathcal{G}_{n,x}^u$. From the independence and the properties of gamma processes, we have on the event $\{S_n(y) - S_n(x) > 0\} \in \mathcal{G}_{n,x}^u$:

$$E(Y_n^u - Y_n^x | \mathcal{H}_{n,x}^u) = \frac{S_n(u) - S_n(x)}{S_n(y) - S_n(x)} (Y_n^y - Y_n^x) \quad \text{a.s.}$$

Therefore, conditioning further to $\mathcal{G}_{n,x}^u$ and using independence, we obtain

$$E(Y_n^u - Y_n^x | \mathcal{G}_{n,x}^u) = \frac{Y_n^y - Y_n^x}{S_n(y) - S_n(x)} E(S_n(u) - S_n(x) | S_n(x) - S_n(y)) = \frac{u - x}{y - x} (Y_n^y - Y_n^x) \quad \text{a.s.,}$$

the last equality because the conditional distribution of $S_n(u) - S_n(x)$ given $S_n(x)$ and $S_n(y)$ is the binomial distribution with parameters $S_n(y) - S_n(x)$ and $(u - x)/(y - x)$. The proof is complete.

Properties (a) and (b) in Lemma 1 mean that the process $Y$ satisfies the so-called hypotheses $(H_1)$ and $(H_2)$ introduced in [8]. Thus, we are in a position to apply the results and techniques given in [7,8]. First, we can assert the following.
THEOREM 1. Let \( n \geq 1 \) be fixed. We have:

(a) \( D_n \) preserves monotonicity.
(b) \( D_n \) preserves convexity.
(c) If \( f \in \text{Lip}(A, \mu) \), then \( D_n f \in \text{Lip}(A_n, \mu) \), where

\[
A_{n,\mu} := A \left( \frac{n}{n+2} \right)^\mu .
\]

PROOF. Property (a) directly follows from (4) and Lemma 1(a). In view of Lemma 1(a,b), property (b) is a particular case of [7, Theorem 2]. Finally, property (c) is a consequence of (4), Lemma 1(a) and the fact that

\[
E Y_n^x = \frac{nx + 1}{n + 2}, \quad x \in [0,1],
\]

as it follows from [7, Theorem 1].

REMARK 1. It should be observed that

\[
D_n f = B_n(A_n f),
\]

where \( B_n \) is the Bernstein operator and \( A_n \) is the Lupas beta operator defined by

\[
A_n(f, x) := \int_0^1 f(u) \frac{u^{nx}(1 - u)^n(1-x)}{B(nx + 1, n(1-x) + 1)} \, du,
\]

where \( B(., .) \) is the beta function (cf. [2]). In view of (9), properties (a)-(c) in Theorem 1 also follow from the corresponding properties for \( B_n \) and \( A_n \) (cf. [7,15]).

REMARK 2. A third proof of properties (a) and (b) in Theorem 1 is the following: Let \( f \) be a real-valued nondecreasing (respectively, convex) function defined on \([0,1]\). Then, for each \( m \geq 1 \), the Bernstein polynomial \( B_m f \) is nondecreasing (respectively, convex). Moreover, it is well known that \( B_m f(u) \) converges to \( f(u) \), as \( m \to \infty \), provided that \( u \in [0,1] \) is a continuity point of \( f \), and, therefore, \( D_n(B_m f, x) \) converges to \( D_n(f, x) \), as \( m \to \infty \), for all \( x \in [0,1] \). Thus, it suffices to show that the polynomials \( D_n(B_m f) \) are nondecreasing (respectively, convex), but this easily follows from the formulae for the derivatives of Bernstein-Durrmeyer polynomials given in [3, p. 334]. Observe that this procedure permits to show that the operator \( D_n \) preserves convexity of any order.

To obtain estimates for the moduli of continuity of \( D_n f \) in terms of the moduli of continuity of \( f \), we need some quantities depending on the process \( Y \) (cf. [8]). Thanks to Lemma 1(c), the computation of these quantities is considerably simplified. We give without proof the following:

LEMMA 2. Let \( n \geq 1 \) and \( x, y, h \in [0,1] \). Then:

(a) \[
E(Y_n^x - x)^2 = \frac{(2n - 6)x(1 - x) + 2}{(n + 2)(n + 3)} .
\]
(b) \[
P(Y_n^h - Y_n^0 > 0) = 1 - (1 - h)^n .
\]
(c) \[
E(Y_n^h - Y_n^0 - h)^2 = \frac{2nh(1 - h) + 6h^2}{(n + 2)(n + 3)} .
\]
(d) \[
E \left( Y_n^y + Y_n^0 - 2Y_n^{(x+y)/2} \right)^2 = \frac{2n|x - y|}{(n + 2)(n + 3)} .
\]
The following theorem gives estimates for the first modulus of continuity of $D_n f$.

