Applied Mathematics Letters 24 (2011) 936–942

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Global existence and blow-up solutions for quasilinear reaction–diffusion equations with a gradient term

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ARTICLE INFO

Article history:
Received 29 June 2010
Received in revised form 31 December 2010
Accepted 31 December 2010

Keywords:
Reaction–diffusion equation
Blow-up solution
Global solution
Gradient term

ABSTRACT

In this work, we study the blow-up and global solutions for a quasilinear reaction–diffusion equation with a gradient term and nonlinear boundary condition:

$$\begin{cases} (g(u))_t = \Delta u + f(x, u, |\nabla u|^2, t) & \text{in } D \times (0, T), \\ \dfrac{\partial u}{\partial n} = r(u) & \text{on } \partial D \times (0, T), \\ u(x, 0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ is a bounded domain with smooth boundary ∂D. Through constructing suitable auxiliary functions and using maximum principles, the sufficient conditions for the existence of a blow-up solution, an upper bound for the "blow-up time", an upper estimate of the "blow-up rate", the sufficient conditions for the existence of the global solution, and an upper estimate of the global solution are specified under some appropriate assumptions on the nonlinear system functions f, g, r, and initial value u_0.

1. Introduction

Global and blow-up solutions for quasilinear reaction–diffusion equations are discussed by many authors (see e.g., [1–6]). In this work, we study the blow-up and global solutions for the following initial-boundary-value problem of quasilinear reaction–diffusion equation with a gradient term and nonlinear boundary condition:

$$\begin{cases} (g(u))_t = \Delta u + f(x, u, q, t) & \text{in } D \times (0, T), \\ \dfrac{\partial u}{\partial n} = r(u) & \text{on } \partial D \times (0, T), \\ u(x, 0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases} \tag{1.1}$$

where $q = |\nabla u|^2$, $D \subset \mathbb{R}^N$ is a bounded domain with smooth boundary ∂D, $\partial/\partial n$ represents the outward normal derivative on ∂D, u_0 is the initial value, T the maximal existence time of u, and $\overline{D}$ the closure of D. Set $\mathbb{R}^+ = (0, +\infty)$. We assume, throughout the work, that $f(x, s, d, t)$ is a nonnegative $C^1(\overline{D} \times \mathbb{R}^+ \times \overline{\mathbb{R}^+} \times \overline{R^+})$ function, $g(s)$ is a $C^2(\mathbb{R}^+)$ function, $g'(s) > 0$ for any $s > 0$, $r(s)$ is a positive $C^2(\mathbb{R}^+)$ function, and u_0 is a positive $C^2(\overline{D})$ function. Under these assumptions, the classical

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doi:10.1016/j.aml.2010.12.052

parabolic equation theory [7] ensures that there exists a unique classical solution $u(x,t)$ for the problem (1.1) with some $T > 0$, and the solution is positive over $\overline{D}\times[0,T)$. Moreover, by the regularity theorem [8], $u \in C^3(D\times(0,T))\cap C^2(\overline{D}\times[0,T))$.

The problems of the blow-up and global solutions for reaction–diffusion equations with gradient term have been investigated extensively by many authors. Souplet et al. [9] deal with the blow-up and global solutions of initial value problems for the reaction–diffusion equations with a gradient term. Chen [10], Chipot and Weissler [11], Fila [12], and Souplet et al. [13–15], and Ding [16] study the existence of blow-up and global solutions for the reaction–diffusion equations with a gradient term and initial-Dirichlet boundary-value. Ding and Guo [17] and Zhang [18] investigate the blow-up and global solutions for the reaction–diffusion equations with gradient terms and initial-Neumann boundary-values. Some special cases of (1.1) are also treated. Walter [19] studies the following problem:

$$\begin{cases} u_t = \Delta u & \text{in } D\times(0,T),\\ \dfrac{\partial u}{\partial n} = r(u) & \text{on } \partial D\times(0,T),\\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. The sufficient conditions characterized by function r are given for the existence of blow-up and global solutions. Amann [20] considers the following problem:

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } D\times(0,T),\\ \dfrac{\partial u}{\partial n} = r(u) & \text{on } \partial D\times(0,T),\\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. The sufficient conditions are obtained for the existence of a blow-up solution. Zhang [21] discusses the following problem:

$$\begin{cases} (g(u))_t = \Delta u + f(u) & \text{in } D\times(0,T),\\ \dfrac{\partial u}{\partial n} = r(u) & \text{on } \partial D\times(0,T),\\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. The sufficient conditions are obtained there for the existence of a global solution and a blow-up solution. Meanwhile, the upper estimate of the global solution, the upper bound of the "blow-up time", and the upper estimate of the "blow-up rate" are also given.

