Schützenberger’s Jeu de Taquin and Plane Partitions*  
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We modify Schützenberger’s “jeu de taquin” and Knuth’s generalization of the Robinson-Schensted correspondence to apply to unrestricted rather than just column-strict plane partitions. The “jeu de taquin,” their modifications, and the Hillman-Grassl mapping are essentially equivalent. We extend the combinatorial methods of Bender and Knuth to give an extension of an elegant, unpublished result of Stanley. Our main result is equivalent to the evaluation of the generating function for column-strict plane partitions of fixed shape with parts less than or equal to $m$. We prove MacMahon’s “box” theorem and give a generating function for plane partitions with parts less than or equal to $m$ and the parts below row $r$ form a column-strict plane partition with at most $c$ columns.

1. INTRODUCTION AND SUMMARY

In an elegant and difficult paper [Schü1], Schützenberger introduced his “jeu de taquin,” related it to the Robinson–Schensted correspondence [Ro1, Sche1], and used it to prove the Littlewood–Richardson rule [LR1]. This provides the combinatorial substructure for the representation theory of $S_n$, Young tableaux, Schur functions, and plane partitions. See Frobenius [Fr1], Young [Yo1], Macdonald [Macd1], and Stanley [St1, St2]. Kung [Ku1] gives extensive references.

MacMahon [MacM1, Arts. 429–435] obtained the generating function

$$g^{m}_{r, c} = \prod_{i=1}^{r} \left( \frac{q^{i+c}}{q^{i}} \right)$$

(1.1)

for plane partitions with parts less than or equal to $m$, at most $r$ rows, and at most $c$ columns. Here, $q$ is fixed with $|q| < 1$ and $(x)_n = \prod_{i=0}^{n-1} (1 - xq^i)$, $n \geq 1$.

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We refer to (1.1) as MacMahon’s “box” theorem. It is not obvious but it is fairly easy to show that $g^m r_c$ is symmetric in $m$, $r$, and $c$.

Let $C^m r_c$ be the set of plane partitions with parts less than or equal to $m$ and the parts below row $r$ form a column-strict plane partition with at most $c$ columns. Let $\lambda$ be the shape of the column-strict part and set

$$f^m r_c = \sum_{\pi \in C^m r_c} q^{[m] - r |\lambda|}. \quad (1.2)$$

Gordon and Houten [GH1], who treated the case $m = \infty$, Andrews [An2] and Macdonald [Macd1] have evaluated the generating function

$$f^m 0_c = \prod_{i=1}^{m} \frac{(q^{i+c})}{(q^i)}, \quad (1.3)$$

for column-strict plane partitions with parts less than or equal to $m$ and at most $c$ columns. This was conjectured by Bender and Knuth [BK1]. Observe that $f^m 0_c$ has only two parameters $m$ and $c$. The following theorem incorporates the parameter $r$ into (1.3).

**Theorem 1.**

$$f^m r_c = g^m r_c = f^m 0_c. \quad (1.4)$$

Bender and Knuth [BK1] gave the first simple, combinatorial proofs of the case $c = \infty$ of (1.1) and (1.3) and, using determinants, Gordon and Houten’s generating function [GH1] for column-strict plane partitions of fixed shape. They used a fundamental extension of the Robinson–Schensted correspondence due to Knuth [Kn1]. Hillman and Grassl [HG1] proved Stanley’s’ theorem [St2, Proposition 18.3] on reverse plane partitions of fixed shape by generalizing the hooks into zigzag paths.

We shall see that all of these combinatorial mappings are essentially equivalent to the “jeu de taquin,” extend the analysis to treat MacMahon’s “box” theorem (1.1), and prove Theorem 1. Our main result is equivalent to the evaluation of the generating function for column-strict plane partitions of fixed shape and parts less than or equal to $m$.

In Section 2, we modify Schützenberger’s “jeu de taquin” to apply to unrestricted rather than column-strict plane partitions. To account for ties, we subtract one from each part that is moved upward. This gives a basic process which, like the “jeu de taquin,” involves underlying paths that can be reversed and, in a certain sense, cannot cross.

In Section 3, we give a simple proof of the case $c = \infty$ of (1.1). We obtain a common refinement of a result due to Bender and Knuth [BK1] and Stanley’s theorem [St3] involving the trace of a plane partition. Using the
combinatorial representation of the Schur functions, this result is a refor-
mulation of the Cauchy identity.

In Section 4, we see that our basic process, Knuth's extension DELETE
of the Robinson–Schensted correspondence, and the Hillman–Grassl map-
ing are essentially equivalent to the “jeu de taquin.” We obtain a further
refinement of the case \( c = \infty \) of (1.1) which gives Littlewood's identity
[Li1] for Schur functions.

In Section 5, we obtain another Littlewood identity [Li2, (1.9; 6)]. This
gives Bender and Knuth's generating function [BK1] for column-strict
plane partitions with parts in the set \( S \). We modify Knuth's DELETE in
the same way that we modified the “jeu de taquin.” This allows us to
combine our proofs to establish the case \( c = \infty \) of Theorem 1.

In Section 6, we apply our basic process to skew plane partitions. We
obtain MacMahon's "box" theorem (1.1) in the course of extending an
elegant, unpublished result of Stanley.

In Section 7, we use simple transformations to show that our main result
is equivalent to the evaluation of the generating function for column-strict
plane partitions of fixed shape and parts less than or equal to \( m \). We give
an equivalent extension (implicit in the bijection given in [St1]) of
Stanley's theorem on reverse plane partitions of fixed shape. We close with
a proof of Theorem 1.

2. THE BASIC PROCESS

A plane partition \( \pi \) of \( n \) is an array
\[
\begin{array}{cccccc}
n_{1,1} & n_{1,2} & n_{1,3} & \cdots \\
n_{2,1} & n_{2,2} & n_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]
(2.1)
of nonnegative integers with
\[
n = \sum_{i,j \geq 1} n_{i,j},
\]
(2.2)
which is nonincreasing along each row and column. Thus,
\[
n_{i,j} \geq n_{i,j+1}, \quad n_{i,j} \geq n_{i+1,j},
\]
(2.3)
holds for all \( i,j \geq 1 \). The nonzero entries \( n_{i,j} > 0 \) of the array (2.1) are
called the parts of \( \pi \). We let \(|\pi|\) denote the sum (2.2) of the parts of \( \pi \). We
say that \( \pi \) is column-strict if
\[
n_{i,j} > n_{i+1,j} \quad \text{whenever} \quad n_{i,j} > 0.
\]
(2.4)
We must deal with a more general type of plane partition known as a skew plane partition. Let \( \mu = (\mu_1, \mu_2, \ldots) \) be a linear partition. Thus \( \mu_1 \geq \mu_2 \geq \cdots \geq 0 \) and the parts of \( \mu \) may be arrayed on a line rather than over the plane (2.1). We set

\[
n_{i,j} = \infty \quad \text{if} \quad \{1 \leq i, 1 \leq j \leq \lambda_i\},
\]

and

\[
|\pi| = \sum_{i \geq 1, j \geq \mu_i} n_{i,j}.
\]

The shape \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of \( \pi \) is the linear partition defined by

\[
n_{i,j} > 0, \quad n_{i,j+1} = 0,
\]

for all \( i \geq 1 \). We use the conventions

\[
n_{i,0} = n_{0,i} = \infty \quad \text{for all} \quad i \geq 1; \quad \infty > \infty.
\]

Clearly \( \mu_i \leq \lambda_i \) for all \( i \geq 1 \). We say that \( \pi \) is a skew plane partition of shape \( \lambda/\mu \). We omit \( \mu \) when \( \mu = (0, 0, \ldots) \).