**Theorem 2.** Let $f \in C[0, 1]$, $n \geq 1$ and $h \in [0, 1]$. We have:

(a) If $\omega(f; .)$ is concave then

$$\omega(D_n f; h) \leq \omega \left( f; \frac{nh}{n+2} \right).$$

(b) \[
\omega(D_n f; h) \leq \left( \frac{2(n+1)}{n+2} - (1-h)^n \right) \omega(f; h),
\]

(c) \[
\omega(D_n f; h) \leq \omega(f; h) + \omega(f; a(n,h)), \text{ where } a(n,h) := \sqrt{\frac{2nh(1-h) + 6h^2}{(n+2)(n+3)}}.
\]

(d) \[
|\omega(D_n f; h) - \omega(f; h)| \leq 4\omega(f; b(n)), \text{ where } b(n) := \frac{n+1}{2(n+2)(n+3)}.
\]

**Proof.** Property (a) follows from (8) and [8, Corollary 1(a)]. The bound in (b) follows from (8), Lemma 2(b) and [8, Corollary 2]. The estimate in (c) is a consequence of [8, Corollary 3] and Lemma 2(c). Finally, (d) follows from [8, Theorem 3] and Lemma 2(a).

Next, we obtain analogous estimates for the second modulus of continuity. In the case under consideration, a direct application of [8, Theorem 4] yields

$$\omega_2(D_n f; h) \leq 2E\omega_2(f; Y_n^h - Y_n^0),$$

but the interesting point is that this inequality can be strengthened in the following way.

**Theorem 3.** For $f \in C[0, 1]$, $n \geq 1$ and $h \in [0, 1]$, we have

$$\omega_2(D_n f; h) \leq 2E\omega_2(f; Y_n^h - Y_n^0)I(S_n(h) \geq 2).$$

**Proof.** The validity of the result is readily seen by looking at the proof of Theorem 4 in [8], taking into account that the random variables $S_n(x)$ and $S_n(y)$ are measurable with respect to the $\sigma$-algebra $G_{n,x}$ appearing in Lemma 1(b) above.

**Corollary 1.** Let $f$ and $n$ as in Theorem 3. Then, for $h \in (0, 1]$,

$$\omega_2(D_n f; h) \leq 2c(n,h) \omega_2(f; h),$$

where

$$c(n,h) := E \left( 1 + \frac{Y_n^h - Y_n^0}{h} \right)^2 I(S_n(h) \geq 2) \leq \frac{6n^2 + 8n + 6}{(n+2)(n+3)}.$$

**Proof.** For the first inequality, use the well-known property

$$\omega_2(f; ah) \leq (1+a)^2 \omega_2(f; h), \quad a, h > 0.$$
THEOREM 4. Let $f$, $n$ and $h$ be as in Theorem 3. We have:
(a) If $f \in \text{Lip}_2(A, \mu)$, with $\mu \in (0, 2]$, then $D_n f \in \text{Lip}_2(2A_{n, \mu}, \mu)$, where

$$A_{n, \mu}' := A \left( \frac{3n(n - 1)}{(n + 2)(n + 3)} \right)^{\mu/2}.$$  

In the particular case $\mu \in (0, 1]$, we also have $D_n f \in \text{Lip}_2(2A_{n, \mu}, \mu)$, where $A_{n, \mu}$ is defined in (7).

(b) If $\omega_2(f; \cdot)$ is concave, then

$$\omega_2(D_n f; h) \leq 2 \omega_2 \left( f; \frac{nh}{n + 2} \right).$$

(c) $\omega_2(D_n f; h) \leq \omega_2(f; h) + 3\omega(f; a(n, h)) + 3\omega(f; d(n, h))$, where $a(n, h)$ is defined in (10) and

$$d(n, h) := \frac{2nh}{(n + 2)(n + 3)}.$$  

(d) $|\omega_2(D_n f; h) - \omega_2(f; h)| \leq 8\omega(f; b(n))$, where $b(n)$ is defined in (11).

PROOF. The first part of (a) follows from Theorem 3 by using Hölder's inequality and taking into account that

$$E(Y_n^h - Y_n^0)^2 I(S_n(h) \geq 2) \leq \frac{3n(n - 1)h^2}{(n + 2)(n + 3)},$$

as it follows from Lemma 2(f). The second part of (a) follows from (12), (8) and Jensen's inequality. Part (b) follows from (8) and [8, Corollary 4]. By virtue of Lemma 2(c,d), part (c) is a particular case of [8, Theorem 5]. Finally, part (d) follows from Lemma 2(a) and [8, Theorem 7].