In this work, we study the problem (1.1). Through technical construction of suitable auxiliary functions and using maximum principles, the sufficient conditions for the existence of a blow-up solution, an upper bound for the "blow-up time", an upper estimate of the "blow-up rate", the sufficient conditions for the global solution, and an upper estimate of the global solution are specified under some appropriate assumptions on the functions f, g, r, and initial data u_0. Our results extend and supplement those obtained in [19–21].

We proceed as follows. In Section 2 we give the proofs for the main results. A few examples are presented in Section 3 to illustrate the applications of the abstract results.

2. The main results

Our first result Theorem 2.1 is about the existence of a blow-up solution.

Theorem 2.1. *Let u be a solution of (1.1). Assume that the following conditions* (i)–(iii) *are fulfilled:*

(i) *the initial value condition:*

$$\beta = \min_{\overline{D}} \frac{\Delta u_0 + f(x,u_0,q_0,0)}{r(u_0)g'(u_0)} > 0, \quad q_0 = |\nabla u_0|^2; \tag{2.1}$$

(ii) *further restrictions for functions involved: for any $(x,s,d,t) \in D\times\mathbb{R}^+\times\overline{\mathbb{R}^+}\times\mathbb{R}^+$,*

$$r''(s) + 2r'(s)f_d(x,s,d,t) \ge 0, \qquad \frac{f_t(x,s,d,t)}{r^2(s)} + \beta\left(\frac{f(x,s,d,t)}{r(s)}\right)_s - \beta^2 g''(s) \ge 0; \tag{2.2}$$

(iii) *the integration condition:*

$$\int_{M_0}^{+\infty} \frac{1}{r(s)}\mathrm{d}s < +\infty, \quad M_0 = \max_{\overline{D}} u_0(x). \tag{2.3}$$

Then the solution u of (1.1) must blow up in a finite time T, and

$$T \le \frac{1}{\beta}\int_{M_0}^{+\infty}\frac{1}{r(s)}\mathrm{d}s, \tag{2.4}$$

$$u(x,t) \le H^{-1}\left(\beta(T-t)\right), \tag{2.5}$$

where

$$H(z) = \int_{z}^{+\infty}\frac{1}{r(s)}\mathrm{d}s, \quad z > 0, \tag{2.6}$$

and H^{-1} is the inverse function of H.

Proof. Consider the auxiliary function

$$\Psi(x,t) = -\frac{1}{r(u)}u_t + \beta. \tag{2.7}$$

We find that

$$\nabla\Psi = \frac{r'}{r^2}u_t\nabla u - \frac{1}{r}\nabla u_t, \tag{2.8}$$

$$\Delta\Psi = \left(\frac{r''}{r^2} - \frac{2(r')^2}{r^3}\right)u_t|\nabla u|^2 + \frac{2r'}{r^2}\nabla u\cdot\nabla u_t + \frac{r'}{r^2}u_t\Delta u - \frac{1}{r}\Delta u_t, \tag{2.9}$$

and

$$\begin{aligned}\Psi_t &= \frac{r'}{r^2}(u_t)^2 - \frac{1}{r}(u_t)_t = \frac{r'}{r^2}(u_t)^2 - \frac{1}{r}\left(\frac{\Delta u}{g'} + \frac{f}{g'}\right)_t \\ &= \frac{r'}{r^2}(u_t)^2 - \frac{1}{rg'}\Delta u_t + \frac{g''}{r(g')^2}u_t\Delta u + \frac{g''f}{r(g')^2}u_t - \frac{f_u}{rg'}u_t - \frac{2f_q}{rg'}\nabla u\cdot\nabla u_t - \frac{f_t}{rg'}.\end{aligned} \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\begin{aligned}\frac{1}{g'}\Delta\Psi - \Psi_t = {}&\left(\frac{r''}{r^2g'} - \frac{2(r')^2}{r^3g'}\right)u_t|\nabla u|^2 + \left(\frac{2r'}{r^2g'} + \frac{2f_q}{rg'}\right)\nabla u\cdot\nabla u_t + \left(\frac{r'}{r^2g'} - \frac{g''}{r(g')^2}\right)u_t\Delta u \\ &- \frac{r'}{r^2}(u_t)^2 + \left(\frac{f_u}{rg'} - \frac{g''f}{r(g')^2}\right)u_t + \frac{f_t}{rg'}.\end{aligned} \tag{2.11}$$