We often identify \( \pi \) with its frame \( F_{\pi} = \{(i,j) | 1 \leq i, 1 \leq j \leq n_{i,j}\} \)

and \( \pi \) with its graph

\[
D(\pi) = \{(i,j,k) | 1 \leq i, \mu_i < j, 1 \leq k \leq n_{i,j}\}.
\]

Let \( \pi \) be a column-strict skew plane partition of shape \( \lambda/\mu \). We may view \( \pi \) as having holes on \( F_{\mu} \). Schützenberger’s “jeu de taquin” [Schütz] moves the parts of \( \pi \) around filling all of the holes. Choose \( (i,j) \in F_{\mu} \) with \( (i,j+1) \notin F_{\mu} \) and \( (i+1,j) \notin F_{\mu} \). Move \( n_{i,j+1} \) or \( n_{i+1,j} \) to the square \( (i,j) \) according to whether or not \( n_{i,j+1} > n_{i+1,j} \). This choice is forced by the column-strictness property (2.4). Continuing, we obtain a path starting at \( (i,j) \) which consists of jumps to the right and downward and terminates at \( (r_{i,j}, r_{i,j}) \) when the newly created partition \( \pi' \) is of skew shape. If \( r_{i,j} \) is known, then we can reverse the path. If we fill a hold \( (i',j') \) of \( \pi' \) in the same manner, then the path generated will not cross the first.

If (2.3) rather than column-strictness (2.4) is required and \( n_{i,j+1} = n_{i+1,j} \), then we can fill the hole at \( (i,j) \) by moving \( n_{i,j+1} \) left or \( n_{i+1,j} \) up. If we subtract one from each part that is moved upward, then (2.3) forces the choice of direction. This suggests that we define our basic process by

\[
n_{i,j}=n_{i,j+1}, \quad n_{i,j+1}=0, \quad \text{if} \quad n_{i,j+1} \geq n_{i+1,j},
\]

\[
n_{i,j}=n_{i+1,j}-1, \quad n_{i+1,j}=0, \quad \text{if} \quad n_{i+1,j} > n_{i,j+1}.
\]
Observe that while $|\pi|$ is preserved in case (2.11a), it is decreased by one in case (2.11b). Set $(i_0, j_0) = (i, j)$ and let $h = \lambda_{i, j} - j$ and $v = r_{i, j} - i$ be the number of horizontal and vertical steps, respectively, in our underlying path. Observe that $|\pi|$ is decreased by $v$. Let $i_1, i_2, \ldots, i_s, (j_1, j_2, \ldots, j_s)$ be the consecutive row (column) numbers of the squares which are filled by horizontal (2.11a) (vertical (2.11b)) type moves. Both sequences are non-decreasing. Set $(i_{h+1}, j_{v+1}) = (r_{i, j}, \lambda_{i, j})$, the ending point of our path. We use primes to indicate the statistics for the path generated when the hole $(i', j')$ of $\pi'$ is filled.

**Lemma 2.** The inverse of our basic process (2.11a), (2.11b) is given by

\[ n_{h+1, i} = n_{h, i-1}, \quad n_{h, i-1} = 0, \quad \text{if} \quad n_{h+1, i-1} \leq n_{h, i-1}, \quad (2.12a) \]

\[ n_{h, i} = n_{h-1, i} + 1, \quad n_{h-1, i} = 0, \quad \text{if} \quad n_{h-1, i} < n_{h, i-1}. \quad (2.12b) \]

and we have

\[ i' \geq i \Leftrightarrow j' < j \]

\[ \Leftrightarrow i'_{h+1} \geq i_{h+1} \Leftrightarrow j'_{v+1} < k_{v+1}. \quad (2.13) \]

**Proof.** It is easy to check for our basic process (2.11a), (2.11b) and its inverse (2.12a), (2.12b) that the choice of a vertical or horizontal jump is forced by (2.3). If we set $(s, t) = (i_{h+1}, j_{v+1})$, then (2.12a), (2.12b) must retrace our underlying path back to $(i, j)$ since it can do so.

Since $(i, j) \in F_p$ and $(i, j+1) \notin F_p$, we have $j = \mu_j$. Since $(i+1, j) \notin F_p$, we have $\mu_{i+1} < j$. If $i' = i$, then $j' = j - 1$. For $i' \neq i$, we have $j' = \mu_{j'}$. Observe that $i' \geq i$ implies $j' = \mu_{j'} \leq \mu_{i+1} < j$. Hence $i' \geq i$ implies $j' < j$. If $i' < i$, then $j' = \mu_{j'}$. The contrapositive is that $j' < j$ implies $i' \geq i$. This establishes the equivalence in (2.13a). Since the ending points of our paths are chosen from $F_j$ in a similar manner, we also obtain the equivalence in (2.13b).

Assume that $i' \geq i$ and $j' < j$. Our path starting at $(i, j)$ enters row $i'$ (if at all) along column $j < j'$. Thus (2.14) holds (if it is not vacuously true) for $k = i'$. We proceed by induction on $k$. We assume that (2.14) holds for some $k$ with $k < i_{h+1}$. The path from $(i', j')$ enters row $k$ along column $j'_{k-1} < j_{k-1} \leq j_{k-1}$. Since $k < i_{h+1}$, the path from $(i, j)$ turns down at $(k, j_{k-1} + 1)$. Thus

\[ n_{k+1, i_{h+1}, i_{h+1} - 1} \geq n_{k+1, i_{h+1}, i_{h+1} - 1} > 0 \quad (2.16) \]
and

\[ n'_k, i' = n_{k+1, i'-1} - 1, \quad (2.17a) \]
\[ n'_{k+1, i'} = n_{k+1, j_{k+1}-1}, \quad (2.17b) \]

Observe that (2.16) guarantees that the path from \((i', j')\) cannot terminate on row \(k\) and must turn down if it reaches \((k, j_{k-1}+1)\) since it can do so. Thus \(i'_{k+1} > k\) and \(j'_{k-1}+1 < j_{k-1}+1\). This completes the proof of (2.14) and shows that \(i'_{k+1} \geq i_{k+1}\). Thus (2.13a) implies (2.13b). The easy proof of (2.15) (use contradiction) is left to the reader. If we let the inverse (2.12a), (2.12b) of our basic process retrace our paths, then the same argument shows that (2.13b) implies (2.13a), as required.