REMARK 3. In the estimates given in Theorem 3, Corollary 1 and Theorem 4(a,b), the coefficient 2 can be dropped if the function $f$ is convex (cf. [8, Section 5]).

Before closing this section, we briefly consider the problem of monotonic convergence under convexity. Many sequences $\{H_n : n \geq 1\}$ of Bernstein-type operators satisfy the monotonicity property

$$H_n f \geq H_{n+1} f, \quad n = 1, 2, \ldots,$$

whenever the function $f$ is convex. The sequence $\{D_n : n \geq 1\}$ does not (take, for instance, the function $f(u) := u$). Consider, however, the following modification of $D_n$ proposed by Goodman and Sharma [9,16]:

$$D_n^*(f, x) := f(0)p_{n,0}(x) + (n - 1) \sum_{k=1}^{n-1} \int_0^1 p_{n-2,k-1}(u)f(u) du + f(1)p_{n,n}(x).$$

It is shown in [9,16] that the operator $D_n^*$ shares, in some sense, the advantages of both the operator $D_n$ and the Bernstein operator $B_n$. In particular, we have

$$D_n^* f \geq D_{n+1}^* f, \quad n = 1, 2, \ldots,$$

whenever $f$ is a convex function on $[0, 1]$. A simple way to show this is the following: First, we observe that

$$D_n^* f = B_n(A_n^* f),$$

where $B_n$ is the Bernstein operator and $A_n^*$ is the beta operator (cf. [7]) defined by

$$A_n^*(f, x) := \begin{cases} f(x), & \text{if } x = 0, 1, \\ \int_0^1 f(u) \frac{u^{nx-1}(1-u)^n(1-x)^{-1}}{B(nx, n(1-x))} du, & \text{if } x \in (0, 1). \end{cases}$$
If \( f \) is convex, then \( A_{n}^{*} f \geq A_{n+1}^{*} f \) (cf. [17]), and \( A_{n+1}^{*} f \) is also convex (cf. [15]). Therefore, by the positivity and monotonicity of the Bernstein operator, we conclude

\[
D_{n}^{*} f = B_{n} (A_{n}^{*} f) \geq B_{n} (A_{n+1}^{*} f) \geq B_{n+1} (A_{n+1}^{*} f) = D_{n+1}^{*} f.
\]

Finally, a probabilistic representation for \( D_{n}^{*} \) is given by

\[
D_{n}^{*} (f, x) = Ef \left( \frac{U_{S_{n}(x)}}{U_{n}} \right),
\]

where \( U_{n} \) and \( S_{n}(x) \) are the same as above. Using this representation, we can establish properties for \( D_{n}^{*} \) in the same way as for \( D_{n} \). Details are omitted.

### 3. THE SZÁSZ-DURRMEYER OPERATORS

The classical Szász-Mirakyan operator \( S_{t} \) is defined by

\[
S_{t}(f, x) := \sum_{k=0}^{\infty} f \left( \frac{k}{t} \right) \pi_{t,k}(x), \quad x \geq 0, \; t > 0,
\]

where

\[
\pi_{t,k}(x) := e^{-tx} \left( \frac{tx}{k!} \right)^{k}, \quad k = 0, 1, \ldots
\]

Mazhar and Totik [10] introduced the following Durrmeyer-type modification of \( S_{t} \):

\[
L_{t}(f, x) := \int_{0}^{\infty} f(u)H_{t}(x, u) \, du, \quad x \geq 0, \; t \geq 0,
\]

where

\[
H_{t}(x, u) := t \sum_{k=0}^{\infty} \pi_{t,k}(x) \pi_{t,k}(u).
\]

It is not hard to see that \( L_{t} f \) is well-defined whenever \( f \) is either a real-valued bounded measurable function on \([0, \infty)\), or a continuous function on \([0, \infty)\) such that \( f(x) = O(x^{r}) \), \( x \to \infty \), for some \( r > 0 \). A suitable probabilistic representation for \( L_{t} \) can be constructed as follows: Let \( \{N(t) : t \geq 0\} \) be a standard Poisson process and let \( \{U_{t} : t \geq 0\} \) be a gamma process, independent of the former and defined on the same probability space. Set

\[
Z_{t}^{x} := \frac{U_{N(tx)+1}}{t}, \quad x \geq 0, \; t > 0.
\]

It is straightforward to check that the random variable \( Z_{t}^{x} \) has the probability density \( H_{t}(x, \cdot) \) given in (14), and, therefore, we can write

\[
L_{t}(f, x) = Ef(Z_{t}^{x}).
\]

The following result is analogous to Lemma 1 above and it is shown in a similar way.

