
1. INTRODUCTION

This paper is concerned with the Cauchy problem associated to Kadomtsev-Petviashvili (KP) equations having higher order dispersion in the main direction of propagation. Such equations occur naturally in the modeling of certain long dispersive waves (cf. [2, 15, 16]). The study of their solitary wave solutions was done in [6, 7]. Thus we consider the Cauchy problems in \( \mathbb{R}^d, d = 2, 3 \)

\[
\begin{align*}
\{(u_t + xu_{xxx} + \beta u_{xxxxx} + uu_x)_x + u_{yy} = 0, \\
u(0, x, y) = \phi(x, y)
\}\tag{1}
\end{align*}
\]

in the two dimensional case and

\[
\begin{align*}
\{(u_t + xu_{xxx} + \beta u_{xxxxx} + uu_x)_x + u_{yy} + u_{zz} = 0, \\
u(0, x, y, z) = \phi(x, y, z)
\}\tag{2}
\end{align*}
\]

in the three dimensional case. The “usual” KP equations correspond to \( \beta = 0 \) and \( z = -1 \) (KP-I) or \( z = +1 \) (KP-II). We are interested in the local well-posedness of the Cauchy problems (1) and (2). Following [4] and [13] we introduce the notion for well-posedness which will be convenient for our purposes.
Definition 1. The Cauchy problem (1) (resp. (2)) is locally well-posed in the space \( X \) if for any \( \phi \in X \) there exists \( T = T(\|\phi\|_X) > 0 \) (\( T \) is a non-decreasing continuous function such that \( \lim_{\|\phi\|_X \to 0} T(\|\phi\|_X) = \infty \)) and a map \( F \) from \( X \) to \( C([0, T]; X) \) such that \( u = F(\phi) \) solves the equation (1) (resp. (2)) and \( F \) is continuous in the sense that

\[
\|F(\phi_1) - F(\phi_2)\|_{L^\infty([0, T], X)} \leq M(\|\phi_1 - \phi_2\|_X, R)
\]

for some locally bounded function \( M \) from \( \mathbb{R}^+ \times \mathbb{R}^+ \) to \( \mathbb{R}^+ \) such that \( M(\|\phi_1\|_X + \|\phi_2\|_X, R) \leq R \).

We introduce the nonisotropic Sobolev spaces \( H^{s_1, s_2}(\mathbb{R}^d) \) equipped with the norm

\[
\|u\|_{H^{s_1, s_2}}^2 = \int \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{u}(\xi, \eta)|^2 \, d\xi \, d\eta,
\]

where \( \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2} \) and \( \eta = (\eta^1, \eta^2, \eta^3) \) in the case of three space dimensions. By \( H^{s_1, s_2}(\mathbb{R}^d) \) we shall denote the homogeneous nonisotropic Sobolev spaces equipped with the norm

\[
\|u\|_{\tilde{H}^{s_1, s_2}}^2 = \int \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{u}(\xi, \eta)|^2 \, d\xi \, d\eta.
\]

Taking into account the specific structure of the KP-type equations (cf. [18]) we introduce the modified Sobolev space \( \tilde{H}^{s_1, s_2}(\mathbb{R}^d) \) equipped with the norm

\[
\|u\|_{\tilde{H}^{s_1, s_2}}^2 = \int \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} (1 + |\xi|^{-1})^2 |\hat{u}(\xi, \eta)|^2 \, d\xi \, d\eta.
\]

Note that any \( u \in \tilde{H}^{s_1, s_2}(\mathbb{R}^d) \) has (formally) a zero \( x \) mean value. We shall denote \( \tilde{H}^{b, s_1, s_2}(\mathbb{R}^d) \) by \( \tilde{L}^2 \). For \( b, s_1, s_2 \in \mathbb{R} \) we define \( X^{b, s_1, s_2}(\mathbb{R}^{d+1}) \) to be the completion of the functions of \( C_0^\infty \) with zero \( x \) mean value with respect to the norm

\[
\|u\|_{X^{b, s_1, s_2}}^2 = \int \langle \tau + p(\xi, \eta) \rangle^{2b} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} (1 + |\xi|^{-1})^2 |\hat{u}(\tau, \xi, \eta)|^2 \, d\tau \, d\xi \, d\eta,
\]

where

\[
p(\xi, \eta) = b\xi^3 - a\xi^3 + \frac{|\eta|^2}{\xi}.
\]

If \( a \) or \( b \) vanishes then the equations (1) and (2) are scale invariant. If \( u(t, x, y) \) is a solution of (1) with \( \beta = 0 \) then so is \( u(t, x, y) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y) \).
and \(|u(t, \cdot)\|_{\dot{H}^{s_1}} = \lambda^{s_1 + 2s_2 + 1/2} |u(\lambda^{2s_2} t, \cdot)|_{\dot{H}^{s_1}}.| Hence one may expect local well-posedness (resp. ill-posedness) in \(H^{s_1,s_2}\) of (1) for \(b = 0\) when \(s_1 + 2s_2 > -1/2\) (resp. \(s_1 + 2s_2 < -1/2\)). Similarly if \(u(t, x, y, z)\) is a solution of (2) with \(b = 0\) then so is \(u(t, x, y, z) = \lambda^{2s_2} u(\lambda^{2s_2} t, \lambda x, \lambda^2 y, \lambda^2 z)\) and \(|u(t, \cdot)\|_{\dot{H}^{s_1}} = \lambda^{s_1 + 2s_2 - 1/2} |u(\lambda^{2s_2} t, \cdot)|_{\dot{H}^{s_1}}.| Hence one may expect local well-posedness (resp. ill-posedness) in \(H^{s_1,s_2}\) of (2) for \(b = 0\) when \(s_1 + 2s_2 > 1/2\) (resp. \(s_1 + 2s_2 < 1/2\)).

If \(u(t, x, y)\) is a solution of (1) with \(s = 0\) then so is \(u(t, x, y) = \lambda^{4s_2} u(\lambda^{2s_2} t, \lambda x, \lambda^2 y)\) and \(|u(t, \cdot)\|_{H^{s_1}} = \lambda^{s_1 + 3s_2 + 2} |u(\lambda^{2s_2} t, \cdot)|_{H^{s_1}}.| Hence one may expect local well-posedness (resp. ill-posedness) in \(H^{s_1,s_2}\) of (1) for \(s = 0\) when \(s_1 + 3s_2 > -2\) (resp. \(s_1 + 3s_2 < -2\)). Similarly if \(u(t, x, y, z)\) is a solution of (2) with \(s = 0\) then so is \(u(t, x, y, z) = \lambda^{4s_2} u(\lambda^{2s_2} t, \lambda x, \lambda^2 y, \lambda^2 z)\) and \(|u(t, \cdot)\|_{H^{s_1}} = \lambda^{s_1 + 3s_2 + 1/2} |u(\lambda^{2s_2} t, \cdot)|_{H^{s_1}}.| Hence one may expect local well-posedness (resp. ill-posedness) in \(H^{s_1,s_2}\) of (2) for \(s = 0\) when \(s_1 + 3s_2 > -1/2\) (resp. \(s_1 + 3s_2 < -1/2\)).

Now we state the well-posedness result. The proof uses the Fourier transform restriction norms introduced by Bourgain [8–10], in the context of Schrödinger, KdV or KP equations. In the nonlinear estimates we shall make an essential use of the smoothing effects for the linear group associated to (1) or (2) established in [3].

**Theorem 1.** Let \(d = 2\) and \(b > 0\), \(s \in \mathbb{R}\). Then for any \(s_1 \geq -1/4\), \(s_2 \geq 0\) the Cauchy problem (1) is locally well-posed in \(\dot{H}^{s_1,s_2}\). Moreover there exists \(b > 1/2\) such that the solution \(u(t, x, y)\) satisfies \(u \in X^{b,s_1,s_2}\).

Let \(d = 3\) and \(b < 0\), \(s \in \mathbb{R}\). Then for any \(s_1 \geq -1/8\), \(s_2 \geq 0\) the Cauchy problem (2) is locally well-posed in \(\dot{H}^{s_1,s_2}\). Moreover there exists \(b > 1/2\) such that the solution \(u(t, x, y, z)\) satisfies \(u \in X^{b,s_1,s_2}\).

If \(b = 0\) and \(s > 0\) (the “usual” KP-II equation) local well-posedness of (1) in \(L^2\) is established in [10]. The proof uses Fourier transform restriction norms, a dyadic decomposition related to the structure of the symbol of the linearized operator and can be performed for periodic initial data. Local well-posedness of KP-II in \(\dot{H}^{s_1,s_2}\), \(s_1 > -1/4\), \(s_2 \geq 0\) is established in [20].

The sign of \(b\) is crucial in the proof of Theorem 1. We do not know of a similar result when \(b > 0\) (KP-I type equations). In this case, however, it is easy to obtain global weak solutions by energy methods (cf. [19]). The uniqueness of such solutions is unknown.

Note that \(L^2\) norms of the local solutions of (1) and (2) are conserved.

**Lemma 1.** If \(u\) is a local solution of (1) or (2) in the time interval \([0, T]\) then for any \(t \in [0, T]\)

\[\|u(t, \cdot)\|_{L^2} = \|\phi\|_{L^2}.\]
Proof. It suffices to multiply (1) or (2) with \(u\) and integrate over \(\mathbb{R}^d\).

Using Theorem 1 with \(s_1 = s_2 = 0\) and Lemma 1 we obtain

**Theorem 2.** Let \(\beta < 0, \alpha \in \mathbb{R}\). Then for any \(\phi \in L^2(\mathbb{R}^2)\) (resp. \(L^2(\mathbb{R}^3)\)) there exists a unique global solution of the Cauchy problem (1) (resp. (2)).