The negation of (2.13a), (2.13b) gives

\[ i' < i \iff j' \geq j \]

\[ \iff i'_{k+1} < i_{k+1} \iff j'_{k+1} \geq j_{k+1}. \]

We find that in this case

\[ i'_{r-j} < i_{r-j}, \quad j' \leq \ell \leq j_{r+1}, \]

and

\[ j'_{k-r} \geq j_{k-r}, \quad i \leq k \leq i'_{m+1}. \]

The choice of direction for the “jeu de taquin” and its inverse is forced by the column-strictness property (2.4). Thus Lemma 2 and the properties above also apply to the “jeu de taquin.”

3. THE CASE \(c = \infty\) OF (1.1)

Set

\[ B^m_{r,c} = \{ (i, j, k) | 1 \leq i \leq r, 1 \leq j \leq c, 1 \leq k \leq m \}. \]

(3.1)

The generating function for plane partitions with parts less than or equal to \(m\), at most \(r\) rows and at most \(c\) columns is

\[ g^m_{r,c} = \sum_{D \in B^m_{r,c}} q^{|D|}. \]

(3.2)

We omit any parameters which are equal to \(\infty\). Thus \(g = g^\infty_{\infty, \infty} \cdot g^m_{r,c} = g^m_{r,c}\)

and \(B^m_{r,c} = B^\infty_{r,c}\). The following theorem gives the case \(c = \infty\) of MacMahon’s “box” theorem (1.1).
Theorem 3.

\[ g_m = \prod_{i=1}^{m} \frac{1}{(q^i)^r}. \]  

(3.3)

Proof. This follows from the difference equation

\[ g_m = \frac{1}{(q^m)^r} g_m^{r-1}, \quad m \geq 1, \]  

(3.4)

and the initial condition \( g_0 = 1 \). Fix \( \ell \geq 0 \) and let \( \pi \) be a plane partition with

\[ D(\pi) \subseteq B_{\ell}^r, \quad n_{1,1} = \ldots = n_{1,\ell} = m > n_{1,\ell+1}. \]  

(3.5)

We prove (3.4) by giving a bijection \( \pi \leftrightarrow \langle v, \pi^* \rangle \), where \( v \) is a linear partition with

\[ v_j = 0 \quad \text{whenever} \quad j > \ell, \]
\[ m \leq v_j \leq m + r - 1 \quad \text{whenever} \quad 1 \leq j \leq \ell; \]  

(3.6)

\( \pi^* \) is a plane partition with

\[ D(\pi) \subseteq B_{\ell}^{r-1} \]  

(3.7)

and

\[ |\pi| = |v| + |\pi^*|. \]  

(3.8)

Let \( \pi \) have shape \( \lambda \) and set \( \mu = (\ell, 0, 0, \ldots) \) and \( v_j = 0 \) for all \( j > \ell \). Let \( \pi^* \) be the skew plane partition of shape \( \lambda / \mu \) obtained by removing the \( \ell \) occurrences of the part \( m \) from the first row of \( \pi \). We have \( |\pi^*| = |\pi| - m \ell \). For \( j \) decreasing from \( \ell \) to 1, we use our basic process (2.11a), (2.11b) to fill the hole of \( \pi^* \) at \((1, j)\) and set \( v_j = m + v \). Observe that (3.8) holds since each step a linear partition. For each \( j, 1 \leq j \leq \ell \), our basic process satisfies \( 0 \leq v \leq r - 1 \) and all of the parts of \( \pi^* \) left in column \( j \) are less than or equal to \( m - 1 \). This gives (3.6) and (3.7).

Given \( \langle v, \pi^* \rangle \), we can recover \( \pi \) as follows. Set \( \pi = \pi^* \). For \( j \) increasing from 1 to \( \ell \), we apply the inverse (2.12a), (2.12b) of our basic process with \( s = v_j + 1 - m \) and \( t \) the smallest value for which \((s, t)\) is outside the shape of \( \pi \). Observe that (2.8) guarantees that the first hole is created at \((1, 1)\). By (2.13a), the succeeding holes form \( F_{\mu} \). We set \( n_{1,j} = m \) for \( 1 \leq j \leq \ell \) and we are done. \( \square \)
We have the well-known $q$-binomial theorem (see Andrews [An1, (2.2.1)])

$$\frac{(at)}{(t)^{\infty}} = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} r^n, \quad |r| < 1. \quad (3.9)$$

For $a = q^r$, $t = q^m$, this becomes

$$\frac{1}{(q^m)^r} = \sum_{\ell=0}^{\infty} \frac{(q^\ell)_r}{(q)_r} q^{m\ell}. \quad (3.10)$$

The summand on the right side of (3.10) is the generating function for linear partitions $v$ satisfying (3.6). Since $\ell \geq 0$ is fixed in the proof of Lemma 3, we have

$$\sum_{n_1,1 = \cdots = n_{i,j} = m > m_{i,j+1}} q^{\alpha(i)} = q^{m\ell}(q^\ell)/(q)_\ell g_{\ell}^{m-1}. \quad (3.11)$$

This result is due to Bender and Knuth [BK1].

For $i,j \geq 1$, set

$$\alpha_{i,j} = \max(n_{i,j} - i + 1, 0). \quad (3.12)$$

$\alpha = (\alpha_{i,j})$ is a column-strict plane partition. Let $\lambda$ be the shape of $\alpha$.

We may remove $n_{1,1}$ from the corner of $\alpha$ and apply our basic process, repeating until $\pi$ is empty. A part of $\pi$ can be changed only if it is moved upward, in which case we subtract one. Observe that $n_{i,j}$ survives to reach the corner if $\alpha_{i,j} > 0$, in which case $\alpha_{i,j}$ is removed from the corner. This happens

$$\text{trace}(\pi) = |\lambda| \quad (3.13)$$

times. Let $x = (x_1, x_2, \ldots)$ and set

$$W_x(\alpha) = \prod_{(i,j) \in F_{\alpha}} x_{\alpha_{i,j}}. \quad (3.14)$$

The proof of Theorem 3 gives

$$\sum_{D(x) \subseteq B_{\alpha}^n} W_x(\alpha) q^{|\alpha| - |\alpha|} = \prod_{i=1}^{\infty} \frac{1}{(x_i)_{\ell_i}}. \quad (3.15)$$
For $x_i = tq^i$, $1 \leq i \leq m$, and $x_j = 0$, $i > m$, we have $W_{x}(\mathbf{x}) = t^{\text{trace}(\mathbf{x})} q^{\mathbf{x}}$. We see that (3.15) becomes
\[
\sum_{D(\mathbf{x}) \in B_{\mathbf{r}}^m} t^{\text{trace}(\mathbf{x})} q^{\mathbf{x}} = \prod_{i=1}^{m} \frac{1}{(tq^i)^{r_i}},
\]
which is due to Stanley [St3].