**Lemma 3.** Let \( t > 0 \) be fixed. Then

(a) \( Z_{t}^{x} \leq Z_{t}^{y} \) a.s., \( 0 \leq x < y \).

(b) \( E(Z_{t}^{x} - Z_{t}^{y} | G_{n,x}^{y}) = \frac{y-x}{y-x} (Z_{t}^{y} - Z_{t}^{x}) \) a.s., \( 0 \leq x < u < y \),

where \( G_{n,x}^{y} \) denotes the \( \sigma \)-algebra generated by the random variables \( Z_{t}^{x}, Z_{t}^{y}, N(tx) \) and \( N(ty) \).

(c) The process \( \{Z_{t}^{x} : x \geq 0\} \) has stationary increments.

Following the same lines as in the preceding section, it can be shown from Lemma 3 that \( L_{t} \) satisfies the analogous properties to those established for \( D_{n} \). We shall mention without proof some of the most significant results.
THEOREM 5. Let \( t > 0 \) be fixed. We have:

(a) \( L_t \) preserves monotonicity and convexity.

(b) If \( f \in \text{Lip}(A, \mu) \), then \( L_t f \in \text{Lip}(A, \mu) \).

(c) If \( f \in C[0, \infty) \) satisfies \( \omega(f; h) < \infty, h > 0 \), then

\[
\omega(L_t f; h) \leq (2 - e^{-th}) \omega(f; h), \quad h \geq 0.
\]

(d) If \( f \in C[0, \infty) \) satisfies \( \omega_2(f; h) < \infty, h > 0 \), then

\[
\omega_2(L_t f; h) \leq 2 k(t, h) \omega_2(f; h), \quad h > 0,
\]

where

\[
k(t, h) := E \left( 1 + \frac{Z^h_t - Z^0_t}{h} \right)^2 I(N(th) \geq 2)
= 4 - (3 + th) e^{-th} + \frac{2(1 - e^{-th})}{th} \leq 6.
\]

(e) If \( f \in \text{Lip}_2(A, \mu) \), then \( L_t f \in \text{Lip}_2(2A, \mu) \) or \( L_t f \in \text{Lip}_2(2\sqrt{3}A, \mu) \) according to \( \mu \in (0, 1] \) or \( \mu \in (1, 2] \).

Next, we deal with the property of monotonic convergence under convexity for \( L_t \).

THEOREM 6. Let \( x > 0 \) and let \( f \) be a convex nondecreasing function defined on \([0, c_\infty)\) such that \( L_t(|f|, x) < \infty \) and \( L_r(|f|, x) < \infty \), with \( r > t > 0 \). Then

\[
L_t(f, x) \geq L_r(f, x).
\]  (16)

PROOF. It is an easy exercise in probability calculus to show that

\[
E(Z^x_t \mid Z^x_r, N(tx), N(rx)) = \frac{r(1 + N(tx))}{t(1 + N(rx))} Z^x_r
\]
a.s. and, hence,

\[
E(Z^x_t \mid Z^x_r, N(rx)) = \frac{(r/t) + N(rx)}{1 + N(rx)} Z^x_r \geq Z^x_r
\]
a.s. Therefore, the conclusion follows from (15) and the conditional version of Jensen’s inequality (cf. [7, Theorem 5]).

In general, the inequality (16) does not hold if the nondecreasing character of the function \( f \) is dropped (take \( f(u) := -u \)). However, let us consider the following modification of \( L_t \) also introduced by Mazhar and Totik in [10]:

\[
L^*_t(f, x) := f(0) \pi_{t,0} + t \sum_{k=1}^{\infty} \pi_{t,k}(x) \int_0^\infty \pi_{t,k-1}(u)f(u) \, du.
\]
The operator \( L^*_t \) reproduces linear functions and, moreover, we have

\[
L^*_t(f, x) \geq L^*_r(f, x), \quad r > t > 0,
\]
for any convex function defined on \([0, \infty)\) satisfying the necessary integrability conditions. Actually, using the same notations as above, we have the representation

\[
L_t^*(f, x) = Ef(Q^x_t),
\]  (17)

where

\[
Q^x_t := U_{N(tx)} \frac{t}{N(tx)},
\]
and the claim follows from the equality

\[
E(Q^x_r \mid Q^x_t) = Q^x_t \quad \text{a.s.} \quad r > t > 0,
\]
and the conditional version of Jensen’s inequality.