In [4] it is shown that the solitary wave solutions can be used to construct examples proving local ill-posedness of the (generalized) KdV equation. One considers the limit of the solitary wave solutions as the propagation speed tends to infinity. In the favorable cases the initial data tends weakly to a nonzero distribution (for example the Dirac delta function for the KdV equation with cubic nonlinearity) while the solution at time \(t > 0\) tends weakly to zero. Using the idea of [4, 5] together with the properties of the solitary waves of KP-I type equations (cf. [6, 7]), we can prove the next Theorem.

**Theorem 3.** Let \(d = 2\). If \(\alpha < 0\) and \(\beta = 0\) then (1) is locally ill-posed in \(H^{s,0}\) for \(s < -1/2\). Let \(d = 3\). If \(\alpha < 0\) and \(\beta = 0\) then (1) is locally ill-posed in \(H^{s,0}\) for \(s_1 + 2s_2 = 1/2, s_1 \geq 0, s_2 \geq 0\). If \(\alpha = 0\) and \(\beta > 0\) then (2) is locally ill-posed in \(H^{s,0}\) for \(s < -1/2\).

Note that our well-posedness or ill-posedness results do not contradict with the scaling argument. In fact the region \(\{(s_1, s_2): s_1 \geq -1/4, s_2 \geq 0\}\) is a subset of the region suggested by the scaling argument \(\{(s_1, s_2): s_1 + 3s_2 + 2 > 0\}\), where well-posedness is expected to hold in the case of two space dimensions when \(\alpha = 0, \beta < 0\). The same holds for the region \(\{(s_1, s_2): s_1 \geq -1/8, s_2 \geq 0\}\) which is a subset of \(\{(s_1, s_2): s_1 + 3s_2 + 1/2 > 0\}\) in the case of three space dimensions. If \(d = 2\), \(\alpha < 0\) and \(\beta = 0\) then ill-posedness holds for the pairs \((s, 0), s < -1/2\) which can achieve the “critical” line \(s_1 + 2s_2 + 1/2 = 0\), suggested by the scaling argument. If \(d = 3\), \(\alpha < 0\) and \(\beta = 0\) then ill-posedness holds for the pairs \((s_1, s_2): s_1 + 2s_2 = 1/2, s_1 \geq 0, s_2 \geq 0\) which are a part of the line suggested by the scaling argument. If \(d = 3\), \(\alpha = 0\) and \(\beta > 0\) then the pairs \((s, 0), s < -1/2\) can achieve the line \(s_1 + 3s_2 + 1/2 = 0\), suggested by the scaling argument.

The method for obtaining local well-posedness works just for KP-II type equations (i.e., \(\beta < 0\) or \(\beta = 0, \alpha > 0\)) since one can recuperate the derivative \(\partial_x\) in the non-linear term by the aid of an algebraic relation for the symbol of the linearized operator. (cf. (16) below). In the case of KP-I type equations (\(\beta > 0\) or \(\beta = 0, \alpha < 0\)) we can not gain the \(\partial_x\) derivative using the relation (16). However, using the parabolic regularisation method it is possible to prove local well-posedness of \((1)\) in \(H^1, s > 2\) and local well-posedness of \((2)\) in \(H^s, s > 5/2\) (cf. for instance [14] for an illustration of this method). The proof does not make use of the specific structure of the
KP-type equations and could be performed for quite general evolution
equations. The conditions for $s$ are in order to control the $L^\infty$ norm of the
gradient of the solution and they seem to be very restrictive. Hence there
is still no satisfactory theory for the local well-posedness of the Cauchy
problem for KP-I type equations.

On the other hand our ill-posedness results are only valid for KP-I type
equations since the existence of solitary wave solutions is used in an essen-
tial way. In [6] using Pohojaev type identities it is shown that the KP-II
type equations do not possess localized solitary wave solutions. Note also
that we are forced to take $\alpha$ or $\beta$ zero in order to keep the scale invariance
of the equation.

The paper is organized as follows. In Section 2 we recall the global
smoothing effects for (1) and (2) established in [3]. Then we inject these
estimates into the framework of the Bourgain spaces associated to (1) and
(2). In Section 3 we prove the crucial nonlinear estimate. In Section 4 we
perform a fixed point argument to complete our well-posedness result.
Section 5 is devoted to the ill-posedness of (1) and (2). In Section 6 we
remark that the ill-posedness of the (generalized) KdV equations does not
depend on the special form of the solitary waves. Finally in an appendix we
give the proof of a decay estimate for a fractional derivative of the “lump”
solitary wave of KP-I equation.

2. ESTIMATES FOR THE LINEAR EQUATION

Let us consider the linear initial value problem associated to (1) or (2)

\[
\begin{cases}
  i\partial_t u = p(D) u \\
  u(0, x, y) = \phi,
\end{cases}
\]  

(3)

where $D = (D_1, D_2)$, when $d = 2$, $D = (D_1, D_2, D_3)$, when $d = 3$ $D_1 = -i\partial_x$, $D_2 = -i\partial_y$, and $D_3 = -i\partial_z$. We recall that $p(\xi, \eta) = \beta \xi^2 - \alpha \xi^3 + (|\eta|^2/\xi)$. We
shall denote by $U(t) = \exp(-ip(D))$ the unitary group which generates the
solutions of (3). We have the following representation for the solutions of
(3), when $d = 3$

\[
U(t, x, y, z) = U(t)\phi = G_t \ast \phi,
\]

where $G_t$ is the oscillatory integral

\[
G_t(x, y, z) = c \int_{\mathbb{R}^3} \exp(itp(\xi, \eta) + i(x\xi + y\eta + z\eta^2)) \, d\xi \, d\eta.
\]
A similar representation for the solutions of (3) holds when \( d = 2 \). We have in fact a global smoothing effect for the solutions of (3).

**Lemma 2** (cf. [3], Theorem 4.2). Let \( \delta(r) = \frac{d}{2} - (1/r) \). Then the following estimates hold

\[
\| D_x \|^{d(r)/2} U(t) \phi \|_{L_t^2(L_x^\infty)} \leq c \| \phi \|_{L^2}, \quad (d = 2),
\]
\[
\| D_x \|^{d(r)/6} U(t) \phi \|_{L_t^6(L_x^\infty)} \leq c \| \phi \|_{L^2}, \quad (d = 3),
\]

provided

\[
\frac{2}{q} = \delta(r).
\]

Next we shall prove an inequality which will be used in the proof of the nonlinear estimates.

**Lemma 3.** Let \( d = 2 \) and

\[
0 < \theta \leq 1, \quad \epsilon_1 > 0, \quad 0 < b \leq 1/2 + \epsilon_1.
\]

Then the following inequality holds

\[
\| \mathcal{F}^{-1} |\xi|^b \langle \tau + p(\xi, \eta) \rangle^{-b} |\hat{u}(\tau, \xi, \eta)| \rangle; L_t^q(L_x^\infty) \| \leq c \| u \|_{L^2}, \quad (4)
\]

where

\[
\frac{2}{q} = 1 - \frac{\theta b}{1/2 + \epsilon_1}, \quad \delta(r) = \frac{(1 - \theta) b}{1/2 + \epsilon_1}, \quad \lambda = \frac{\delta(r) \hat{b}}{2(1/2 + \epsilon_1)}.
\]

Let \( d = 3 \) and

\[
0 < \theta \leq 1, \quad \epsilon_1 > 0, \quad 0 < b \leq 1/2 + \epsilon_1.
\]

Then the following inequality holds

\[
\| \mathcal{F}^{-1} |\xi|^b \langle \tau + p(\xi, \eta) \rangle^{-b} |\hat{u}(\tau, \xi, \eta)| \rangle; L_t^q(L_x^\infty, L_y^\infty) \| \leq c \| u \|_{L^2}, \quad (5)
\]

where

\[
\frac{2}{q} = 1 - \frac{\theta b}{1/2 + \epsilon_1}, \quad \delta(r) = \frac{(1 - \theta) b}{1/2 + \epsilon_1}, \quad \lambda = \frac{\delta(r) \hat{b}}{6(1/2 + \epsilon_1)}.
\]

**Proof.** Let \( d = 3 \). Using Lemma 2 and [12], Lemma 3.3 we obtain

\[
\| D_x \|^{d(r)/6} u; L_t^q(L_x^\infty, L_y^\infty) \| \leq c \| u; X_t^{1/2 + \epsilon_1, 0, 0} \|,
\]

(6)
provided
\[
\frac{2}{q} = \delta(r).
\]
Interpolating between (6) and
\[
\|u; L^2_t(L^2_x)\| \leq \|u; X^{0,0}\|,
\]
we obtain
\[
\|D_x^k u; L^2_t(L^2_x)\| \leq c \|u; X^{b,0}\|, \tag{7}
\]
provided
\[
\frac{2}{q} = 1 - \frac{\theta b}{1/2 + \varepsilon_1}, \quad \delta(r) = \frac{(1 - \theta)b}{1/2 + \varepsilon_1}, \quad \lambda = \frac{\delta(r)b}{6(1/2 + \varepsilon_1)}.
\]
But (7) is equivalent to (5) which completes the proof of Lemma 3 when \(d = 3\). If \(d = 2\) then the arguments are the same.