Let $M \subseteq \{1, ..., m\}$ and let $r_i \geq 0$ for all $i \in M$. Let $'$ denote the restriction that for all $i \in M$, $i$ occurs exactly $r_i$ times as a part of $\mathbf{x}$. We obtain
\[
\sum_{D(\mathbf{x}) \in B_{\mathbf{r}}^m} W_{x}(\mathbf{x}) q^{\mathbf{x}} = \prod_{i=1}^{m} x_i^{r_i} \prod_{i \in M} (q_i)^{r_i} \prod_{i \notin M} (x_i)^{r_i},
\]
which is a common refinement of (3.11) and (3.16). The reader may treat the details when $m$, $r$, or both tend to $\infty$.

4. A FURTHER REFINEMENT

The Frobenius decomposition [Fr1] of $\rho = (6, 6, 6, 5, 4, 2)$ is the pair of distinct partitions $(6, 5, 4, 2, 1)$ and $(6, 5, 3, 2, 0)$ obtained graphically:

\[
\begin{align*}
6 & \quad 6 \\
6 & \quad 5 \\
6 & \quad 4 \\
6 & \quad 3 \\
5 & \quad 1 \\
4 & \quad 0 \\
2 & \quad 0 \\
6 & \quad 5 \quad 3 \quad 2 \quad 0
\end{align*}
\]

We say that $\rho$ has order 5, the number of squares of $F_{\rho}$ on the diagonal $\{(i, i) \mid i \geq 1\}$.

Bender and Knuth [BK1] and Stanley [St1, St2] both rely on this essential construct to study plane partitions. Recombining the horizontal and vertical parts of the Frobenius decompositions of the columns of $\mathbf{x}$, we obtain $\mathbf{x}$ and another column-strict plane partition $\mathbf{\beta} = (\beta_{i,j})$. For $i, j \geq 1$, we have
\[
\mathbf{\sigma}_{i,j} = \text{card} \{(k \mid k \geq i \text{ and } (i, j, k) \in D(\mathbf{x})\},
\]

\[
4.1
\]

\[
4.2
\]
in agreement with (3.12), and
\[ \beta_{i,j} = \text{card}(\{k \mid k > i \text{ and } (k, j, i) \in D(\pi)\}). \] (4.3)

Since each \((i, j, k) \in D(\pi)\) contributes to \(\alpha\) or \(\beta\), we have
\[ |\pi| = |\alpha| + |\beta|. \] (4.4)

For all \(j \geq 1\), \(\lambda_j\) is the order of the \(j\)th column of \(\pi\) and, allowing for at most one zero part at the bottom of each column of \(\beta\), \(\alpha\) and \(\beta\) both have shape \(\lambda\). Given such \(\lambda\), \(\alpha\) and \(\beta\), we can recover \(\pi\) using (4.2) and (4.3).

The conjugate \(\pi' = (n'_{i,j})\) of \(\pi\) is obtained by conjugating the columns of \(\pi\). It is given by
\[ D(\pi') = \{(i, j, k) \mid (k, j, i) \in D(\pi)\}. \] (4.5)

For \(i, j \geq 1\), we have
\[ \beta_{i,j} = \max(n'_{i,j} - i, 0). \] (4.6)

We can let \(S_3\) act on \(D(\pi)\). This action is generated by the transposition in (4.5) and the 3-cycle in
\[ D(\pi^{r_3}) = \{(i, j, k) \mid (j, k, i) \in D(\pi)\}. \] (4.7)

For the example below, we let \(m = 6, \ell = 2\). In (4.8), we superimpose the underlying paths for the two applications of our basic process (2.11a), (2.11b) in the proof of Theorem 3. In (4.9), (4.10), and (4.11), we give the \(\pi, \beta, \text{ and } \pi^{r_3}\), respectively, corresponding to the plane partitions in (4.8).
We are led to the following theorem.
Theorem 4. If we remove $m=n_{1,1}$ from $\pi$ and apply our basic process, then

1. $\alpha$ is transformed by Schützenberger’s “jeu de taquin” [Schu1],
2. $\beta$ is transformed by Knuth’s extension DELETE [Kn1] of the Robinson–Schensted correspondence [Ro1, Sche1],
3. $\pi^\triangleright$ is essentially transformed by the Hillman–Grassl mapping [HG1], and
(1) starts by removing $m=n_{1,1}$ from $\pi$, (2) starts where (1) ends and (2) ends with bouncing $v=\beta_{1,h+1}$ from $\beta$, and the zigzag path of (3) starts at $(m,1)$ and ends at $(1,v+1)$.

Proof. By (3.12), $m=n_{1,1}$ and moving the parts of $\pi$ to the left or upward correspond to moving the parts of $\pi$ and subtracting one from each part that is moved upward. Observe that (1) holds since the “jeu de taquin” applied to $\pi$ and our basic process applied to $\pi$ are forced to follow the same path.

We say that $(i,j,k)$ is in row $i$, column $j$, and on level $k$ of $\pi$. It is easy to see that our basic process is as follows. We start at $(1,1,m)$ on level $m$ and let the level decrease to 1. Assume we enter level $k$ at $(s,t,k)$. Let $t'$ be the largest value for which $(s,t',k) \in D(\pi)$ and let $s'$ be the largest value for which $(s',t',k) \in D(\pi)$ and enter level $k-1$ at $(s',t',k-1)$. Our underlying path is the projection of this three-dimensional path onto level one.

Suppose $(i,j)$ is the last square of $\pi$ to be emptied in (1). Since $\pi'$ and $\beta'$ have the same shape, we have $\beta'_{i,j}=0$. Let $1 \leq k < i$. Since we enter level $k$ at $(s,t,k)$, we have $\beta'_{k+1,i}=s-k-1$. Observe that

$$\{(i,j,k) \in D(\pi) : \beta'_{k+1,i} \geq s-k \geq \beta'_{k+1,i} \}$$

and removing $\{(i,t',k) \mid s \leq i \leq s'\}$ from $D(\pi)$ results in

$$\beta'_{k+1,i} = s-k-1 = \beta'_{k+1,i}.$$  \hfill (4.12)

We see that (2) follows since Knuth’s process DELETE finds the largest $t'$ satisfying (4.12) and “bumps” $\beta'_{k+1,i}$ out of row $k$, replacing it as in (4.13) with $\beta'_{k+1,i}$. This implies our underlying path ends at $(v+1,h+1)$, we see that (2) ends by “bumping” $v=\beta_{1,h+1}$ from $\beta$.