In view of (2.8), we have

$$\nabla u_t = -r\nabla\Psi + \frac{r'}{r}u_t\nabla u. \tag{2.12}$$

Substitute (2.12) into (2.11) to obtain

$$\begin{aligned}\frac{1}{g'}\Delta\Psi + \frac{2(r'+rf_q)}{rg'}\nabla u\cdot\nabla\Psi - \Psi_t = {}&\left(\frac{r''}{r^2g'} + \frac{2r'f_q}{r^2g'}\right)u_t|\nabla u|^2 + \left(\frac{r'}{r^2g'} - \frac{g''}{r(g')^2}\right)u_t\Delta u \\ &- \frac{r'}{r^2}(u_t)^2 + \left(\frac{f_u}{rg'} - \frac{g''f}{r(g')^2}\right)u_t + \frac{f_t}{rg'}.\end{aligned} \tag{2.13}$$

By (1.1), we have

$$\Delta u = g'u_t - f. \tag{2.14}$$

Substitute (2.14) into (2.13), to get

$$\frac{1}{g'}\Delta\Psi + \frac{2(r'+rf_q)}{rg'}\nabla u\cdot\nabla\Psi - \Psi_t = \left(\frac{r''}{r^2g'} + \frac{2r'f_q}{r^2g'}\right)u_t|\nabla u|^2 - \frac{g''}{rg'}(u_t)^2 + \left(\frac{f_u}{rg'} - \frac{fr'}{r^2g'}\right)u_t + \frac{f_t}{rg'}. \tag{2.15}$$

With (2.7), we have

$$u_t = -r\Psi + r\beta. \tag{2.16}$$

Substitution of (2.16) into (2.15) gives

$$\frac{1}{g'}\Delta\Psi + \frac{2(r'+rf_q)}{rg'}\nabla u\cdot\nabla\Psi + \left\{\frac{r''+2r'f_q}{rg'}|\nabla u|^2 + \frac{r}{g'}\left[(\Psi-2\beta)g'' + \left(\frac{f}{r}\right)_u\right]\right\}\Psi - \Psi_t$$
$$= \frac{r''+2r'f_q}{rg'}\beta|\nabla u|^2 + \frac{r}{g'}\left[\frac{f_t}{r^2} + \beta\left(\frac{f}{r}\right)_u - \beta^2 g''\right]. \tag{2.17}$$

From assumptions (2.1) and (2.2), the right-hand side of (2.17) is nonnegative, i.e.

$$\frac{1}{g'}\Delta\Psi + \frac{2(r'+rf_q)}{rg'}\nabla u\cdot\nabla\Psi + \left\{\frac{r''+2r'f_q}{rg'}|\nabla u|^2 + \frac{r}{g'}\left[(\Psi-2\beta)g'' + \left(\frac{f}{r}\right)_u\right]\right\}\Psi - \Psi_t \geq 0. \tag{2.18}$$

Now by (2.1), we have

$$\max_{\overline{D}}\Psi(x,0) = \max_{\overline{D}}\left(-\frac{\Delta u_0 + f(x,u_0,q_0,0)}{r(u_0)g'(u_0)} + \beta\right) = 0. \tag{2.19}$$

It follows from (1.1) that, on $\partial D\times(0,T)$,

$$\frac{\partial\Psi}{\partial n} = \frac{r'}{r^2}u_t\frac{\partial u}{\partial n} - \frac{1}{r}\frac{\partial u_t}{\partial n} = \frac{r'}{r^2}u_t r - \frac{1}{r}\left(\frac{\partial u}{\partial n}\right)_t = \frac{r'}{r}u_t - \frac{1}{r}(r)_t = \frac{r'}{r}u_t - \frac{r'}{r}u_t = 0. \tag{2.20}$$