**Corollary 1.** Let \(d = 2\). Then the following inequalities hold
\[
\|F^{-1}(|\xi|^{1/4} \langle \tau + p(\xi, \eta) \rangle^{-b} |\hat{u}(\tau, \xi, \eta)|); L^2_t(L^2_x)\| \leq c \|u\|_{L^2}, \tag{8}
\]
where \(b > 1/2\).
\[
\|F^{-1}(|\xi|^{1/4} \langle \tau + p(\xi, \eta) \rangle^{-b} |\hat{u}(\tau, \xi, \eta)|); L^4_t(L^2_x)\| \leq c \|u\|_{L^2}, \tag{9}
\]
where \(2 \leq q \leq \infty, \ b \geq 0 \) and \((bq)/(q - 2) > \frac{1}{2}\).
Let \(d = 3\). Then the following inequalities hold
\[
\|F^{-1}(|\xi|^{1/6} \langle \tau + p(\xi, \eta) \rangle^{-b} |\hat{u}(\tau, \xi, \eta)|); L^6_t(L^2_x)\| \leq c \|u\|_{L^2}, \tag{10}
\]
where \(b > 1/2\).
\[
\|F^{-1}(|\xi|^{1/6} \langle \tau + p(\xi, \eta) \rangle^{-b} |\hat{u}(\tau, \xi, \eta)|); L^4_t(L^2_x)\| \leq c \|u\|_{L^2}, \tag{11}
\]
where \(2 \leq q \leq \infty, \ b \geq 0 \) and \((bq)/(q - 2) > \frac{1}{2}\).

**Proof:** The proof of (8) follows from Lemma 3 (\(d = 2\)) applied with \(\theta = 1/2\) and \(b = 1/2 + \varepsilon_1\). Lemma 3 (\(d = 3\)) applied with \(\theta = 1/4\) and \(b = 1/2 + \varepsilon_1\) yields (10). To prove (9) and (11) we apply Lemma 3 with \(\theta = 1\).
3. AN ESTIMATE FOR THE NONLINEAR TERM

Lemma 4. Let \( d = 2 \) and \( s_1 \geq -1/4, s_2 \geq 0 \). Then for sufficiently small \( \varepsilon \) we have

\[
\|uu\|_{X^{1/2 + s_1, s_2}} \leq C \|u\|_{X^{1/2, s_1, s_2}}^2.
\]

(12)

Let \( d = 3 \) and \( s_1 \geq -1/8, s_2 \geq 0 \). Then for sufficiently small \( \varepsilon \) we have

\[
\|uu\|_{X^{1/2 + s_1, s_2}} \leq C \|u\|_{X^{1/2, s_1, s_2}}^2.
\]

(13)

Proof. We shall give the proof only when \( d = 3 \). In the case of two space dimensions the arguments are similar. We set

\[
\begin{align*}
\hat{w}(\tau, \zeta, \eta) &= \langle \tau + p(\zeta, \eta) \rangle^{1/2 + 2\varepsilon} \langle \zeta \rangle^{s_1} \langle \eta \rangle^{s_2} \hat{u}(\tau, \zeta, \eta), \\
\sigma := \sigma(\tau, \zeta, \eta) &= \tau + p(\zeta, \eta), \quad \sigma_1 := \sigma(\tau_1, \zeta_1, \eta_1), \\
\sigma_2 := \sigma(\tau - \tau_1, \zeta - \zeta_1, \eta - \eta_1).
\end{align*}
\]

Let \( \zeta = (\zeta, \eta) \). Then (13) is equivalent to

\[
\left\| \langle \sigma \rangle^{-1/2 + s_1} \langle \zeta \rangle^{1 + s_1} \langle \eta \rangle^{s_2} \right\|_{L^2_{\tau, \zeta, \eta}}^2
\times \int \int K(\tau, \zeta, \tau_1, \zeta_1) \hat{w}(\tau_1, \zeta_1) \hat{w}(\tau - \tau_1, \zeta - \zeta_1) \, d\tau_1 \, d\zeta_1 \\
\leq C \left\{ \|w\|_{L^2}^2 + \|w\|_{L^2} \|D_x\|^{-1} w \|L^2 \| \|D_x\|^{-1} w \|L^2 \| \right\}.
\]

(14)

where

\[
K(\tau, \zeta, \tau_1, \zeta_1) = \frac{\langle \zeta_1 \rangle^{-s_1} \langle \zeta - \zeta_1 \rangle^{-s_1} \langle \eta_1 \rangle^{-s_2} \langle \eta - \eta_1 \rangle^{-s_2}}{\langle \sigma_1 \rangle^{1/2 + 2\varepsilon} \langle \sigma_2 \rangle^{1/2 + 2\varepsilon} \langle \sigma \rangle^{1/2 - s_2} \langle \eta_1 \rangle^{s_2} \langle \eta - \eta_1 \rangle^{s_2}}.
\]

Hence by the self-duality of \( L^2 \) we obtain that (14) is equivalent to

\[
\left\| \langle \sigma \rangle^{-1/2 + s_1} \langle \zeta \rangle^{1 + s_1} \langle \eta \rangle^{s_2} \right\|_{L^2_{\tau, \zeta, \eta}}^2
\times \int \int K_1(\tau, \zeta, \tau_1, \zeta_1) \hat{w}(\tau_1, \zeta_1) \hat{w}(\tau - \tau_1, \zeta - \zeta_1) \, d\tau_1 \, d\zeta_1 \\
\leq C \left\{ \|w\|_{L^2}^2 + \|w\|_{L^2} \|D_x\|^{-1} w \|L^2 \| \|D_x\|^{-1} w \|L^2 \| \right\} \|v\|_{L^2},
\]

(15)

where

\[
K_1(\tau, \zeta, \tau_1, \zeta_1) = \frac{\langle \zeta_1 \rangle^{1 + s_1} \langle \zeta - \zeta_1 \rangle^{-s_1} \langle \eta_1 \rangle^{-s_2} \langle \eta - \eta_1 \rangle^{-s_2}}{\langle \sigma_1 \rangle^{1/2 + 2\varepsilon} \langle \sigma_2 \rangle^{1/2 + 2\varepsilon} \langle \sigma \rangle^{1/2 - s_2} \langle \eta_1 \rangle^{s_2} \langle \eta - \eta_1 \rangle^{s_2}}.
\]
Since for \( s_2 \geq 0 \)
\[
\frac{\langle \eta \rangle^{s_2}}{\langle \eta_1 \rangle^{s_2} \langle \eta - \eta_1 \rangle^{s_2}} \leq \text{const},
\]
we shall suppose that \( s_2 = 0 \) from now on. Without loss of generality we can assume that \( \tilde{w} \geq 0 \) and \( \tilde{v} \geq 0 \). We have the following relation, where \( \eta = (\eta^1, \eta^2) \) and \( \eta_1 = (\eta_1^1, \eta_1^2) \)
\[
\sigma_1 + \sigma_2 - \sigma = -5 \beta \xi_1 \tilde{z}(\xi - \xi_1)(\xi^2 - \xi_1^2 + \xi_2^2) + 3 \alpha \xi_1 \tilde{z}(\xi - \xi_1)
\]
\[
+ \frac{(\xi_1 \eta^1 - \xi_1 \eta_1^1)^2}{\xi_1 \tilde{z}(\xi - \xi_1)} + \frac{(\xi_1 \eta^2 - \xi_1 \eta_1^2)^2}{\xi_1 \tilde{z}(\xi - \xi_1)}. \tag{16}
\]

**Lemma 5.** There exists a positive constant \( c_0 \) such that
\[
\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq c |\xi_1(\xi - \xi_1) \tilde{z}^3|, \quad \text{for} \quad |\xi| \geq c_0. \tag{17}
\]

**Proof.** Since \( \beta < 0 \) we have that
\[
3 \max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq |\xi_1 \tilde{z}(\xi - \xi_1)| \left( -\frac{12}{5} \beta \xi^2 + 3 \alpha \right),
\]
where we used the elementary inequality \( \xi^2 - \xi_1^2 + \xi_2^2 \geq \frac{1}{2} \xi^2 \). If \( \alpha > 0 \) then we obtain (17) with \( c_0 = 0 \). If \( \alpha < 0 \) then for \( -\frac{12}{5} \beta \xi^2 + 3 \alpha \geq 0 \) we have
\[
3 \max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq |\xi_1 \tilde{z}(\xi - \xi_1)| \left( -\frac{12}{5} \beta \xi^2 \right),
\]
which completes the proof of Lemma 5. We now continue the proof of Lemma 4.

We shall denote by \( J \) the integral in the left-hand side of (15). We shall divide the domain of integration taking into account which term dominates in the left-hand side of (17). By symmetry we can assume that
\[
|\sigma_1| \geq |\sigma_2|.
\]

**Case 1.** \( |\xi| \leq \max(2, c_0) \). We denote by \( J_1 \) the restriction of \( J \) on this region. In this case we have that
\[
K_1(\tau, \zeta, \tau_1, \zeta_1) \leq \frac{\langle \xi_1 \rangle^{1/2} \langle \xi - \xi_1 \rangle^{1/2}}{\langle \sigma_1 \rangle^{1/2 + \tau} \langle \sigma_2 \rangle^{1/2 + \tau} \langle \sigma \rangle^{1/2 - \tau}}.
\]
Case 1.1. $|\xi_1| \gtrsim 1$, $|\xi - \xi_1| \gtrsim 1$. We denote by $J_{11}$ the restriction of $J$ on this region. In this case we have that
\[
K_1(\tau, \zeta, \tau_1, \zeta_1) \leq \frac{|\xi_1|^{1/8} |\xi - \xi_1|^{1/8}}{(\sigma_1)^{1/2 + 2\varepsilon} (\sigma_2)^{1/2 + 2\varepsilon} (\sigma)^{1/2 - \varepsilon}}.
\]
Hence using Hölder inequality (10) and (11) we obtain
\[
J_{11} \lesssim \|F\| \langle \langle \sigma \rangle \rangle^{-1/2 + 3\varepsilon} \|\tilde{\phi}(\tau, \zeta)\|_{L^4_x(L^4_{t,x,y})}^2
\times \|F\|^{-1} |\xi_1|^{1/8} \langle \sigma_1 \rangle^{-1/2 - 2\varepsilon} \|\tilde{u}(\tau_1, \zeta_1)\|_{L^4_x(L^4_{t,x,y})}
\times \|F\|^{-1} |\xi - \xi_1|^{1/8} \langle \sigma_2 \rangle^{-1/2 - 2\varepsilon} \|\tilde{u}(\tau - \tau_1, \zeta - \zeta_1)\|_{L^4_x(L^4_{t,x,y})}
\leq c \|\|w\|_{L^2} \|\|v\|_{L^2}.
\]