On the other hand, the representation (17) can be used for studying \( L_t^* \). As a matter of fact, it can be shown that Theorem 5 above also holds if \( L_t \) is replaced by \( L_t^* \).
The Baskakov operator $B^*_t$ is defined by

$$B^*_t(f, x) := \sum_{k=0}^{\infty} f\left(\frac{k}{t}\right) b_{t,k}(x), \quad x \geq 0, \; t > 0,$$

where

$$b_{t,k}(x) := \binom{t + k - 1}{k} \frac{x^k}{(1 + x)^{t+k}}, \quad k = 0, 1, \ldots$$

Sahai and Prasad [11] introduced the following Durrmeyer-type modification of $B^*_t$:

$$M_t(f, x) := \int_0^\infty f(u) J_t(x, u) \, du, \quad x \geq 0, \; t > 1,$$

where

$$J_t(x, u) := (t-1) \sum_{k=0}^{\infty} b_{t,k}(x) b_{t,k}(u).$$

It can be shown that $M_t f$ is well-defined whenever $f$ is either a real-valued bounded measurable function on $[0, \infty)$, or a continuous function on $[0, \infty)$ such that $f(x) = O(x^r)$, $x \to \infty$, for some $0 < r < t - 1$.

Similarly to $D_n$ and $L_t$, the operator $M_t$ allows for an interesting probabilistic representation. Let $\{N(t) : t \geq 0\}$, $\{U_t : t \geq 0\}$, $\{U'_t : t \geq 0\}$ and $\{U''_t : t \geq 0\}$ be four mutually independent stochastic processes defined on the same probability space, where the first one is a standard Poisson process and the last three are gamma processes. Then, for $x \geq 0$ and $t > 1$, the random variable $W^x_t$ given by

$$W^x_t := \frac{U'_t(U_t + xU'_t)}{U''_{t-1}}$$

has the probability density $J_t(x, \cdot)$ defined in (19) and, therefore, we can write

$$M_t(f, x) = Ef(W^x_t).$$

Moreover, it is not hard to see that, for $t > 1$, Lemma 3 above also holds true if $Z^x_t$ is replaced by $W^x_t$ and $G^x_t$ is assumed to be the $\sigma$-algebra generated by the random variables $W^x_t, W^y_t, N(xU_t), N(yU_t), U_t$ and $U''_{t-1}$.

This shows that the operator $M_t$ can be studied by applying the same techniques we used in the preceding sections. Specific results are omitted. However, it is interesting to observe the following: For every real-valued function $f$ satisfying the adequate integrability conditions, we have

$$M_t f = B^*_t(T_t f),$$

where $B^*_t$ is the Baskakov operator and $T_t$ is the integral operator defined by

$$T_t(f, x) := \frac{1}{B(tx + 1, t - 1)} \int_0^\infty f(u) \frac{u^{tx}}{(1 + u)^{tx+1}} \, du, \quad x \geq 0, \; t > 1,$$

that is, using the same notations as above,

$$T_t(f, x) = Ef\left(\frac{U'_t + 1}{U''_{t-1}}\right).$$

Note that $T_t$ is a variant of the inverse beta operator introduced in [18]. It can be easily shown that $T_t$ also preserves monotonicity and convexity. Moreover, proceeding as in [18] for the inverse beta operator, we have

$$T_t f \geq T_r f, \quad r > t > 2,$$
whenever \( f \) is nondecreasing and convex on \([0, \infty)\) (and both \( T_t f \) and \( T_r f \) are well-defined). Since \( B_t \) is positive and satisfies the property of monotonic convergence under convexity, we conclude from (20)

\[
M_t f \geq M_r f, \quad r > t > 2,
\]

if \( f \) is nondecreasing and convex.

## 5. \( L_t \) AS LIMIT OF \( D_n \)

In this section, we show the following result which is analogous to Theorem 3(a) in [12].

**Theorem 7.** Let \( m \) be a fixed positive integer and let \( f \) be a real-valued bounded measurable function on \([0, \infty)\). Then, for \( n \geq x > 0 \), we have

\[
|D_{mn} \left( f(nu), \frac{x}{n} \right) - L_m(f, x) | \leq \|f\| \frac{5m^2x^2 + 18mx + 8}{2mn},
\]

where \( \|\cdot\| \) denotes sup-norm. As a consequence, we have uniform convergence, as \( n \to \infty \), on every bounded interval \([0, a] \).