Combining (2.18)–(2.20), and applying the maximum principles [22], we know that the maximum of Ψ in $\overline{D}\times[0,T)$ is zero. Thus

$$\Psi \leq 0 \quad \text{in } \overline{D}\times[0,T),$$

and

$$\frac{1}{\beta r(u)}u_t \geq 1. \tag{2.21}$$

At the point $x_0\in\overline{D}$ where $u_0(x_0) = M_0$, integrate (2.21) over $[0,t]$ to produce

$$\frac{1}{\beta}\int_0^t \frac{1}{r(u)}u_t\mathrm{d}t = \frac{1}{\beta}\int_{M_0}^{u(x_0,t)}\frac{1}{r(s)}\mathrm{d}s \geq t.$$

This together with assumption (2.3) shows that u must blow up in the finite time T and

$$T \leq \frac{1}{\beta}\int_{M_0}^{+\infty}\frac{1}{r(s)}\mathrm{d}s.$$

By integrating the inequality (2.21) over $[t,s]$ $(0<t<s<T)$, one has, for each fixed x, that

$$H(u(x,t)) \geq H(u(x,t)) - H(u(x,s)) = \int_{u(x,t)}^{u(x,s)}\frac{1}{r(s)}\mathrm{d}s \geq \beta(s-t).$$

Passing to the limit as $s\to T$ yields

$$H(u(x,t)) \geq \beta(T-t),$$

which implies that

$$u(x,t) \leq H^{-1}(\beta(T-t)).$$

The proof is complete. □

The result on the global solution is stated as Theorem 2.2 below.

Theorem 2.2. *Let u be a solution of (1.1). Assume that the following conditions are satisfied:*

(i) *the initial value condition:*

$$\alpha = \max_{\overline{D}}\frac{\Delta u_0 + f(x,u_0,q_0,0)}{r(u_0)g'(u_0)} > 0, \quad q_0 = |\nabla u_0|^2; \tag{2.22}$$

(ii) *further restrictions on functions involved: for any* $(x, s, d, t) \in D \times \mathbb{R}^+ \times \overline{\mathbb{R}^+} \times \mathbb{R}^+$,

$$r''(s) + 2r'(s)f_d(x, s, d, t) \le 0, \qquad \frac{f_t(x, s, d, t)}{r^2(s)} + \alpha \left(\frac{f(x, s, d, t)}{r(s)}\right)_s - \alpha^2 g''(s) \le 0; \tag{2.23}$$

(iii) *the integration condition:*

$$\int_{m_0}^{+\infty} \frac{1}{r(s)} \mathrm{d}s = +\infty, \quad m_0 = \min_{\overline{D}} u_0(x). \tag{2.24}$$

Then the solution u of (1.1) must be a global solution and

$$u(x, t) \le G^{-1} \left(\alpha t + G(u_0(x))\right), \tag{2.25}$$

where

$$G(z) = \int_{m_0}^{z} \frac{1}{r(s)} \mathrm{d}s, \quad z \ge m_0, \tag{2.26}$$

and G^{-1} is the inverse function of G.

Proof. Construct an auxiliary function

$$\Phi(x, t) = -\frac{1}{r(u)} u_t + \alpha. \tag{2.27}$$

Replacing Ψ and β with Φ and α in (2.17), we have

$$\begin{aligned} &\frac{1}{g'}\Delta\Phi + \frac{2(r' + rf_q)}{rg'}\nabla u \cdot \nabla\Phi + \left\{\frac{r'' + 2r'f_q}{rg'}|\nabla u|^2 + \frac{r}{g'}\left[(\Psi - 2\alpha)g'' + \left(\frac{f}{r}\right)_u\right]\right\}\Phi - \Phi_t \\ &\quad = \frac{r'' + 2r'f_q}{rg'}\alpha|\nabla u|^2 + \frac{r}{g'}\left[\frac{f_t}{r^2} + \alpha\left(\frac{f}{r}\right)_u - \alpha^2 g''\right]. \end{aligned} \tag{2.28}$$

It is seen from assumptions (2.22) and (2.23) that the right-hand side of (2.28) is nonpositive, i.e.