Case 1.2. $|\xi_1| \lesssim 1$. We denote by $J_{12}$ the restriction of $J$ on this region. In this case we have that
\[
K_1(\tau, \zeta, \tau_1, \zeta_1) \leq \frac{|\xi_1|^{1/8} |\xi - \xi_1|^{1/8}}{(\sigma_1)^{1/2 + 2\varepsilon} (\sigma_2)^{1/2 + 2\varepsilon} (\sigma)^{1/2 - \varepsilon}}.
\]
Hence using Hölder inequality (10) and (11) we obtain
\[
J_{12} \lesssim \|F\| \langle \langle \sigma \rangle \rangle^{-1/2 + 3\varepsilon} \|\tilde{\phi}(\tau, \zeta)\|_{L^4_x(L^4_{t,x,y})}^2
\times \|F\|^{-1} |\xi_1|^{1/8} \langle \sigma_1 \rangle^{-1/2 - 2\varepsilon} |\xi_1|^{-1} \|\tilde{u}(\tau_1, \zeta_1)\|_{L^4_x(L^4_{t,x,y})}
\times \|F\|^{-1} |\xi - \xi_1|^{1/8} \langle \sigma_2 \rangle^{-1/2 - 2\varepsilon} \|\tilde{u}(\tau - \tau_1, \zeta - \zeta_1)\|_{L^4_x(L^4_{t,x,y})}
\leq c \|\|w\|_{L^2} \|\|D_x\|^{-1} \|\|v\|_{L^2} \|\|w\|_{L^2}.
\]

Case 1.3. $|\xi - \xi_1| \lesssim 1$. This case can be treated similarly to Case 1.2.

Case 2. $|\xi| \gtrsim \max(2, c_9)$, $|\sigma| \gtrsim |\sigma_1|$. We denote by $J_2$ the restriction of $J$ on this region. In this case we have that
\[
|\sigma|^{1/4 - 4\varepsilon} \lesssim c |\xi_1|^{1/4 - 12\varepsilon} |\xi - \xi_1|^{1/4 - 4\varepsilon}.
\]
Hence
\[
K_1(\tau, \zeta, \tau_1, \zeta_1) \lesssim \frac{\langle \xi_1 \rangle^{1/4 + 2\varepsilon} |\xi - \xi_1|^{1/4} \langle \sigma_1 \rangle^{1/2 + 2\varepsilon} \langle \sigma \rangle^{1/2 + 2\varepsilon}}{|\xi_1|^{1/4} |\xi - \xi_1|^{1/4} \langle \sigma \rangle^{1/2 + 2\varepsilon} \langle \sigma \rangle^{1/2 + 2\varepsilon}}
\]

Case 2.1. $|\xi_1| \gtrsim 1$, $|\xi - \xi_1| \gtrsim 1$. We denote by $J_{21}$ the restriction of $J$ on this region. In this case we have that
\[
K_1(\tau, \zeta, \tau_1, \zeta_1) \lesssim \frac{\langle \xi_1 \rangle^{1/4} |\xi - \xi_1|^{1/4}}{|\sigma_1|^{1/2 + 2\varepsilon} \langle \sigma_2 \rangle^{1/2 + 2\varepsilon} \langle \sigma \rangle^{1/2 + 2\varepsilon}}.
\]
Using (10), (11) and Hölder inequality we obtain

\[ J_{21} \leq \| F^{-1}(\langle \sigma \rangle^{-1/4-\varepsilon} \hat{\rho}(\tau, \zeta)) \|_{L^2_t(L^4, x, n)} \]
\[ \times \| F^{-1}(|\xi_1|^{1/8} \langle \sigma_1 \rangle^{-1/2-2\varepsilon} \hat{w}(\tau_1, \zeta_1)) \|_{L^2_t(L^4, x, n)} \]
\[ \times \| F^{-1}(|\xi - \xi_1|^{1/8} \langle \sigma_2 \rangle^{-1/2-2\varepsilon} \hat{w}(\tau - \tau_1, \zeta - \zeta_1)) \|_{L^2_t(L^4, x, n)} \]
\[ \leq c \| w \|_{L^2} \| v \|_{L^2}. \]

**Case 2.2.** \(|\xi_1| \leq 1\). We denote by \( J_{22} \) the restriction of \( J \) on this region. In this case we have that

\[ K_1(\tau, \zeta, \tau_1, \zeta_1) \leq \frac{|\xi_1|^{1/4} |\xi - \xi_1|^{1/8}}{\| \xi_1 \|_{L^4} \| \langle \sigma_1 \rangle \|_{L^4} \| \langle \sigma_2 \rangle \|_{L^4}}. \]

Using (10), (11) and Hölder inequality we obtain

\[ J_{22} \leq \| F^{-1}(\langle \sigma \rangle^{-1/4-\varepsilon} \hat{\rho}(\tau, \zeta)) \|_{L^2_t(L^4, x, n)} \]
\[ \times \| F^{-1}(|\xi_1|^{1/8} \langle \sigma_1 \rangle^{-1/2-2\varepsilon} |\xi_1|^{-1} \hat{w}(\tau_1, \zeta_1)) \|_{L^2_t(L^4, x, n)} \]
\[ \times \| F^{-1}(|\xi - \xi_1|^{1/8} \langle \sigma_2 \rangle^{-1/2-2\varepsilon} \hat{w}(\tau - \tau_1, \zeta - \zeta_1)) \|_{L^2_t(L^4, x, n)} \]
\[ \leq c \| w \|_{L^2} \| D_x^{-1} w \|_{L^2} \| v \|_{L^2}. \]

**Case 2.3.** \(|\xi - \xi_1| \leq 1\). This case can be treated similarly to Case 2.2.

**Case 3.** \(|\xi| \geq \max(2, c_0), |\sigma| \geq |\sigma|\). We denote by \( J_3 \) the restriction of \( J \) on this region. In this case we have that

\[ |\sigma_1|^{1/4-4\varepsilon} \geq c |\xi_1|^{1/4-2\varepsilon} |\xi - \xi_1|^{1/4-4\varepsilon}. \]

Hence

\[ K_1(\tau, \zeta, \tau_1, \zeta_1) \leq \frac{\langle \xi \rangle^{1/4+4\varepsilon} \langle \xi_1 \rangle^{-n_1+4\varepsilon} \langle \xi - \xi_1 \rangle^{-n_1+4\varepsilon}}{|\xi_1|^{1/4} |\xi - \xi_1|^{1/4} \| \langle \sigma_1 \rangle \|_{L^4} \| \langle \sigma_2 \rangle \|_{L^4} \| \langle \sigma \rangle \|_{L^4}}. \]

**Case 3.1.** \(|\xi_1| \geq 1, |\xi - \xi_1| \geq 1\). We denote by \( J_{31} \) the restriction of \( J \) on this region. In this case we have that

\[ K_1(\tau, \zeta, \tau_1, \zeta_1) \leq \frac{\langle \xi \rangle^{12\varepsilon} \langle \xi_1 \rangle^{4\varepsilon} \langle \xi - \xi_1 \rangle^{4\varepsilon}}{|\sigma_1|^{1/4+6\varepsilon} \| \langle \sigma_1 \rangle \|_{L^4} \| \langle \sigma_2 \rangle \|_{L^4} \| \langle \sigma \rangle \|_{L^4}}. \]
Using Lemma 3 and Hölder inequality we obtain
\[ J_{31} \leq \|F_1\|^{1/2} \|\xi\|^{(1/2 - 3\varepsilon)/(1/2 + \varepsilon)} \|\sigma\|^{1/2 + 3\varepsilon} \|\tilde{\phi}(\tau, \zeta)\|_{L^1_t(L^1_x)} \]
\[ \times \|F_2\|^{1/2} \|\xi/\zeta\|^{(1/4 + 6\varepsilon)/(1/2 + \varepsilon)} \|\sigma_1\|^{1/2 + 6\varepsilon} \|\tilde{w}(\tau_1, \zeta_1)\|_{L^2_t(L^2_x)} \]
\[ \times \|F_3\|^{1/2} \|\xi - \zeta\|^{(1/2 + 2\varepsilon)/(1/2 + \varepsilon)} \|\sigma_2\|^{1/2 + 2\varepsilon} \|\tilde{w}(\tau_2, \zeta_2)\|_{L^2_t(L^2_x)} \]
\[ \leq \|w\|^2_{L^2} \|v\|_{L^2}, \]
provided
\[ \frac{2}{q_1} = 1 - \frac{\theta(1/2 - 3\varepsilon)}{1/2 + \varepsilon}, \quad \delta(r_1) = \frac{(1 - \theta)(1 - 3\varepsilon)}{1/2 + \varepsilon}, \]
\[ \frac{2}{q_2} = 1 - \frac{\theta(1/4 + 6\varepsilon)}{1/2 + \varepsilon}, \quad \delta(r_2) = \frac{(1 - \theta)(1/4 + 6\varepsilon)}{1/2 + \varepsilon}, \]
\[ \frac{2}{q_3} = 1 - \frac{\theta(1/2 + 2\varepsilon)}{1/2 + \varepsilon}, \quad \delta(r_3) = \frac{(1 - \theta)(1 - 2\varepsilon)}{1/2 + \varepsilon}, \]
\[ \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1, \quad \delta(r_1) + \delta(r_2) + \delta(r_3) = \frac{3}{2}. \]
Now it is sufficient to take \( \varepsilon = 2\varepsilon \) and \( \theta = 2/5 \) to ensure the restrictions of the Hölder inequality.