Consider the part of our three-dimensional path which is removed from $D(\pi)$. If we apply the 3-cycle in (4.7) and project onto level one, we obtain a zigzag path which starts at $(m,1)$ and ends at $(1,v+1)$. For each $k$, $1 \leq k \leq m$, we enter row $k$ at $(k,s)$. Since $t'$ is the largest value satisfying (4.12), we have $n'_{k,s} = t'$ and

$$\{(i,t',k) \in D(\pi) : n'_{k,s} = t' = n'_{k,s} \}$$

\hfill (4.14)
We subtract one from $n^x_{s,i}$ for all $i$, $s \leq i \leq s'$, where $s'$ is the largest value satisfying (4.14) and enter row $k$ at $(k-1, s')$. We obtain the Hillman–Grassl mapping if $x'$ is "flipped" in a straightforward manner to give a reverse plane partition.

Let $y = (y_0, y_1, \ldots)$. Since $v$ is the part "bumped" from $\beta$, we obtain

$$\sum_{D(x) \in B^p} W_x(\alpha) W_y(\beta) = \prod_{i=1}^{m-1} \prod_{j=0}^{r-1} \frac{1}{(1-x_i y_j)}. \quad (4.15)$$

For a fixed $\lambda$, we may choose $\alpha$ and $\beta$ independently. The Schur function $s_\lambda(x)$ is given by

$$s_\lambda(x) = \sum_{x} W_x(\alpha), \quad (4.16)$$

where $\alpha$ is a column-strict plane partition of shape $\lambda$. See Stanley [St2]. Setting $m = r = \infty$ in (4.15) and using (4.16), we have

$$\prod_{i \geq 1} \frac{1}{(1-x_i y_j)} = \sum_{\lambda} s_\lambda(x) s_\lambda(y), \quad (4.17)$$

which is the celebrated Cauchy identity. See Littlewood [Li1], Stanley [St2], and Macdonald [Macd1, Chap. I]. Knuth [Kn1] gives essentially the same combinatorial proof.

5. COLUMN–STRICT PLANE PARTITIONS

A result of Littlewood [Li1, (11.9; 6)] is the case $t = 1$ of

$$\prod_{i \geq 1} \frac{1}{(1-t x_i)} \prod_{j \geq 1} \frac{1}{(1-x_i x_j)} = \sum_{\lambda} t^{\text{odd}(\lambda)} s_\lambda(x), \quad (5.1)$$

where odd$(\lambda)$ is the number of columns of $\lambda$ of odd length. We may prove this as follows. By (4.16), we have

$$\sum_{\lambda} t^{\text{odd}(\lambda)} s_\lambda(x) = \sum_{x} W_x(\alpha). \quad (5.2)$$

Remove $m = x_{1,1}$ from $x$ and apply the "jeu de taquin." Apply Knuth’s DELETE to the zero in the last emptied square of $x$. This "bumps" $v < m$ from the first row of $x$. Since the parts "bumped" from the rows of $x$ are strictly increasing, we can combine the inverses of our two processes. We see that odd$(x)$ is unchanged unless $v = 0$, in which case it decreases by one.
Letting \( x_0 = t \), we have identified a factor \( x_m x_v \) of the product in (5.1). Since the columns in Knuth's process DELETE (4.10) coincide with those of our underlying path, they satisfy (2.14). Hence, for fixed \( m \), the \( v \)'s are nondecreasing. The result (5.1) follows by induction on \( m \).

Our plane partition generating functions in Section 3 can be obtained by specializing \( x \) and \( y \) in (4.17). Let \( 0 \notin S \subseteq \mathbb{N} \) and set \( x_i = s q^i \) for \( i \in S \) and \( x_i = 0 \) otherwise. We have \( W'(x) = 0 \) unless all of the parts of \( \pi \) are in \( S \). Taking \( x_0 = t \) as before, (5.1) and (5.2) yield

\[
\sum_{n_i \geq 0} s^{x_i} q^{\text{odd}(x)} = \prod_{i \in S} \frac{1}{1 - st q^i} \prod_{j > i} \frac{1}{1 - s t q^{j+q^r}}. \tag{5.3}
\]

Let \( S = \{ 1, 2, \ldots, m \} \), \( s = t = 1 \), and \( \ell \geq 0 \). We obtain

\[
\sum_{n_i \geq 0} q^{x_i} = q^m (q^m)_{\ell} \prod_{0 \leq i < j \leq m - 1} \frac{1}{1 - q^{j+q^r}}. \tag{5.4}
\]

Bender and Knuth [BK1] give essentially the same results.

There is a parallel between our results on unrestricted and column-strict plane partitions. Since our proofs of Lemma 3 and (5.1) both use the "jeu de taquin" followed by DELETE, we suspect that we can incorporate them into the proof of a more general theorem. To do this we modify DELETE as follows to apply to unrestricted plane partitions. Let \( \ell' \) be the largest value satisfying

\[
n_{k, \ell'} \geq n_{k+1, \ell'}, \tag{5.5a}
\]

\[
n_{k, \ell'}^* = n_{k+1, \ell'} - 1 \tag{5.5b}
\]

and decrease \( k \). It is clear that our basic process (2.11a), (2.11b) and (5.5a), (5.5b), which we call mod-DELETE, express in terms of \( \pi = (n_i, j) \) the results of applying the "jeu de taquin" and DELETE, respectively, to \( \pi \). Observe by (3.12) that while the parts of \( \pi \) are moved around, we must subtract one, as in (2.11b) and (5.5b), from each part of \( \pi \) that is moved upward. The inverse mod-INSERT of (5.5a), (5.5b) is as follows. Let \( \ell' \) be the smallest value satisfying

\[
n_{k+1, \ell'} \leq n_{k, \ell'} \tag{5.6a}
\]

Set

\[
n_{k+1, \ell'}^* = n_{k, \ell'} + 1 \tag{5.6b}
\]

and increase \( k \).
Subtracting one from each part of \( \pi \), adding one to each part of \( \beta \), and interchanging gives the Frobenius decomposition of the conjugate \( \pi' \) of \( \pi \). The additions and subtractions have no effect on the inequalities in any of our combinatorial processes. By Theorem 4, mod-DELETE and its inverse (5.6a), (5.6b) also arise by applying our basic process and its inverse (2.12a), (2.12b) to \( \pi' \).