**Proof.** Let \( f, m, n, \) and \( x \) be fixed and satisfying the assumptions in the statement of Theorem 7. By a change of variable, we can write

\[
D_{mn} \left( f(nu), \frac{x}{n} \right) = \int_0^\infty f(u)K_{mn} \left( \frac{x}{n}, \frac{u}{n}, \frac{1}{n} \right) du,
\]

where \( K_{mn}(x/n, u/n) \) is defined in (2) and it is assumed to be zero for \( u \geq n \). Therefore, the conclusion will follow as soon as we show that the next theorem holds.

**Theorem 8.** Let \( m, n \) and \( x \) be as in Theorem 7. Then

\[
\int_0^\infty \left| K_{mn} \left( \frac{x}{n}, \frac{u}{n}, \frac{1}{n} \right) - H_m(x, u) \right| du < \frac{5m^2x^2 + 18mx + 8}{2mn}, (21)
\]

where \( H_m(x, u) \) is defined in (14).

**Remark 4.** Observe that the left member in (21) is just the total variation distance between the probability distributions of the random variables \( N_m^{x/n} \) and \( Z_m \) introduced in Sections 2 and 3, respectively.

In order to prove Theorem 8, we shall need two auxiliary results. The first one gives a bound for the total variation distance between the binomial distribution and the Poisson distribution with the same mean, and it follows from a more general result by Barbour and Hall [19, Theorem 1].

**Lemma 4.** Let \( n \) be a natural number and let \( p \in (0, 1) \). Then

\[
\sum_{k=0}^\infty \left| \binom{n}{k} p^k (1-p)^{n-k} - e^{-np} \frac{(np)^k}{k!} \right| \leq 2p.
\]

**Lemma 5.** Let \( m, n \) and \( x \) be as in Theorem 8. Then

\[
m \sum_{k=0}^{mn} p_{mn,k} \left( \frac{x}{n} \right) \int_0^n \left| p_{mn,k} \left( \frac{u}{n} \right) - \pi_{mn,k}(u) \right| du < \frac{5m^2x^2 + 12mx + 4}{2mn},
\]

where \( p_{mn,k}(x) \) and \( \pi_{mn,k}(x) \) are defined in (3) and (13), respectively.

**Proof of Lemma 5.** Let \( k = 0, 1, \ldots, mn \) and \( u \in (0, n) \). We have

\[
\left| p_{mn,k} \left( \frac{u}{n} \right) - \pi_{mn,k}(u) \right| = \frac{(mu)^k}{k!} \left| e^{-mu} - \left( 1 - \frac{u}{n} \right)^{mn-k} \prod_{i=0}^{k-1} \left( 1 - \frac{i}{mn} \right) \right| \leq \frac{(mu)^k}{k!} (A + B + C),
\]

where

\[
A = \frac{x}{n} \left[ 1 - \left( 1 - \frac{1}{n} \right)^{mn} \right],
\]

\[
B = \frac{x}{n} \sum_{i=1}^{mn} \frac{(mu)^i}{i!},
\]

and

\[
C = \frac{x}{n} \sum_{i=1}^{mn} \frac{(mu)^i}{i!} - \frac{x}{n} \sum_{i=1}^{mn} \frac{(mu)^i}{i!}.
\]
where
\[
A := e^{-mu} \left| 1 - \prod_{i=0}^{k-1} \left( 1 - \frac{i}{mn} \right) \right| \leq e^{-mu} \sum_{i=0}^{k-1} \frac{i}{mn} = e^{-mu} \frac{k(k-1)}{2mn},
\]
\[
B := \left| 1 - \left( 1 - \frac{u}{n} \right)^k \left( 1 - \frac{u}{n} \right)^{mn-k} \prod_{i=0}^{k-1} \left( 1 - \frac{i}{mn} \right) \right| \leq \frac{ku}{n} \left( 1 - \frac{u}{n} \right)^{mn-k} \prod_{i=0}^{k-1} \left( 1 - \frac{i}{mn} \right)
\]
and
\[
C := e^{-mu} - \left( 1 - \frac{u}{n} \right)^{mn-k} \prod_{i=0}^{k-1} \left( 1 - \frac{i}{mn} \right) \leq e^{-mu} \left( \frac{mu}{mn} \right)^2.
\]
Therefore,
\[
m \int_0^n \left| p_{mn,k} \left( \frac{u}{n} \right) - \pi_{m,k}(u) \right| du \leq \frac{k(k-1)}{2mn} \int_0^\infty e^{-mu} \frac{(mu)^k}{k!} mdu + mk \left( \frac{mn}{k} \right) \int_0^n \left( \frac{u}{n} \right)^{k+1} \left( 1 - \frac{u}{n} \right)^{mn-k} du
\]
\[
+ \frac{1}{mn} \int_0^\infty e^{-mu} \frac{(mu)^{k+2}}{k!} mdu
\]
\[
\leq \frac{k(k-1)}{2mn} + \frac{k(k+1)}{mn} + \frac{(k+1)(k+2)}{mn}
\]
\[
= \frac{5k^2 + 7k + 4}{2mn}.
\]
Since
\[
\sum_{k=0}^{mn} \frac{5k^2 + 7k + 4}{2mn} p_{mn,k} \left( \frac{x}{n} \right) = \frac{5m^2x^2 + 12mx + 4}{2mn} - \frac{5x^2}{2n^2},
\]
the conclusion follows.