Case 3.2. \(|\xi_1| \leq 1\). We denote by \( J_{32} \) the restriction of \( J \) on this region. Using the arguments of Case 3.1, we similarly obtain
\[ J_{32} \leq \|w\|^2_{L^2} \|v\|_{L^2}. \]

Case 3.2. \(|\xi - \zeta| \leq 1\). This case can be treated similarly to Case 3.2. This completes the proof of Lemma 4.

4. PROOF OF THEOREM 1

In this section we shall give the proof of Theorem 1. Note that the Eqs. (1) or (2) are equivalent to the integral equation
\[ u(t) = U(t)\phi - \int_0^t U(t - t') u(t') \, dt'. \]
Let \( \psi \) be a cut-off function such that
\[
\psi \in C_0^\infty(\mathbb{R}), \quad \text{supp } \psi \subset [-2, 2], \quad \psi = 1 \text{ over the interval } [-1, 1].
\]
We truncate (18)
\[
\begin{align*}
  u(t) &= \psi(t) \left( U(t) u(0) - \psi(t/T) \int_0^t U(t-t') u(t') \partial_x u(t') \, dt' \right).
\end{align*}
\]  
(19)
We shall solve (19) globally in time. To the solutions of (19) will correspond local solutions of (18) in the time interval \([-T, T]\). We shall apply a fixed point argument to solve (19) in
\[
X^{1/2+\varepsilon, s_1, s_2} = \mathbb{R}^{s_1, s_2},
\]
for sufficiently small \( \varepsilon > 0 \).

**Lemma 6.** The following estimates hold for sufficiently small \( \varepsilon > 0 \)
\[
\begin{align*}
  &\|\psi(t) U(t) \phi\|_{X^{1/2+\varepsilon, s_1, s_2}} \leq c \|\phi\|_{B^{s_1, s_2}}, \quad (20) \\
  &\|\psi(t/T) \int_0^t U(t-t') u(t') \partial_x u(t') \, dt'\|_{X^{1/2+\varepsilon, s_1, s_2}} \\
  &\leq cT^\varepsilon \|u\|_{X^{1/2+\varepsilon, s_1, s_2}}. \quad (21)
\end{align*}
\]

**Proof.** To prove (20) it is sufficient to note that
\[
\|u\|_{X^{1/2+\varepsilon, s_1, s_2}} = \|U(-t)u\|_{B^{s_1, s_2}},
\]
where \( \hat{B}^{s_1, s_2} \) is equipped with the norm
\[
\|u\|^2_{\hat{B}^{s_1, s_2}} = \int \langle \tau \rangle^b \langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} (1 + |\xi|^{-1})^2 |\hat{u}(\tau, \xi, \eta)|^2 \, d\tau \, d\xi \, d\eta.
\]
The proof of (21) is a direct consequence of [12], Lemma 3.2 applied with \( b = 1/2 + 2\varepsilon \) and \( b' = -1/2 + 3\varepsilon \). This completes the proof of Lemma 6.

Now we define an operator \( L \)
\[
L u(t) := \psi(t) \left( U(t) u(0) - \psi(t/T) \int_0^t U(t-t') u(t') \partial_x u(t') \right) dt'.
\]
We obtain from Lemmas 5 and 6
\[
\|Lu\|_{X^{1/2+\varepsilon, s_1, s_2}} \leq c \|\phi\|_{B^{s_1, s_2}} + cT^\varepsilon \|u\|^2_{X^{1/2+\varepsilon, s_1, s_2}}. \quad (22)
\]
Similarly we obtain
\[
\|Lu - L\| \leq cT^\varepsilon \|u + v\|_{X^{1/2+\varepsilon, s_1, s_2}} \|u - v\|_{X^{1/2+\varepsilon, s_1, s_2}}. \quad (23)
\]
Using (22) and (23) we can apply the contraction mapping principle for sufficiently small $T$ which completes the proof of Theorem 1.

5. PROOF OF THEOREM 3

It is known (cf. [6]) that in some cases (1) (resp. (2)) possesses solitary wave solutions. We recall that a solitary wave is a “localized” solution of (1) (resp. (2)) of the form $u(x - ct, y)$ (resp. $u(x - ct, y, z)$). More precisely we have the following theorem.

**Theorem 5.1.** (cf. [6, 7]). Let $\beta > 0$ and $\alpha \leq 0$. Then (1) (resp. (2)) possesses a solitary wave solution $u(x - ct, y)$ (resp. $u(x - ct, y, z)$) such that

$$u \in L^2_y(L^1_x)(\mathbb{R}^d), \quad \left(\text{resp. } u \in L^2_{y,z}(L^1_x)(\mathbb{R}^d)\right).$$

The same statement holds when $\beta = 0$ and $\alpha < 0$.

We have (cf. [7]) that $r u \in L^2(R^d)$, for $\delta < d/2$, where $r^2 = x^2 + y^2$, when $d = 2$ and $r^2 = x^2 + y^2 + z^2$, when $d = 3$. Hence using Cauchy–Schwarz inequality we obtain immediately

$$u \in L^2_y(L^1_x) \quad (d = 2), \quad u \in L^2_{y,z}(L^1_x) \quad (d = 3).$$

The rest of the proof of Theorem 5.1. is contained in [6, 7]. Note that when $\beta = 0$, $\alpha < 0$ an explicit form of a solitary wave can be derived by the inverse scattering method and we will use it below. In the 3D case we will use essentially scaling arguments and no explicit form of the solitary waves is needed. Actually, solitary waves for (1) or (2) (cf. [6, 7]) can be obtained by the aid of the concentration compactness principle for some pure power nonlinearities and the method for obtaining ill-posedness could be extended to these cases (see the end of the section).

- Let $d = 2$, $\beta = 0$ and $\alpha < 0$. Then (1) has a solitary wave solution of the form

$$u_\delta(t, x, y) = \phi(x - ct, y), \quad c > 0,$$

where

$$u_\delta(0, x, y) = \phi(x, y) = c\phi(t^{1/2}x, cy) := c\phi(t^{1/2}x, cy).$$

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Here $u_c$ is the “lump” solitary wave of KP-I equation. Recall that

$$u_c(t, x, y) = \frac{8c \left( 1 - \frac{c}{3} (x - ct)^2 + \frac{c^2}{3} y^2 \right)}{\left( 1 + \frac{c}{3} (x - ct)^2 + \frac{c^2}{3} y^2 \right)^2}$$

is a solitary wave solution of (1) with $\alpha = -1$ and $\beta = 0$. We will need a decay estimate of a fractional derivative of $\phi_c$. Obviously $x\phi_c$ does not belong to $L^2(\mathbb{R}^2)$, but

**Lemma 7 (cf. Appendix).** Let $\varepsilon > 0$. Then

$$|D_x|^\varepsilon (x\phi_c) \in L^2(\mathbb{R}^2).$$

We shall denote by $\mathcal{F}_x$ the partial Fourier transform with respect to $x$. One has

$$\mathcal{F}_x(\phi_c)(\xi, y) = e^{1/2} \mathcal{F}_x(\phi) \left( \frac{\xi}{c^{1/2}}, c y \right).$$

Further we have

$$\|\phi_c\|_{H^s}^2 = \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} \left| \mathcal{F}_x(\phi) \left( \frac{\xi}{c^{1/2}}, c y \right) \right|^2 d\xi dy.$$

Hence for $s < -1/2$ Lebesgue theorem yields

$$\lim_{\varepsilon \to \infty} \|\phi_c\|_{H^s}^2 = c_s \|\mathcal{F}_x(\phi)(0, .)\|_{L^2}^2 = c_s \|\phi\|_{L^2(\mathbb{R}^2)}^2,$$

where

$$c_s = \int_{-\infty}^{\infty} \langle \xi \rangle^{2s} d\xi.$$

Similarly we obtain

$$(\phi_{c_1}, \phi_{c_2})_s = \left( \frac{c_2}{c_1} \right)^{1/2} \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} \mathcal{F}_x(\phi) \left( \frac{\xi}{c_1^{1/2}}, c y \right) \overline{\mathcal{F}_x(\phi) \left( \frac{\xi}{c_2^{1/2}}, c y \right)} d\xi dy,$$

where $(., .)_s$ stays for the $H^{s, 0}$ scalar product. Hence

$$\lim_{c_1, c_2 \to \infty, c_1/c_2 \to 1} (\phi_{c_1}, \phi_{c_2})_s = c_s \|\mathcal{F}_x(\phi)(0, .)\|_{L^2}^2,$$
\[
\lim_{c_1, c_2 \to -\infty, c_1/c_2 \to 1} |\phi_{c_1} - \phi_{c_2}|_{H^s\alpha} = 0.
\]

Now we have that
\[
\mathcal{F}(u_c)(t, \xi, y) = c^{1/2} \exp(-itc_1^2) \mathcal{F}(\phi)(\xi/c^{1/2}, cy).
\]

The presence of the oscillatory term \(\exp(-itc_1^2)\) makes the expression \((u_{c_1}(t, .), u_{c_2}(t, .))_s\) tend to zero as \(n \to \infty\) by taking \(c_1 = n^2\) and \(c_2 = (n + 1)^2\).

