Continuous Dependence in Rational
Chebyshev Approximation

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The dependence of best Chebyshev approximation by generalized abstract rational functions on the function being approximated is studied.

Let \( X \neq \emptyset \) be a compact topological space and

\[
\| g \| = \max \{ \| g(x) \| : x \in X \}.
\]

Let \( \{ \phi_1, \ldots, \phi_n \} \), \( \{ \psi_1, \ldots, \psi_m \} \) be linearly independent subsets of \( C(X) \). Define

\[
R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k \phi_k(x) / \sum_{k=1}^{m} a_{n+k} \psi_k(x).
\]

The conventions of Boehm (assuming his dense nonzero property is satisfied) or of Goldstein (stabilized rational functions) [14, pp. 84–89] can be used to give \( R(A, x) \) a value when \( Q(A, x) = 0 \). Let \( K \) be a subset of \( (n + m) \)-space.

The problem of \( K \)-rational approximation is given \( f \in C(X) \) to choose \( A^* \) minimizing \( e(A) = \| f - R(A, \cdot) \| \) over \( K \). Such an \( A^* \) is called best and \( R(A^*, \cdot) \) is called a best approximation to \( f \). Denote \( A^* \) by \( C(f) \) and \( R(A^*, \cdot) \) by \( T(f) \).

If \( K \) is closed and nonempty, a best coefficient and approximation exists by generalizations of the arguments of Boehm and Goldstein [Dunham, 9, 11].

\( \{ f_k \} \) will denote any sequence with limit \( f \).

Now \( R(\alpha A, x) = R(A, x) \) for \( \alpha > 0 \). Hence for convenience in existence and convergence arguments, we normalize rational functions \( R(A, \cdot) \) so that

\[
\sum_{k=1}^{m} |a_{n+k}| = 1.
\]
Let

\[ K_{GE} = \{ A : Q(A, \cdot) \geq 0, Q(A, \cdot) \neq 0 \}, \]
\[ K_0 = \{ A : Q(A, \cdot) \neq 0 \}. \]

Under the normalization (1), \( K_{GE} \) and \( K_0 \) are closed.

**Theorem 1.** Let \( \| R(A, \cdot) \| < \infty \) for some \( A \in K \). Let \( K \) be closed. The sequence \( C(f_k) \) has a limit point \( A \). For any such limit point \( A \), \( R(A, \cdot) \) is a best approximation to \( f \).

**Proof.** Define

\[ \| A \| = \sum_{k=1}^{n} |a_k|. \]

Let \( C(f_k) = A^k \). Suppose \( \{\|A^k\|\} \) is an unbounded sequence. By taking a subsequence if necessary, we can assume \( \|A^k\| > k \). Define \( B^k = A^k/\|A^k\| \).

Then \( \|B^k\| = 1 \) and \( \{B^k\} \) is a bounded sequence with accumulation point \( B \).

Assume without loss of generality that \( \{B^k\} \to B \). Select \( z \in X \) such that \( |P(B, z)| > \epsilon \). Then for all \( k \) sufficiently large \( |P(B^k, z)| > \epsilon \) and \( P(A^k, z) > k\epsilon \). As

\[ |Q(A, z)| \leq \sum_{k=1}^{m} |\psi_k(z)| \]

for \( A \) satisfying (1), \( |R(A^k, z)| \to \infty \) and \( \|f_k - R(A^k, \cdot)\| \to \infty \). This gives a contradiction of \( A^k \) being best and so \( \|A^k\| \) is bounded.

\( \{A^k\} \) has a convergent subsequence, which we assume without loss of generality to be \( \{A^k\} \), with limit \( A \). We claim \( R(A, \cdot) \) is best to \( f \). Suppose not, then there exists a point \( x \) and \( \epsilon > 0 \) such that

\[ |f'(x) - R(A, x)| > \|f - T(f')\| + \epsilon. \] (2)

The first possibility is that \( Q(A, x) \neq 0 \) and \( P(A, x) = 0 \). But in this case \( R(A^k, x) \to \infty \), which is impossible. The second possibility is that \( P(A, x) = Q(A, x) = 0 \). In a Goldstein type theory, \( R(A, x) \) can always be defined equal to \( f'(x) \). In a Boehm-type theory, we can find in a neighbourhood of \( x \) a point at which \( Q(A, \cdot) \) does not vanish and for which an inequality of the type (2) holds. Thus we need only consider the remaining possibility, which is that \( Q(A, x) \neq 0 \). In this case \( f'(x) - R(A^k, x) \to f'(x) - R(A, x) \) and for all \( k \) sufficiently large

\[ |f(x) - R(A^k, x)| > \|f - T(f')\| + \epsilon, \]
hence

\[ |f_k(x) - R(A^k, x)| > \|f_k - T(f)\| + (\varepsilon/2) \]

for all \( k \) sufficiently large. This contradicts optimality of \( A^k \), hence (2) cannot hold, proving the theorem.