Now, we are ready to prove Theorem 8.

**Proof of Theorem 8.** Using the triangle inequality, the left-hand side in (21) is bounded above by the sum \( V + X + Y + Z \), where
\[
V := \sum_{k=0}^{mn} p_{mn,k} \left( \frac{x}{n} \right) \int_0^n p_{mn,k} \left( \frac{u}{n} \right) \left( \frac{1}{n} \right) du = \frac{1}{mn+1} \sum_{k=0}^{mn} p_{mn,k} \left( \frac{x}{n} \right) < \frac{1}{mn},
\]
\[
X := m \sum_{k=0}^\infty \pi_{m,k}(x) \int_0^\infty \pi_{m,k}(u) du \leq \frac{1}{mn} \sum_{k=0}^\infty \pi_{m,k}(x) \int_0^\infty e^{-mu} \frac{(mu)^{k+1}}{k!} mdu
\]
\[
= \frac{1}{mn} \sum_{k=0}^{mn} (k+1) \pi_{m,k}(x) = \frac{mx+1}{mn},
\]
\[
Y := m \sum_{k=0}^\infty \left| p_{mn,k} \left( \frac{x}{n} \right) - \pi_{m,k}(x) \right| \int_0^n \pi_{m,k}(u) du
\]
\[
\leq \sum_{k=0}^\infty \left| p_{mn,k} \left( \frac{x}{n} \right) - \pi_{m,k}(x) \right| \leq \frac{2x}{n},
\]
\[
Z := e^{-mu} \left( 1 - \frac{u}{n} \right)^{mn-k} \prod_{i=0}^{k-1} \left( 1 - \frac{i}{mn} \right) \leq e^{-mu} \left( \frac{mu}{mn} \right)^2.
\]
the last inequality by Lemma 4, and, finally,
\[ Z := m \sum_{k=0}^{mn} p_{mn,k} \left( \frac{x}{t} \right) \int_0^1 |p_{mn,k} \left( \frac{u}{t} \right) - \pi_{m,k}(u)| du < \frac{5m^2x^2 + 12mx + 4}{2mn}, \]
by Lemma 5. The proof of Theorem 8 is complete.

6. \( L_t \) AS LIMIT OF \( M_t \)

In this section, we show the following result which is analogous to Theorem 5 in [13].

**Theorem 9.** Let \( x \geq 0 \) and let \( r, t > 0 \) with \( rt > 1 \). If \( f \) is a real-valued bounded measurable function on \([0, \infty)\), then
\[ |M_{rt} \left( f(tu), \frac{x}{t} \right) - L_r(f, x)| \leq \|f\| C_t(r, x), \]
where
\[ C_t(r, x) := 2 \min \left( \frac{x}{t}, \frac{rx^2}{t} + \frac{r^2x^2 + 2rx + 2}{rt} \right). \] (22)

Therefore, we have uniform convergence, as \( t \to \infty \), on every bounded interval \([0, a]\).

Proceeding as in the proof of Theorem 7, it is clear that Theorem 9 is a consequence of the following result which gives a bound for the total variation distance between the probability distributions of the random variables \( tW_{rt}^{2/t} \) and \( Z_r^t \) introduced in Sections 4 and 3, respectively.

**Theorem 10.** Let \( r, t \) and \( x \) be as in Theorem 9. Then
\[ \int_0^{\infty} \left| J_{rt} \left( \frac{x}{t}, \frac{u}{t} \right) - H_r(x, u) \right| du \leq C_t(r, x), \] (23)
where \( C_t(r, x) \), \( H_r(x, u) \) and \( J_t(x, u) \) are defined in (22), (14) and (19), respectively.

To prove this theorem, we need two lemmas. The first one gives a bound for the total variation distance between the negative binomial distribution and the Poisson distribution with the same mean. It has been shown in [13].