More precisely we have by integration by parts
\[
(u_{n^2}(t, .), u_{(n+1)^2}(t, .))_s
\]
\[
= c(n, t) \int_{\mathbb{R}^2} \partial_\xi (e^{in^2+1})^2ormal \xi \mathcal{F}(\phi)(\xi/n, y)
\]
\[
\times \mathcal{F}(\phi)(\xi/n, y) d\xi dy
\]
\[
+ c(n, t) \int_{\mathbb{R}^2} e^{in^2+1} \xi > 2 \xi \mathcal{F}(\phi)(\xi/n, y)
\]
\[
\times \mathcal{F}(\phi)(\xi/n, y) d\xi dy
\]
\[
= c(n, t) \int_{\mathbb{R}^2} \frac{e^{in^2+1} \xi > 2 \xi \mathcal{F}(\phi)(\xi/n, y)}{\mathcal{F}(\phi)(\xi/n, y)} d\xi dy
\]
\[
+ c(n, t) \int_{\mathbb{R}^2} \frac{e^{in^2+1} \xi > 2 \xi \mathcal{F}(\phi)(\xi/n, y)}{\mathcal{F}(\phi)(\xi/n, y)} d\xi dy
\]
\[
= (1) + (2) + (3),
\]
where
\[
c(n, t) = \frac{n + 1}{im(2n + 1)t}.
\]
Using Cauchy–Schwarz inequality and the estimate $\|\hat{\phi}\|_{L^\infty} \ll \|\phi\|_{L^1}$, we obtain

$$
|(1)| \leq 2s |c(n, t)| \int_{\mathbb{R}^2} \left\langle \xi \right\rangle^{2s} \|\phi(\cdot, y)\|_{L^1} \left\| \phi \left( \frac{(n+1)^2}{n^2} y \right) \right\|_{L^1} d\xi dy
\leq |c(n, t)| \frac{cn}{n+1} \|\phi\|_{L^2}^{1/2} (L^1).
$$

It is clear that the last expression tends to zero as $n$ tends to infinity.

Further we have

$$
|(2)| \leq \frac{n^s |c(n, t)|}{n} \int_{\mathbb{R}^3} \left\langle \xi \right\rangle^{2s} \left\| \mathcal{F}_x(|D_x|^{\alpha} (x\phi)) \left( \frac{\xi}{n} y \right) \right\| \left\| \phi \left( \frac{(n+1)^2}{n^2} y \right) \right\|_{L^1} d\xi dy
\leq \frac{n^s |c(n, t)|}{n} \frac{n}{n+1} \|\phi\|_{L^2}^{1/2} \left\| \mathcal{F}_x(|D_x|^{\alpha} (x\phi)) \left( \frac{\xi}{n} y \right) \right\|_{L^1} \left\| \mathcal{F}_x(|D_x|^{\alpha} (x\phi)) \right\|_{L^1} d\xi dy
\leq \frac{n^s |c(n, t)|}{n} \frac{n^{3/2}}{n+1} \|\phi\|_{L^2}^{1/2} \|\mathcal{F}_x(|D_x|^{\alpha} (x\phi))\|_{L^1},
$$

where we used that $\xi^{-s} \left\langle \xi \right\rangle^{2s} \in L^2_\alpha$ if $\alpha < 1/2$ and Lemma 7. Hence we obtain that for $\alpha < 1/2$, $|(2)|$ tends to zero as $n$ tends to infinity. The term $|(3)|$ can be estimated in a similar fashion. Hence

$$
\lim_{n \to \infty} (u(n+1))_x(t, \cdot) = 0.
$$

Moreover, for any $t > 0$

$$
\lim_{n \to \infty} \|u(n+1)_x(t, \cdot) - u_x(t, \cdot)\|_{H_0} = 2c_2 \|\mathcal{F}_x(\phi)(0, \cdot)\|_{L^2}.
$$

Therefore we obtain that (1) is locally ill-posed in $H^{s,0}(\mathbb{R}^2)$ for $\beta = 0$ and $x < 0$, when $s < -1/2$.

Let $d = 3$, $\beta > 0$ and $x = 0$. Then (2) has a solitary wave solution of the form

$$
u_c(t, x, y, z) = \phi_c(x - ct, y, z), \quad c > 0,
$$

where

$$
u_c(0, x, y, z) = \phi_c(x, y, z) = c\phi(1^{1/4} x, 1^{3/4} y, 1^{3/4} z) := c\phi(1^{1/4} x, 1^{3/4} y, 1^{3/4} z).
$$

One obtains easily that

$$
\mathcal{F}_x(\phi_c)(\xi, y, z) = c^{3/4} \mathcal{F}_x(\phi) \left( \frac{\xi}{c^{3/4}}, 1^{3/4} y, 1^{3/4} z \right).
$$
Further we have

\[ \| \phi \|_{L^6}^2 = \int_{\mathbb{R}^3} \left| \left\langle \xi \right\rangle^{2s} \mathcal{F}_x(\phi) \left( \frac{\xi}{(n+1)^{1/2}}, y, z \right) \right|^2 dy \, dz. \]

Hence for \( s < -1/2 \) Lebesgue theorem yields

\[ \lim_{\epsilon \to \infty} \| \phi_{\epsilon} \|_{L^6}^2 = c_s^2 \| \mathcal{F}_x(\phi)(0, .) \|_{L^2}^2 = c_s \| \phi \|_{L^6}^2(\xi_0^0). \]

Similarly we obtain

\[ (\phi_{\epsilon_1}, \phi_{\epsilon_2})_x = \left( \frac{c_2}{c_1} \right)^{3/4} \int_{\mathbb{R}^3} \left\langle \xi \right\rangle^{2s} \mathcal{F}_x(\phi) \left( \frac{\xi}{(n+1)^{1/2}}, y, z \right) \times \mathcal{F}_x(\phi) \left( \frac{\xi}{c_2(c_1)^{1/4}} y, c_2(c_1)^{1/4} z \right) d\xi \, dy \, dz, \]

where \((., .)_x\) stays for the \( H^{s,0} \) scalar product. Hence

\[ \lim_{\epsilon_1, \epsilon_2 \to \infty, \epsilon_1/\epsilon_2 \to 1} (\phi_{\epsilon_1}, \phi_{\epsilon_2})_x = c_s^2 \| \mathcal{F}_x(\phi)(0, .) \|_{L^2}^2 \]

and

\[ \lim_{\epsilon_1, \epsilon_2 \to \infty, \epsilon_1/\epsilon_2 \to 1} \| \phi_{\epsilon_1} - \phi_{\epsilon_2} \|_{H^{s,0}} = 0. \]

Now we have that

\[ \mathcal{F}_x(u_{\epsilon})(t, \xi, y, z) = e^{3\epsilon^4} \exp(-it\xi) \mathcal{F}_x(\phi)(\xi/c^{1/4}, cy, cz). \]

The presence of the oscillatory term \( \exp(-it\xi) \) makes the expression \( (u_{\epsilon_1}(t, .), u_{\epsilon_2}(t, .))_x \) tend to zero as \( n \to \infty \) by taking \( c_1 = n^2 \) and \( c_2 = (n+1)^2 \).

Let \( x' = (y, z) \). Then

\[ (u_{\epsilon_1}(t, .), u_{(n+1)^2}(t, .))_x = \epsilon(n, t) \int_{\mathbb{R}^3} \partial_x(e^{n(2n+1)\xi} \left\langle \xi \right\rangle^{2s} \mathcal{F}_x(\phi) \left( \frac{\xi}{(n+1)^{1/2}}, x' \right) \times \mathcal{F}_x(\phi) \left( \frac{\xi}{(n+1)^{1/2}}, \frac{(n+1)^2}{n} \right) d\xi \, dx'. \]
\[
\begin{align*}
= 2c(n, t) & \int_{\mathbb{R}^3} e^{i(2n+1)\xi \cdot \eta} \left\langle \frac{\xi}{n^{1/2}}, x' \right\rangle \\
& \times \mathcal{F}_k(\phi) \left( \frac{\xi}{n^{1/2}}, \frac{(n+1)^2}{n^2} x' \right) d\xi' dx' \\
& + \frac{c(n, t)}{in^{1/2}} \int_{\mathbb{R}^3} e^{i(2n+1)\xi \cdot \eta} \left\langle \frac{\xi}{n^{1/2}}, x' \right\rangle \\
& \times \mathcal{F}_k(\phi) \left( \frac{\xi}{n^{1/2}}, \frac{(n+1)^2}{n^2} x' \right) d\xi' dx' \\
& + \frac{c(n, t)}{i(n+1)^{1/2}} \int_{\mathbb{R}^3} e^{i(2n+1)\xi \cdot \eta} \left\langle \frac{\xi}{n^{1/2}}, x' \right\rangle \\
& \times \mathcal{F}_k(\phi) \left( \frac{\xi}{n^{1/2}}, \frac{(n+1)^2}{n^2} x' \right) d\xi' dx' \\
& := (1) + (2) + (3),
\end{align*}
\]

where
\[
c(n, t) = \frac{(n+1)^{3/2}}{in^{1/2}(2n+1)t}.
\]

Using Cauchy–Schwarz inequality and the estimate \( \|\hat{\phi}\|_{L^2} \leq \|\phi\|_{L^1} \), we obtain
\[
|1| \leq 2\|c(n, t)\| \int_{\mathbb{R}^3} \left\langle \frac{\xi}{n^{1/2}}, x' \right\rangle \left\| \phi \left( \frac{(n+1)^2}{n^2} x' \right) \right\|_{L^1} \left\| \mathcal{F}_k(\phi) \left( \frac{\xi}{n^{1/2}}, x' \right) \right\|_{L^1} d\xi' dx'.
\]

It is clear that the last expression tends to zero as \( n \) tends to infinity. Further we have
\[
|2| \leq \frac{|c(n, t)|}{n^{1/2}} \int_{\mathbb{R}^3} \left\langle \frac{\xi}{n^{1/2}}, x' \right\rangle \left\| \mathcal{F}_k(\phi) \left( \frac{\xi}{n^{1/2}}, x' \right) \right\|_{L^1} \left\| \mathcal{F}_k(\phi) \left( \frac{\xi}{n^{1/2}}, x' \right) \right\|_{L^1}^{1/2} d\xi' dx'.
\]

(53)
where we used that $\chi \phi \in L^2(\mathbb{R}^3)$, which follows from the decay properties of the solitary waves. (See [7].) Hence we obtain that $|(2)|$ tends to zero as $n$ tends to infinity. The term $|(3)|$ can be estimated in a similar fashion. Hence

$$\lim_{n \to \infty} (u_{n+1}^2(t, .), u_n^2(t, .))_s = 0.$$ Moreover

$$\lim_{n \to \infty} \|u_{n+1}^2(t, .) - u_n^2(t, .)\|^2_{H^s,0} = 2c_s \|\mathcal{F}_2(\phi)(0, .)\|^2_{L^2}.$$ Therefore we obtain that (2) is locally ill-posed in $H^{s+\alpha}(\mathbb{R}^3)$ for $\beta > 0$ and $s = 0$, when $s < -1/2$.