Let \( C^m_{r,c} \) be the set of plane partitions \( \pi \) satisfying

\[
\begin{align*}
    n_{1,1} &\leq m, \\
    n_{i,j} &> n_{i+1,j} \quad \text{whenever } i > r, \quad n_{i,j} > 0, \\
    n_{r+1,c+1} & = 0.
\end{align*}
\]

The parts of \( \pi \) are less than or equal to \( m \) and the parts below row \( r \) form a column-strict plane partition \( \pi \) with at most \( c \) columns. Thus \( \pi = (n_{i+r,j}) \) has shape \( \lambda \), where \( \lambda_1 \leq c \).

Set

\[
f_{r,c}^{m}(s,t) = \sum_{\pi \in C^m_{r,c}} s^{[\pi]} t^{\text{odd}(\pi)} q^{\pi_1}.
\]

As before, we omit any parameters which are equal to \( \infty \). Thus \( C^m_{r,c} = C^m_{r,\infty} \) and \( f^{m}_{r,c}(s,t) = f^{m}_{r,\infty}(s,t) \). We have the following theorem.

**Theorem 5.**

\[
f_{r,c}^{m}(s,t) = \prod_{i=1}^{m} \frac{1}{(q')_i (1 - stq^{r+1})(s^2q^{r+2})_{i-1}}.
\]

**Proof.** We require the difference equation

\[
f_{r,c}^{m}(s,t) = \frac{1}{(q')_r (1 - stq^{r+1})(s^2q^{r+2})_{m-1}} f_{r-1,c}^{m-1}(s,t).
\]

Remove \( m = n_{1,1} \) from \( \pi \) and apply our basic process. If this ends in one of the first \( r \) rows, then we obtain part of the factor \( 1/(q'^m) \), as in the proof of Lemma 3. If it reaches row \( r+1 \), then we must use the “jeu de taquin.” The concatenation of the underlying paths still satisfies (2.13b). If our path ends in row \( r+1 \), then we obtain a term \( stq^{r+1} \) of the factor \( 1/(1 - stq^{r+1}) \). Observe that in this case the inverse is given by (2.12a), (2.12b). If our path ends in row \( r+2 \) or more, then we apply DELETE to the zero in the last emptied square. This “bumps” a nonzero part out of row \( r+1 \). Use mod-DELETE (5.5a), (5.5b) with \( k \) decreasing from \( r \) to 1.

This “bumps” \( v, 1 \leq v < m \), from the first row of \( \pi \), giving us the term \( s^2q^{m+2} \) of the factor \( 1/(s^2q^{m+2})_{m-1} \). Since the column numbers of
mod-DELETE satisfy (2.14), the v's are nondecreasing. By recombining the inverses of each part of our transformation, we see that (5.10) is established. Observe that we must start (5.6a), (5.6b) with \( k = 0, t = \infty, \ n_{0,-} = v - 1 \).

We may avoid (5.5a), (5.5b) and give a unified proof as follows. Add infinity to \( \pi \) by adding \( r+1 \) to \( n_{i,j} \) throughout the first \( r \) rows of \( \pi \). We treat this column-strict plane partition as in the proof of (5.1). Remove \( m+r \) and apply the "jeu de taquin" followed by DELETE. If the underlying path in the proof of Theorem 5 ends in row \( r+1 \) or less, then we can continue the "jeu de taquin" along an infinite horizontal path and consider DELETE to be performed at infinity with no effect on \( \pi \). Subtracting infinity from \( \pi \) in the obvious way, we obtain the same result as in the proof of Theorem 5.

We have

\[
\sum_{\pi \in \text{C}_r} q^{\sum_{i=1}^m \left( \begin{array}{c} n_i \end{array} \right)} = \prod_{i=1}^m \frac{1}{(q')_{r+i}}.
\]

We can keep track of the number of m's in the first row of \( \pi \) in (5.11), thus generalizing (3.11) and (5.4). We obtain

\[
\sum_{\pi \in \text{C}_r} q^{\sum_{i=1}^m \left( \begin{array}{c} n_i \end{array} \right)} = n^m (q^m+r)^\ell \prod_{i=1}^m \frac{1}{(q')_{r+i}}.
\]

We omit \( s \) and \( t \) when \( s = q^{-r}, t = 1 \). We have

\[
f^m_r(q^{-r/2}, q^{r/2}) = \sum_{\pi \in \text{C}_r} q^{\sum_{i=1}^m \left( \begin{array}{c} n_i \end{array} \right)} = n^m f^m_r(q^{-r/2}, q^{r/2}).
\]

which is the case \( c = \infty \) of Theorem 1.

6. A PROOF OF MAC MAHON'S "BOX" THEOREM

For \((i, j) \in F_\mu\), the hook number \( h_{i,j} \) is the number of cells in the hook of \((i, j)\). This consists of \((i, j)\) together with the points of \(F_\mu\) which are directly to the right of or directly below \((i, j)\). For \((i, j) \notin F_\mu\), we can define \( h_{i,j} \) by letting the hook of \((i, j)\) consist of \((i, j)\) and all of the points outside \(F_\mu\) which are directly to the left of or directly above \((i, j)\). For \(i, j \geq 1\), we have

\[
h_{i,j} = \begin{cases} 
\mu_i - j + \mu_j - i + 1 & \text{if } (i, j) \in F_\mu, \\
-j - i + \mu_j + 1 & \text{if } (i, j) \notin F_\mu.
\end{cases}
\]
where the conjugate \( \mu' \) of \( \mu \) is given by
\[
F_{\mu'} = \{(i, j) \mid (j, i) \in F_\mu\}.
\] (6.2)

A reverse plane partition (rpp) of shape \( \mu \) is defined on \( F_\mu \) and is non-decreasing along each row and down each column. Our proof of MacMahon’s “box” theorem (1.1) relies on
\[
\mu, g = \sum_{\pi \text{ rpp of shape } \mu} q^{1n} = \prod_{(i, j) \in F_\mu} \frac{1}{(1 - q^{h_{ij}})},
\] (6.3)

which is due to Stanley [St2]. He gave a bijection [St1] which proves the equivalence of (6.3) and
\[
\sum_{\pi \text{ column-strict of shape } \mu} q^{1n} = q^{\sum h_{ij}} \prod_{(i, j) \in F_\mu} \frac{1}{(1 - q^{h_{ij}})}.
\] (6.4)

which is due to Gordon and Houten [GH1]. Hillman and Grassl [Hi-Gr1] gave a combinatorial proof of (6.3) which bends the hooks into the zigzag paths encountered in (4.11).