**Lemma 6.** Let \( x \geq 0 \) and let \( r, t > 0 \). Then
\[ \sum_{k=0}^{\infty} b_{rt,k} \left( \frac{x}{t} \right) - \pi_{r,k}(x) \leq 2 \min \left( \frac{x}{t}, \frac{rx^2}{t} \right), \]
where \( b_{rt,k}(x) \) and \( \pi_{r,k}(x) \) are defined in (18) and (13), respectively.

**Lemma 7.** Let \( r \) and \( t \) be as in Theorem 9. For \( k = 0, 1, \ldots \), we have
\[ \int_0^{\infty} \left| r - \frac{1}{t} \right| b_{rt,k} \left( \frac{u}{t} \right) - r\pi_{r,k}(u) \right| du \leq \frac{k^2 + k + 2}{rt}. \] (24)

**Proof of Lemma 7.** Consider the function \( h(u) \) defined, for \( u \geq 0 \), by
\[ h(u) := 1 - \left( r - \frac{1}{t} \right) \frac{b_{rt,k} \left( \frac{u}{t} \right)}{r\pi_{r,k}(u)} \\
= 1 - e^{ru} \left( 1 + \frac{u}{t} \right)^{-rt-k} \prod_{i=-1}^{k-1} \left( 1 + \frac{i}{rt} \right). \]
From
\[ h'(u) = \frac{k - ru}{t} e^{ru} \left( 1 + \frac{u}{t} \right)^{-rt-k-1} \prod_{i=1}^{k-1} \left( 1 + \frac{i}{rt} \right), \]
we have
\[ \sup_{u \geq 0} h(u) = h\left( \frac{k}{r} \right) = 1 - e^{k} \left( 1 + \frac{k}{rt} \right)^{-rt} \left( 1 - \frac{1}{rt} \right) \prod_{i=1}^{k} \left( 1 - \frac{i}{rt+k} \right) \]
\[ \leq 1 - \left( 1 - \frac{1}{rt} \right) \prod_{i=1}^{k} \left( 1 - \frac{i}{rt+k} \right) \]
\[ \leq \frac{1}{rt} + \sum_{i=1}^{k} \frac{i}{rt+k} \]
\[ \leq \frac{k^2 + k + 2}{2rt}. \]
Since both functions \( r\pi_{r,k}(\cdot) \) and \( (r - 1/t)b_{r,t,k}(\cdot/t) \) are probability densities on \([0, \infty)\), the left-hand side in (24) is equal to
\[ 2 \int_{h(u) \geq 0} h(u) r\pi_{r,k}(u) \, du \leq 2 \sup_{u \geq 0} h(u) \leq \frac{k^2 + k + 2}{rt}. \]
The proof of Lemma 7 is complete.

**Proof of Theorem 10.** The left-hand side in (23) is bounded above by the sum \( F + G \), where
\[ F := \sum_{k=0}^{\infty} \left| b_{r,t,k} \left( \frac{x}{t} \right) - \pi_{r,k}(x) \right| \int_{0}^{\infty} \left( r - \frac{1}{t} \right) b_{r,t,k} \left( \frac{u}{t} \right) \, du \]
\[ = \sum_{k=0}^{\infty} \left| b_{r,t,k} \left( \frac{x}{t} \right) - \pi_{r,k}(x) \right|, \]
and
\[ G := \sum_{k=0}^{\infty} \pi_{r,k}(x) \int_{0}^{\infty} \left| \left( r - \frac{1}{t} \right) b_{r,t,k} \left( \frac{u}{t} \right) - r \pi_{r,k}(u) \right| \, du. \]
Therefore, the conclusion follows from Lemmas 6 and 7.

7. **Concluding Remark**

An interesting consequence of the probabilistic representations constructed in Sections 2–4 above is that all the operators considered in the present paper diminish both the \( \phi \)-variation and the fine \( \phi \)-variation. We refer to [20] for a more detailed information on this subject.

**Error Correction**

We have found an error in [8] concerning the estimate of the second modulus of continuity of \( B_n f \), where \( B_n \) is the Bernstein operator. Actually, in example (A) in [8, Section 6], the correct value of the expectation
\[ E \left( 1 + \frac{S_n(h)}{nh} \right)^2 I(S_n(h) \geq 2) \]
is the following:
\[ 4 + \frac{1 - (1 - h)^{n-1}}{nh} - \frac{1}{n} - (3 + (n - 1)h)(1 - h)^{n-1}, \]
and this quantity is bounded above by 5.
REFERENCES