- Let $d = 3, \beta = 0$ and $s = -1/2$. Then (2) has a solitary wave solution of the form

$$u_s(t, x, y, z) = \phi_s(x - ct, y, z),$$

where

$$u_s(0, x, y, z) = \phi_s(x, y) = c\phi(c^{1/2}x, cy, cz).$$ We have that

$$\phi_s'(\xi, \eta) = e^{-3/2} \phi\left(\frac{\xi}{c^{1/2}}, \frac{\eta}{c}\right).$$ Further we have

$$\|\phi_s\|^2_{\dot{H}^{s_1, s_2}} = e^{s_1 + 2s_2 - 1/2} \int_{\mathbb{R}^3} |\xi|^{2s_1} |\eta|^{2s_2} |\phi(\xi, \eta)|^2 d\xi d\eta = e^{s_1 + 2s_2 - 1/2} \|\phi\|^2_{\dot{H}^{s_1, s_2}}.$$ Hence when $s_1 + 2s_2 = 1/2$, $s_1 \geq 0$, $s_2 \geq 0$ we have that $\|\phi_s\|_{\dot{H}^{s_1, s_2}} = \|\phi\|_{\dot{H}^{s_1, s_2}}$. Similarly we obtain

$$(\phi_{c_1}, \phi_{c_2})_{s_1, s_2} = \left(\frac{c_2}{c_1}\right)^{-3/2} \int_{\mathbb{R}^3} |\xi|^{2s_1} |\eta|^{2s_2} \phi(\xi, \eta) \phi\left(\frac{c_1}{c_2} \xi, \frac{c_1}{c_2} \eta\right) d\xi d\eta,$$ where now $(..)_{s_1, s_2}$ stays for the $\dot{H}^{s_1, s_2}$ scalar product. Hence using Lebesgue theorem we obtain

$$\lim_{c_1, c_2 \to \infty, c_1/c_2 \to 1} (\phi_{c_1}, \phi_{c_2})_{s_1, s_2} = \|\phi\|^2_{\dot{H}^{s_1, s_2}}.$$
and
\[ \lim_{\varepsilon_1, \varepsilon_2 \to \infty, \varepsilon_1/\varepsilon_2 \to 1} \| \phi_{\varepsilon_1} - \phi_{\varepsilon_2} \|_{H^{1/2}} = 0. \]

Now we have that
\[ \hat{u}_c(t, \xi, \eta) = e^{-3/2} \exp(-itc\xi) \hat{\phi}\left( \frac{\xi}{c^{3/4}}, \frac{\eta}{c^{3/4}} \right), \]
and furthermore
\[ (u_{c_1}(t, .), u_{c_2}(t, .))_{s_1, s_2} = \left( \frac{c_2}{c_1} \right)^{-3/2} \int_{\mathbb{R}^2} \exp(-it\xi(c_1 - c_2)) |\xi|^{2s_1} |\eta|^{2s_2} \hat{\phi}(\xi, \eta) \]
\[ \times \hat{\phi}\left( \frac{c_1}{c_2}, \frac{\xi}{c_1}, \frac{\eta}{c_2} \right) d\xi d\eta. \]

Hence using Riemann–Lebesgue lemma arguments we obtain as above
\[ \lim_{n \to \infty} (u_{(n+1)c}(t, .), u_{nc}(t, .))_{s_1, s_2} = 0. \]

Moreover
\[ \lim_{n \to \infty} \| u_{(n+1)c}(t, .) - u_{nc}(t, .) \|_{H^{1/2}} = \| \phi \|_{H^{1/2}}^2. \]

Therefore we obtain that (1) is locally ill-posed in \( H^{s_1, s_2}(\mathbb{R}^2) \) for \( \beta = 0 \) and \( \alpha < 0 \), when \( s_1 + 2s_2 = 1/2 \) and \( s_1 \geq 0, s_2 \geq 0 \). This completes the proof of Theorem 3.

There are some difficulties to extend the results of Theorem 3 to the two dimensional case with higher order term. If \( d = 2, \beta > 0 \) and \( \alpha = 0 \) then (1) has a solitary wave solution of the form
\[ u_c(t, x, y) = \phi_c(x - ct, y), \quad c > 0, \]
where
\[ u_c(0, x, y) = \phi_c(x, y) = c\phi(e^{1/4}x, e^{3/4}y) := c\phi(e^{1/4}x, e^{3/4}y). \]

One has
\[ \hat{\phi}_c(\xi, \eta) = \hat{\phi}\left( \frac{\xi}{c^{3/4}}, \frac{\eta}{c^{3/4}} \right). \]
and furthermore
\[
\| \phi_c \|_{H^{n-1}, \xi}^2 = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \phi \left( \frac{\xi}{c^{n/4}}, \frac{\eta}{c^{3/4}} \right) d\xi \right)^2 d\eta. 
\]  
(24)

It is easy to see (cf. [7]) that \( \phi \) does not belong to \( L^1_{x,y} \). Hence one can not define the trace of \( \phi(\xi, \eta) \) at zero and therefore it is impossible to pass to the limit as \( c \) tends to infinity in (24).

We conclude this section with some remarks for the generalized KP-I equation
\[
\begin{aligned}
(u_t + xu_{xxx} + \beta u_{xxxxx} + u^p u_x)_x + u_{yy} + u_{xx} &= 0 \\
0(0, x, y, z) &= \phi(x, y, z).
\end{aligned}
\]  
(25)

If \( \beta = 0 \) and \( u(t, x, y, z) \) is a solution of (25) then so is
\[
u(t, x, y, z) = \lambda^{2/p} u(\lambda^3 t, \lambda x, \lambda^3 y, \lambda^3 z)
\]
and
\[
\| u(t, .) \|_{H^{n-1}, \xi} = \lambda^{n + 2s_2 -(4p-4)/2p} \| u(\lambda^3 t, .) \|_{H^{n-1}, \xi}.
\]

In [6] it is shown that for \( 1 \leq p < 4/3 \) and \( \alpha < 0 \) (25) possesses nontrivial localized solitary waves of the form
\[
u(t, x, y, z) = \phi_\infty(x - ct, y, z), \quad c > 0,
\]
where \( \phi_\infty(x, y, z) = \epsilon^{1/p} \phi(\epsilon^{1/2} x, \epsilon y, \epsilon^3 z) \). Using the arguments of the proof of Theorem 3 one can prove that (25) is locally ill-posed in \( H^{n-1} \) for \( s_1 + 2s_2 - (5p - 4)/2p = 0, s_1 > 0, s_2 \geq 0 \). If \( \alpha = 0 \) and \( u(t, x, y, z) \) is a solution of (25) then so is
\[
u(t, x, y, z) = \lambda^{4/p} u(\lambda^3 t, \lambda x, \lambda^3 y, \lambda^3 z)
\]
and
\[
\| u(t, .) \|_{H^{n-1}, \xi} = \lambda^{n + 3s_2 + (8 - 7p)/2p} \| u(\lambda^3 t, .) \|_{H^{n-1}, \xi}.
\]

In [6] it is shown that for \( 1 \leq p < 8/3 \) and \( \beta > 0 \) (25) possesses nontrivial localized solitary waves of the form
\[
u(t, x, y, z) = \phi_\infty(x - ct, y, z), \quad c > 0,
\]
where \( \phi_\infty(x, y, z) = \epsilon^{1/p} \phi(\epsilon^{1/4} x, \epsilon^{3/4} y, \epsilon^{3/4} z) \). Using the arguments of the proof of Theorem 3 one can prove that (25) is locally ill-posed in \( H^{n-1} \) for
\[ s_1 + 2s_2 + (8 - 7p)/2p = 0, \quad s_1 \geq 0, \quad s_2 \geq 0. \] In particular when \( p = 8/7 \) (25) is locally ill-posed in \( L^2 \).

Finally we note that when \( x < 0 \) and \( \beta > 0 \), the existence of solitary waves is known for a suitable range of \( p \)'s (cf. [6]) but the scaling argument leading to ill-posedness does not work.