$$\frac{1}{g'}\Delta\Phi + \frac{2(r' + rf_q)}{rg'}\nabla u \cdot \nabla\Phi + \left\{\frac{r'' + 2r'f_q}{rg'}|\nabla u|^2 + \frac{r}{g'}\left[(\Psi - 2\alpha)g'' + \left(\frac{f}{r}\right)_u\right]\right\}\Phi - \Phi_t \le 0. \tag{2.29}$$

By (2.22), we have

$$\min_{\overline{D}} \Phi(x, 0) = \min_{\overline{D}} \left(-\frac{\Delta u_0 + f(x, u_0, q_0, 0)}{r(u_0)g'(u_0)} + \alpha\right) = 0. \tag{2.30}$$

It follows from (1.1) and (2.20) that

$$\frac{\partial\Phi}{\partial n} = \frac{\partial\Psi}{\partial n} = 0 \quad \text{on } \partial D \times (0, T). \tag{2.31}$$

Combining (2.29)–(2.31) and applying the maximum principles, we know that the minimum of Φ in $\overline{D} \times [0, T)$ is zero. Hence

$$\Phi \ge 0 \quad \text{in } \overline{D} \times [0, T),$$

i.e.

$$\frac{1}{\alpha r(u)} u_t \le 1. \tag{2.32}$$

For each fixed $x \in \overline{D}$, integrate (2.32) over $[0, t]$ to get

$$\frac{1}{\alpha}\int_0^t \frac{1}{r(u)} u_t \mathrm{d}t = \frac{1}{\alpha}\int_{u_0(x)}^{u(x,t)} \frac{1}{r(s)} \mathrm{d}s \le t.$$

This together with (2.24) shows that u must be a global solution. Moreover, (2.32) implies that

$$G(u(x, t)) - G(u_0(x)) = \int_{m_0}^{u(x,t)} \frac{1}{r(s)} \mathrm{d}s - \int_{m_0}^{u_0(x)} \frac{1}{r(s)} \mathrm{d}s = \int_{u_0(x)}^{u(x,t)} \frac{1}{r(s)} \mathrm{d}s = \int_0^t \frac{1}{r(u)} u_t \mathrm{d}t \le \alpha t.$$

Therefore

$$u(x,t) \le G^{-1}\left(\alpha t + G(u_0(x))\right).$$

The proof is complete. $\square$

3. Applications

When $g(u) \equiv u$, $f(x,u,q,t) \equiv 0$ or $g(u) \equiv u$, $f(x,u,q,t) \equiv f(u)$ or $f(x,u,q,t) \equiv f(u)$, the conclusions of Theorems 2.1 and 2.2 still hold true. In this sense, our results extend and supplement the results of [19–21].

In what follows, we present several examples to demonstrate the applications of Theorems 2.1 and 2.2.

Example 3.1. Let u be a solution of the following problem:

$$\begin{cases} \left(u^m\right)_t = \Delta u + u^n & \text{in } D \times (0,T), \\ \dfrac{\partial u}{\partial n} = u^p & \text{on } \partial D \times (0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ is a bounded domain with smooth boundary ∂D, $m > 0$, $-\infty < n < +\infty$, $-\infty < p < +\infty$. Here

$$g(u) = u^m, \qquad f(x,u,q,t) = u^n, \qquad r(u) = u^p.$$

By Theorem 2.1, if $n \ge p > 1 \ge m$ and

$$\beta = \min_{\overline{D}} \frac{\Delta u_0 + u_0^n}{m u_0^{p+m-1}} > 0,$$

u must blow up in finite time T and

$$T \le \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{1}{r(s)}\mathrm{d}s = \frac{M_0^{1-p}}{\beta(p-1)}, \qquad M_0 = \max_{\overline{D}} u_0(x),$$

$$u(x,t) \le H^{-1}(\beta(T-t)) = [(1-p)\beta(T-t)]^{\frac{1}{1-p}}.$$

By Theorem 2.2, if $0 \le p \le 1 \le m$, $n \le p$ and

$$\alpha = \max_{\overline{D}} \frac{\Delta u_0 + u_0^n}{m u_0^{p+m-1}} > 0,$$

u must be a global solution and

$$u(x,t) \le G^{-1}(\alpha t + G(u_0(x))) = \begin{cases} \left[(1-p)\alpha t + (u_0(x))^{1-p}\right]^{\frac{1}{1-p}}, & p < 1, \\ u_0(x)\mathrm{e}^{\alpha t}, & p = 1. \end{cases}$$