We say that \( \pi \) is outside \( \mu \) if \( \pi \) has shape \( \lambda/\mu \) for some \( \lambda \). Stanley sparked this paper by asking for a combinatorial proof of
\[
\sum_{\pi \text{ outside } \mu} q^{1n} = g_{\lambda, \mu} (6.5a)
\]
\[
= \prod_{(i, j) \in F_\mu} \frac{1}{(1 - q^{h_{ij}})}. (6.5b)
\]

We leave the reader to ponder (6.5a) and its equivalence with (6.5b). Observe that we can “flip” \( \pi \) (as in the proof of part (3) of Theorem 4) so as to reverse the rows and columns, obtaining a rpp. For example, when \( \mu = (4, 2, 1) \) we obtain the rpp in (6.7) by “flipping” \( \pi \) in (6.6) and vice-versa.

\[
\begin{array}{cccc}
3 & 3 & 2 \\
4 & 3 & 3 & 1 \\
5 & 4 & 3 & 2 \\
5 & 5 & 3 & 2 \\
4 & 3 & 3
\end{array}
\] (6.6)
We assume throughout that $\mu'_1 \leq r$ and $\mu_1 \leq c$. Set

$$\mu^m_{r,c} = \sum_{\text{outside } \mu} q^{|\pi|}. \quad (6.7)$$

As before, we omit any parameters which are equal to $\infty$. Thus $\mu^m_{r,c} = \mu^m_{r,c}$ and $\mu^m_{m,m} = \mu^m_{r,c}$. The following lemma provides the first step in evaluating $\mu^m_{r,c}$.

**Lemma 6.**

$$\mu^m_{r,c} = \prod_{1 \leq i \leq r, \text{ } 1 \leq j \leq c, (i,j) \notin F_\pi} \frac{1}{1 - q^{h_{i,j}}}. \quad (6.9)$$

**Proof.** If we “flip” a plane partition $\pi$ which contributes to $\mu^m_{r,c}$, then the resulting rpp has shape $(c - \mu_i, c - \mu_{i-1}, \ldots, c - \mu_1)$ and vice versa. The result follows by (6.3).

The case $r = c = \infty$ is (6.5b). We require the case $c = \infty$. It is

$$\mu^m_{r,c} = \prod_{j \geq \mu_1} \frac{1}{1 - q^{h_j}}. \quad (6.10)$$

We have the following lemma.

**Lemma 7.**

$$\mu^m_{r} = \frac{1}{(q^m)^r} \mu^m_{r}^{-1}, \quad m \geq 1. \quad (6.11)$$

**Proof.** We modify the proof of (3.4) in Theorem 3 as follows. Let $j_1 < j_2 < \cdots < j_r$ be the values of $j$ satisfying

$$n_{\mu_j' + 1} = m + \mu_j'. \quad (6.12)$$
Set $\pi^* = \pi$ and $v_c = 0$ for all $c > \ell$. Let $c$ decrease from $\ell$ to 1. Set $j = j_c$, $i = \mu'_j + 1$, and remove $n^*_c,j = m + \mu'_j$ from $\pi^*$. Apply our basic process (2.11a), (2.11b) and let $v_c = m + \mu'_j + v$ be the amount by which $|\pi^*|$ is reduced. Let $s = \mu'_j + 1 + v$ be the row number of the last emptied square. Since $s = v_c + 1 - m$, we can invert our mapping $\pi \rightarrow \langle v, \pi^* \rangle$ using (2.12a), (2.12b). By (2.13b), the row numbers are nondecreasing and $s$ is a linear partition. It is clear that our results satisfy (3.6), (3.8), and

$$n^*_c,j \leq m - 1 + \mu'_j,$$  \hspace{1cm} (6.13)

as required. \[\]

For $m = 3$, $r = 5$, $\mu = (4, 2, 1)$, and $\pi$ given by (6.6), we obtain $\ell = 4$, $v = (7, 5, 4, 4)$, and $\pi^*$:

\[
\begin{array}{ccc}
2 & 2 \\
3 & 3 & 2 \\
4 & 3 & 2 \\
5 & 3 & 2 \\
4 & 3
\end{array}
\]  \hspace{1cm} (6.14)

We can evaluate $\mu \mathcal{B}^m_s$ using (6.11) since (6.10) provides the value for $m = \infty$. Repeated use of (6.11) yields

$$\mu \mathcal{B}^m_s = \mathcal{B}^m_s \mu \mathcal{B}^m_s.$$  \hspace{1cm} (6.15)

It will be convenient to compute $\mu \mathcal{B}^m_s$ and use (6.15). Setting $m = \infty$ in (6.15) yields

$$\mu \mathcal{B}^r_s = \mathcal{B}^r_s \mu \mathcal{B}^r_s.$$  \hspace{1cm} (6.16)

Fix $i, 1 \leq i \leq r$. It is well known (see Macdonald [Macd1, p. 9]) that

$$\prod_{1 \leq j \leq \mu_i} \frac{1}{1 - q^ {\mu_i + j - i}} = \frac{1}{(q)_i} \prod_{1 \leq j \leq r} (1 - q^{\mu_i + j - i}).$$  \hspace{1cm} (6.17)

The sets $\{\mu_i - \mu_i + j - i \mid i \leq j \leq r\}$ and $\{h_k, 1 \leq j \leq \mu_i\}$ are disjoint. Thus (6.17) follows by the pigeonhole principle. The same argument also gives

$$\prod_{j > \mu_i} \frac{1}{1 - q^{\mu_i + j - i}} = \frac{1}{(q)_i} \prod_{1 \leq j \leq r} (1 - q^{\mu_i + j - i}).$$  \hspace{1cm} (6.18)
Solving (6.16) for $g^0_r$ and using (6.11) and (6.18) yields

$$
\mu g_r^0 = \frac{g_r}{g_r^m} = \prod_{i=1}^r (q^i) = \prod_{j>\mu_i} \frac{1}{1-q^{h_{i,j}}},
$$

$$
= \prod_{i=1}^r (q^i) \cdot \prod_{1 < j < i} \left(1 - q^{\mu_j + i-j} \right)
= \prod_{1 < j < i < r} \left(1 - q^{h_{i,j}} \right) \left(1 - q^{-j} \right).
$$

(6.19)

Interchange $i$ and $j$ and use (6.17). We obtain

$$
\mu g_r^0 = \prod_{1 < i < j < r} \left(1 - q^{h_{i,j}} \right) \left(1 - q^{-j} \right),
$$

$$
= \prod_{i=1}^r (q^i) \cdot \prod_{1 < j < \mu_i} \left(1 - q^{h_{i,j}} \right)
= \prod_{1 < j < \mu_i} \left(1 - q^{h_{i,j}} \right),
$$

(6.20)

Substituting (6.20) into (6.15) yields

$$
\mu g_r^m = g_r^m \prod_{(i,j) \in F_r} \left(1 - q^{h_{i,j}} \right) \left(1 - q^{-j} \right).
$$

(6.21)

For $m = r = \infty$ this gives (6.5a).

We have evaluated $\mu g_r^m$ for $c = \infty$. We may treat the general case since the parameter $c$ is subsumed by $\mu$. We have the following theorem.