If \( \{A^k\} \to A \) and \( Q(A, \cdot) \) has no zeros, \( \{R(A^k, \cdot)\} \) converges uniformly to \( R(A, \cdot) \). An examination of the previous proof gives

**COROLLARY 1.** Suppose \( f \) has a unique best approximation \( R(A, \cdot) \), with \( Q(A, \cdot) > 0 \) and \( R(A, \cdot) \) having a unique representation under (1). Then \( C(f_k) \to A \) and \( T(f_k) \) converges uniformly to \( R(A, \cdot) \) on \( X \).

**Remark.** In case \( K = K_{GE} \), the uniqueness and unique representation hypotheses are satisfied if the tangent space \( S(A) \) of \( R(A, \cdot) \) is a Haar subspace of dimension \( n + m - 1 \). For the arguments see Dunham [6].

In case \( X = [a, \beta] \) and we approximate by ordinary rationals, the above corollary and remark cannot be improved. The theory of Werner [17] shows that if \( S(A) \) is of dimension \(< n + m - 1 \), discontinuity of \( T \) at \( f \) occurs if \( f \) is not an approximant.

Also of interest is \( K_G = \{A: Q(A, \cdot) > 0\} \). \( K_G \) is not closed in general. Arguments of Dunham [6, Theorem 2] give

**LEMMA.** Let \( R(A, \cdot) \) be a unique best approximation from \( K_G \) and have a unique representation under (1). Then \( R(A, \cdot) \) is a unique best approximation from \( K_{GE} \).

From Corollary 1 we get

**THEOREM 2.** Let \( f \) have a unique best approximation \( R(A, \cdot) \) from \( K_G \) with \( R(A, \cdot) \) having a unique representation under (1). Then for all \( k \) sufficiently large, \( f_k \) has a best coefficient from \( K_G \) and \( R(A^k, \cdot) \to R(A, \cdot) \) uniformly.

The previously cited results of Werner show the necessity of unique representation for uniform convergence.

The case where \( X \) is not compact but \( f \) is bounded as well as continuous on \( X \) is also of interest. Applications include approximation by ordinary rationals on \([0, \infty)\) and \((-\infty, \infty)\). Existence is covered by the author in [11]. Theorem 1 holds. As convergence in coefficients may not imply uniform convergence even when \( Q(A, \cdot) > 0 \), Corollary 1 may not hold. Also the Haar condition does not necessarily imply uniqueness, hence the remark following the corollary does not hold. Examples of discontinuity of \( T \) are given by Blatt [2].

The case where all functions are complex-valued is also of interest, in
which case we use $K_0$. Theorem 1 holds. Corollary 1 holds if $X$ is compact, but may not hold if $X$ is noncompact. There may be no global uniqueness result for rational complex Chebyshev approximation [Saff and Varga, 15, 16]. The real discontinuity result of Werner [171] and the result of Saff and Varga as to when real best approximations are complex best approximations show that nonuniform convergence of $T(f_k)$ to $T(f)$ can occur when functions are real-valued and $X$ is a finite interval.

Let us consider an important case not covered exactly by our theory, the case of Chebyshev approximation with Hermite interpolation [Chalmers and Taylor, 4; Perrie, 13]. In this case $K$ varies with $f$ instead of being fixed. Assume coefficient vectors are selected from $K_0$ or $K_{GE}$. Under Boehm's convention, $K$ is not necessarily closed [Dunham, 9, p. 286]. An extension of Goldstein's convention is to match derivatives of $R(A, x)$ with those of $f(x)$ if $P(A, x) = Q(A, x) = 0$: with this convention, $K$ is closed [Dunham, 11]. We henceforth assume this convention. We assume that $f_k^{(j)}(x) \to f^{(j)}(x)$ for all $x$ and $j$ in the interpolation.

Let us assume that $\|f_k - T(f_k)\|$ is bounded. The proof of Theorem 1 goes through except we need to prove that $R(A, \cdot)$ interpolates $f$. This is true at a point $x$ if $Q(A, x) \neq 0$ since $R^{(j)}(A, x) \to R^{(j)}(A, x)$. If $Q(A, x) = 0$ and $P(A, x) \neq 0$, $|R(A, x)| \to \infty$ and we have a contradiction. If $Q(A, x) = P(A, x) = 0$, we can apply our extension of Goldstein's convention. Corollary 1 holds for compact $X$. Analogous of the Lemma and Theorem 2 hold for compact $X$. If $X$ is noncompact, existence and Theorem 1 still hold. Positive remarks on complex approximation carry over.

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