Example 3.2. Let u be a solution of the following problem:

$$\begin{cases} \left(u + \sqrt{u}\right)_t = \Delta u + \left(tu^2 + q + \displaystyle\sum_{i=1}^{3} x_i^2\right) u^2 & \text{in } D \times (0,T), \\ \dfrac{\partial u}{\partial n} = \dfrac{1}{2}u^2 & \text{on } \partial D \times (0,T), \\ u(x,0) = 1 + \displaystyle\sum_{i=1}^{3} x_i^2 & \text{in } \overline{D}, \end{cases}$$

where $q = |\nabla u|^2$, $D = \left\{x = (x_1,x_2,x_3) \mid \sum_{i=1}^{3} x_i^2 < 1\right\}$ is the unit ball of $\mathbb{R}^3$. Now

$$g(u) = u + \sqrt{u}, \qquad f(x,u,q,t) = \left(tu^2 + q + \sum_{i=1}^{3} x_i^2\right)u^2, \qquad r(u) = \frac{1}{2}u^2,$$

and

$$\beta = \min_{\overline{D}} \frac{\Delta u_0 + f(x,u_0,q_0,0)}{r(u_0)g'(u_0)} = 4\min_{1 \le u_0 \le 2} \frac{6 + 5u_0^3 - 5u_0^2}{2u_0^2 + u_0^{\frac{3}{2}}} = 7.0205.$$

It is easy to check that (2.2) and (2.3) hold. It follows from Theorem 2.1 that u must blow up in a finite time T, and

$$T \le \frac{1}{\beta}\int_{M_0}^{+\infty} \frac{1}{r(s)}\mathrm{d}s = 0.1424,$$

$$u(x,t) \le H^{-1}(\beta(T-t)) = \frac{0.2848}{T-t}.$$

Example 3.3. Let u be a solution of the following problem:

$$\begin{cases} \left(u\mathrm{e}^u\right)_t = \Delta u + \left(\mathrm{e}^{-t-u} + \mathrm{e}^{-q} + \sum_{i=1}^{3} x_i^2\right)\sqrt{u} & \text{in } D\times(0,T),\\ \dfrac{\partial u}{\partial n} = \sqrt{2u} & \text{on } \partial D\times(0,T),\\ u(x,0) = 1 + \sum_{i=1}^{3} x_i^2 & \text{in } \overline{D}, \end{cases}$$

where $q = |\nabla u|^2$, $D = \left\{x = (x_1,x_2,x_3) \mid \sum_{i=1}^3 x_i^2 < 1\right\}$ is the unit ball of $\mathbb{R}^3$. Now we have

$$g(u) = u\mathrm{e}^u, \qquad f(x,u,q,t) = \left(\mathrm{e}^{-t-u} + \mathrm{e}^{-q} + \sum_{i=1}^{3} x_i^2\right)\sqrt{u}, \qquad r(u) = \sqrt{2u},$$

and

$$\alpha = \max_{\overline{D}} \frac{\Delta u_0 + f(x,u_0,q_0,0)}{r(u_0)g'(u_0)} = \max_{1\le u_0\le 2} \frac{6 + \left(\mathrm{e}^{-u_0} + \mathrm{e}^{4(1-u_0)} + u_0 - 1\right)\sqrt{u_0}}{\sqrt{2u_0}\,(1+u_0)\,\mathrm{e}^{u_0}} = \frac{7\mathrm{e}+1}{2\sqrt{2}\mathrm{e}^2}.$$

It is easy to check that (2.23) and (2.24) hold. It then follows from Theorem 2.2 that u must be a global solution and

$$u(x,t) \le G^{-1}\left(\alpha t + G(u_0(x))\right) = \left(\sqrt{u_0(x)} + \frac{7\mathrm{e}+1}{4\mathrm{e}^2}t\right)^2.$$

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Nos. 61074048 and 60974032), the National Basic Research Program of China with grant (No. 2011CB808002), the Natural Science Foundation of Shanxi Province, and the Research Project Supported by Shanxi Scholarship Council of China.

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