6. A REMARK ON THE (GENERALIZED) KDV EQUATION

In this section we shall show that the example providing ill-posedness for the (generalized) KdV equations uses just scaling properties and hence the special form of the solitary waves is not needed. Consider the (generalized) KdV equation

\[ u_t + uu_{xx} + u^pu_x = 0. \tag{26} \]

The local ill-posedness of (26) is studied in [4, 5]. It is known that (26) possesses solitary wave solutions of the form

\[ u_{c, p}(t, x) = \phi_{c, p}(x - ct), \]

where

\[ u_{c, p}(0, x) = c^{1/p} \phi_p(c^{1/2}x), \quad \phi_p(x) = \left( \frac{1}{2}(p + 2) \text{sech}^2 \left( \frac{x}{2} \right) \right)^{1/p}. \]

Since the solitary waves of (26) decays exponentially at infinity one can define the trace of its Fourier transforms at zero and then prove local ill-posedness of (26) for \( p = 2 \) in \( H^s, \quad s < -1/2 \) which is the result of [5]. Actually if \( p = 2 \) then \( \phi_{c, 2}(\xi) = \hat{\phi}_2(\xi/c^{1/2}) \) and one obtains that for \( s < -1/2 \)

\[ \lim_{c \to \infty} \| \phi_{c, 2} \|_{H^s} = \lim_{c \to \infty} \int \langle \xi \rangle^s \hat{\phi}_2(\xi/c^{1/2}) d\xi = c_s \hat{\phi}_2(0) \]

and therefore one can easily obtain

\[ \lim_{c_1, c_2 \to \infty, c_1/c_2 \to 1} (\phi_{c_1}, \phi_{c_2})_2 = c_s \hat{\phi}_2(0) \]

and moreover

\[ \lim_{c_1, c_2 \to \infty, c_1/c_2 \to 1} \| \phi_{c_1, 2} - \phi_{c_2, 2} \|_{H^s} = 0. \]

Further we have

\[ \mathcal{F}(u_{c, 2})(t, \xi) = \exp\left( -itc_2 \xi \right) \hat{\phi}_2(\xi/c^{1/2}). \]
Due to the Riemann–Lebesgue lemma, the presence of the oscillatory term \( \exp(-itc \xi) \) makes the expression \((u_{c_1}, z(t, \cdot), u_{c_2}, z(t, \cdot))\) tend to zero as \( n \to \infty \) by taking \( c_1 = n^2 \) and \( c_2 = (n + 1)^2 \). Therefore we obtain that (26) is locally ill-posed in \( H^s \) for \( p = 2 \) when \( s < -1/2 \). In fact one can easily show that the sequence \( \phi_{c_1, 2} \) converge strongly in \( H^s \), \( s < -1/2 \) to \( \sqrt{2} \pi \delta \) as \( c \) tends to infinity. Here \( \delta \) stays for Dirac delta function. Similarly we can see that \( u_{c_1}(t, ...) \) converge weakly to zero in \( H^s \). Now the boundedness of \( \|\phi_c\|_{H^s} \), \( s < -1/2 \) provides the local ill-posedness.

Similarly when \( p = 1 \) (the “usual” KdV equation) one can prove that \( \phi_{c_1}(x) \) converge weakly in \( H^s \), \( s < -3/2 \) to \( \sqrt{2} \pi \delta(x) \) as \( c \) tends to infinity. We also have that \( u_{c_1}(t, ...) \) converge weakly to zero as \( c \) tends to infinity. We also have that

\[
\|u_{c_1}(t, \cdot)\|_{H^s} = \|\phi_{c_1}\|_{H^s} = c^{1/2} \|\phi_1\|_{H^s}.
\]

Since \( \partial_s \delta(x) \in H^s \), \( s < -3/2 \) the last arguments strongly suggest local ill-posedness of (26) in \( H^s \), \( s < -3/2 \) when \( p = 1 \).

However, as was pointed out to us by the referee, J. Bourgain has shown in [11] that if one requires that the map data solution \( u_0 \mapsto u(t) \) be smooth then the results of Kenig–Ponce–Vega [17] for the KdV (\( p = 1, s \gg 3/4 \)) or the modified KdV (\( p = 2, s \gg 1/4 \)) are optimal.

7. APPENDIX

In this appendix we shall give the proof of Lemma 7.

**Proof of Lemma 7.** By scaling it suffices to prove that \( |D_x|^s (x \phi) \in L^2(\mathbb{R}^2) \), where

\[
\phi(x, y) = \frac{1 - x^2 + y^2}{(1 + x^2 + y^2)^2}.
\]

We have

\[
\phi(x, y) = \frac{1}{1 + x^2 + y^2} - \frac{2x^2}{(1 + x^2 + y^2)^2} := \phi_1(x, y) + \phi_2(x, y).
\]

A simple computation shows that

\[
\hat{\phi}_1(\zeta, y) := \mathcal{F}_x(\phi_1)(\zeta, y) = \frac{1}{(1 + y^2)^{1/2}} \int_{-\infty}^{\infty} e^{-it(1 + y^2)^{1/2} s\zeta} dx
\]

\[
= \frac{1}{(1 + y^2)^{1/2}} \int_{-\infty}^{\infty} e^{-(1 + y^2)^{1/2} |t|} dt
\]
Hence

\[ \frac{\partial \phi_1}{\partial \xi} (\xi, y) = -\text{sgn} \xi \cdot e^{-(1 + y^2)^{1/2} |\xi|} \]

and

\[ \int_{\mathbb{R}^2} |\xi|^{2\alpha} \left| \frac{\partial \phi_1}{\partial \xi} (\xi, y) \right|^2 \, d\xi \, dy = \int_{\mathbb{R}^2} |\xi|^{2\alpha} e^{-2(1 + y^2)^{1/2} |\xi|} \, d\xi \, dy \]

\[ = \int_{\mathbb{R}^2} \frac{|\xi|^{2\alpha}}{(1 + |\xi|)^{1/2 + z}} e^{-2 |\xi|} \, dX \, dy < \infty, \]

since \( \epsilon > 0 \). Let now \( \psi(x, y) = -\frac{1}{2} \phi_3(x, y) \). A simple calculation yields

\[ \hat{\psi}(\xi, y) := \mathcal{F}(\psi)(\xi, y) = \frac{1}{(1 + y^2)^{1/2}} \int_{\mathbb{R}^2} \frac{x^2 e^{-n(1 + y^2)^{1/2} |\xi|}}{(1 + x^2)^2} \, dx. \]

Write \( x^2/(1 + x^2)^2 = (1/1 + x^2) - (1/(1 + x^2)^2) \) and recall that (cf. [1])

\[ \mathcal{F} \left( \frac{1}{(1 + x^2)^{1/2}} \right)(\xi) = e^{-|\xi|} \]

and

\[ \mathcal{F} \left( \frac{1}{(1 + x^2)^{1/2}} \right)(\xi) = c |\xi|^{3/2} K_{3/2}(|\xi|), \]

where \( K_{3/2} \) is the modified Bessel function of order 3/2 (cf. [1])

\[ K_{3/2}(z) = \frac{\sqrt{\pi}}{2z} e^{-z} \left( 1 + \frac{1}{z} \right). \]

Therefore

\[ \\
\hat{\psi}(\xi, y) = \frac{1}{(1 + y^2)^{1/2}} e^{-(1 + y^2)^{1/2} |\xi|} + \frac{1}{(1 + y^2)^{1/2}} \mathcal{F} \left( \frac{1}{(1 + x^2)^{1/2}} \right) (1 + y^2)^{1/2} \xi \]

\[ = \frac{1}{(1 + y^2)^{1/2}} e^{-(1 + y^2)^{1/2} |\xi|} - \frac{\sqrt{\pi}}{2} \frac{c}{(1 + y^2)^{1/2}} (1 + |\xi|) (1 + y^2)^{1/2} e^{-(1 + y^2)^{1/2} |\xi|} \]

\[ = \frac{1 - c}{(1 + y^2)^{1/2}} e^{-(1 + y^2)^{1/2} |\xi|} - c \sqrt{\pi} \frac{|\xi|}{(1 + y^2)^{1/2}} e^{-(1 + y^2)^{1/2} |\xi|} \]

\[ := \hat{\psi}_1(\xi, y) + \hat{\psi}_2(\xi, y). \]
The term $\hat{\psi}_1(\zeta, y)$ leads to a computation similar to that of $\phi_1$. Now

$$\frac{\partial \hat{\psi}_1}{\partial \zeta}(\zeta, y) = -c \sqrt{\frac{\pi}{2}} \text{sgn} \zeta \cdot (e^{-(1 + y^2)^{1/2} |\zeta|} - |\zeta| (1 + y^2)^{1/2} e^{-(1 + y^2)^{1/2} |\zeta|})$$

$$:= \hat{\psi}_{21}(\zeta, y) + \hat{\psi}_{22}(\zeta, y).$$

Now we have that

$$\int_{\mathbb{R}^2} |\xi|^2 |\hat{\psi}_{21}(\zeta, y)|^2 \, d\zeta \, dy = c \int_{\mathbb{R}^2} |\xi|^2 e^{-2(1 + y^2)^{1/2} |\zeta|} |\zeta| \, d\zeta \, dy$$

$$= c \int_{\mathbb{R}^2} \frac{|X|^{2n}}{(1 + y^2)^{1/2} |\zeta|} e^{-2 |X|} \, dX \, dy < \infty,$$

since $c > 0$ and furthermore

$$\int_{\mathbb{R}^2} |\xi|^2 |\hat{\psi}_{22}(\zeta, y)|^2 \, d\zeta \, dy = c \int_{\mathbb{R}^2} |\xi|^{1+2n} (1 + y^2)^{1/2} e^{-2(1 + y^2)^{1/2} |\zeta|} \, d\zeta \, dy$$

$$= c \int_{\mathbb{R}^2} \frac{|X|^{1+2n}}{(1 + y^2)^{1/2} |\zeta|} e^{-2 |X|} \, dX \, dy < \infty.$$

This completes the proof.

**Remark.** We conjecture that the result of Lemma 7 is valid for any localized solitary wave of KP-I type equation.

**REFERENCES**