Theorem 8.

$$
\mu g_{r, n}^m = \prod_{j=1}^n \left(\frac{q^{r+j}_{j}}{q^{r+j}_{j} + \mu_{j}} \right) \prod_{(i,j) \in F_r} \left(1 - q^{h_{i,j}} \right) \left(1 - q^{-j} \right).
$$

(6.22)

Proof. Let $\omega$ be the linear partition defined by

$$
\omega_i = \begin{cases} 
c & \text{if} \ 1 \leq i \leq m, \\
\mu_{i-m} & \text{if} \ i > m.
\end{cases}
$$

(6.23)
Then we have
\[ g^m_{r,c} = \mu g^0_{m+r}. \]  
(6.24)

For \( \mu = (4, 2, 1) \), we see that \( \pi \) (6.6) contributes to \( \mu g^3_{3,7} \). We obtain
\[ \omega = (7, 7, 7, 4, 2, 1) \]  and
\[
\begin{array}{cccccc}
3 & 3 & 2 \\
4 & 3 & 3 & 2 & 1 \\
5 & 4 & 3 & 2 \\
5 & 5 & 3 & 2 \\
4 & 3 & 3 \\
\end{array}
\]  
(6.25)
gives the corresponding contribution to \( g^0_{433} \). By (6.20) and (6.24), we have
\[ g^m_{r,c} = \mu g^0_{m+r} = \prod_{(i,j) \in F_{m+r}} \frac{(1-q^{m+r+j-i})}{(1-q^{r+j})}. \]  
(6.26)
The result (6.22) follows by separately considering the contributions to (6.26) for \( 1 \leq i \leq m \).

MacMahon’s “box” theorem (1.1) is the case \( \mu = (0) \) of Theorem 8 (6.22). Since \( g^m_{r,c} \) is symmetric in \( m, r, \) and \( c \), we obtain
\[ g^m_{r,c} = (\mu g^m_{r,m}) = \prod_{j=1}^{m} (\frac{q^{r+j}}{(q^{r+j+1})}) = \prod_{i=1}^{m} (\frac{q^{r+i}}{(q^{r})}), \]  
(6.27)
as required.

7. COLUMN-STRICK PLANE PARTITIONS

By “flipping” the plane partitions contributing to \( \mu g^m_{r,c} \), we see that Theorem 8 is an extension of Stanley’s theorem (6.3). It also extends Gordon and Houten’s theorem (6.4). To see this let
\[ \mu f^m_{r,c} = \sum_{\pi \in \mathbf{P}^m_{r,c} \atop n_1, \ldots, n_c \geq 0} q^{\pi} \]  
(7.1)
be the generating function for plane partitions with parts less than or equal to \( m \), at most \( c \) columns and the parts below row \( r \) form a column-strict plane partition of shape \( \mu \). The unusual placement of the parameter \( c \) indicates that the column restriction applies to all of \( \pi \) rather than just the column-strict part.
We have the following theorem.

**Theorem 9.**

\[ \mu f_{\nu}^{m,c} = q^{r(|\nu| + 2|m|)} \mu s_{m,c}^{\nu}. \]  

(7.2)

**Proof.** Let \( \pi \in C_{r,m}^{c} \), \( n_{1,c+1} = 0 \), and \( \lambda = \mu \). For \((i,j) \in F_{\mu}\), add \( i - 1 \) to \( n_{i+r,j} \). This adds \( \sum(i-1) \mu_{i} \) to \( |\pi| \) and the result satisfies

\[ n_{1,1} \leq m, \quad n_{1,c+1} = 0, \]
\[ n_{i+r,j} \geq n_{i,j}' \quad \text{whenever} \quad (i,j) \in F_{\mu}, \]
\[ n_{i+r,j} = 0 \quad \text{whenever} \quad i \geq 1, \quad j > \mu. \]

(7.3)

Observe that column-strictness is no longer required. The conjugate \( (4.5) \) satisfies

\[ n_{m+1,1} = n_{1,c+1} = 0, \]
\[ n_{i,j} = r + \mu_{i}' \quad \text{whenever} \quad (i,j) \in F_{\mu}, \]
\[ n_{1,m+1} \leq r. \]

(7.4)

Now consider the skew plane partition obtained by removing \( n_{i,j} = r + u_{i}' \) for all \((i,j) \in F_{\mu}\). This subtracts

\[ r |\mu| + \sum (\mu_{i}')^{2} = r |\mu| + \sum (2i-1) \mu_{i}, \]

(7.5)

from \( |\pi| \) and we have

\[ \pi \text{ outside } \mu, \]
\[ n_{m+1,1} = n_{1,c+1} = 0, \]
\[ n_{i,j} \leq r + \mu_{i}' \quad \text{whenever} \quad (i,j) \not\in F_{\mu}. \]

(7.6)

Comparing with (6.8), we see that this is the condition defining \( \mu s_{m,c}^{\nu}. \) If \( \pi \) satisfies (7.6), then we may easily reverse our mapping, obtaining \( \pi \in C_{r,m}^{c} \), \( n_{1,c+1} = 0 \), and \( \lambda = \mu \), as required.

The case \( r = 0 \) is independent of \( c \gg \mu_{1} \). By (6.20), it is given by

\[ \mu f_{\nu}^{m,c} = q^{\sum_{i} m_{i}} \mu s_{m,c}^{\nu} = q^{\sum_{i} m_{i}} \prod_{(i,j) \in F_{\nu}} \frac{(1 - q^{m_{i} + j-1})}{(1 - q^{m_{i}})}, \]

(7.7)

which is due to Stanley [St1]. Combining Theorem 9 and (6.24), we can evaluate \( \mu f_{\nu}^{m,c} \) using (7.7). Thus \( r \) and \( c \) are both subsumed by \( \mu \). We may now prove Theorem 1.
Proof of Theorem 1.

\[ f_{r,v}^m = \sum_{p \in C_{r,v}} q^{m-p \cdot |p|} \]

by (1.2)

\[ = \sum_{j_1 \leq r} q^{-\frac{|r|}{2}} j f_{r}^m \]

by (7.1)

\[ = \sum_{j_1 \leq r} q^{\frac{X}{j} r} g_{r}^m \]

by Theorem 9

\[ = g_{m}^{r} \sum_{j_1 \leq r} q^{\frac{X}{j} r} \prod_{(i,j) \in F_r} \frac{(1-q^{m-j-1})}{(1-q^{j-1})} \]

by (6.21). (7.8)

Setting \( r = 0 \) gives

\[ f_{0,v}^m = \sum_{j_1 \leq r} q^{\frac{X}{j} r} \prod_{(i,j) \in F_r} \frac{(1-q^{m-j-1})}{(1-q^{j-1})}. \] (7.9)

The result (1.4) now follows by substituting (7.9) into (7.8). □

REFERENCES

SCHÜTZENBERGER’S JEUX DE TAQUIN


