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27 September 2001

PHYSICS LETTERS B

Physics Letters B 517 (2001) 158–166

www.elsevier.com/locate/npe

Parameters' domain in three flavour neutrino oscillations

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Received 28 June 2001; accepted 30 July 2001
Editor: P.V. Landshoff

Abstract

We consider analytically the domain of the three mixing angles Θ_{ij} and the CP phase δ for three flavour neutrino oscillations both in vacuum and matter. Similarly to the quark sector, it is necessary and sufficient to let all the mixing angles $\Theta_{12}, \Theta_{13}, \Theta_{23}$ and δ be in the range $\langle 0, \frac{\pi}{2} \rangle$ and $0 \leqslant \delta < 2\pi$, respectively. To exploit the full range of δ will be important in future when more precise fits are possible, even without CP violation measurements. With the above assumption on the angles we can restrict ourselves to the natural order of masses $m_1 < m_2 < m_3$. Considerations of the mass schemes with some negative δm^2's, though for some reasons useful, are not necessary from the point of view of neutrino oscillation parametrization and cause double counting only. These conclusions are independent of matter effects. © 2001 Published by Elsevier Science B.V.
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1. Introduction

Three flavour neutrino oscillations are considered as a reliable mechanism to explain atmospheric and solar neutrino anomalies. The neutrino flavour eigenstates $\nu_\alpha = (\nu_e, \nu_\mu, \nu_\tau)$ are assumed to be combinations of mass eigenstates $\nu_i = (\nu_1, \nu_2, \nu_3)$

$$\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i} \nu_i . \tag{1}$$

Various parametrizations of the mixing matrix U are possible for Dirac and Majorana neutrinos. All of them use three mixing angles Θ_{ij} $(ij = 12, 13, 23)$ and one (Dirac) or three (Majorana) CP phases. As the neutrino oscillation experiments are not sensitive to Majorana CP phases the same mixing matrix U as in the quark sector [1] is adopted (see, e.g., [2] for discussion of various parametrizations)

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{2}$$

where $c(s)_{ij} \equiv \cos(\sin)\Theta_{ij}$.

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PII: S0370-2693(01)00962-5

The mixing angles Θ_{ij} can be defined to lie in the first quadrant by appropriately adjusting the neutrino and charged lepton phases, analogously to the quark sector [3]. To exhaust the full parameter space the CP phase δ must be taken in the range $0 \leqslant \delta \leqslant 2\pi$. There is only one important difference between quark and neutrino sectors: alignment of absolute neutrino masses is unknown and, among others, normal and inverse neutrino mass hierarchy schemes are considered. It is also true that neutrino oscillations in vacuum will never be able to distinguish these two schemes. The argument is that in vacuum, without CP violation measurements, the oscillation probability depends only on $\sin^2\left(\delta m_{ij}^2 \frac{L}{4E}\right)$, and the sign of δm_{ij}^2, which decide about the mass scheme, is unmeasurable. Neutrino oscillations in matter would only give the chance to measure the sign of δm_{ij}^2. We would like to clarify the notion of using δm_{ij}^2 signs for neutrinos mixing parametrization [4]. We find the full domain of the three mixing angles Θ_{ij} and δ phase in the mixing matrix U in matter. It appears that parameter space in the matter case is exactly the same as in vacuum

$$0 \leqslant \Theta_{ij} \leqslant \frac{\pi}{2}, \qquad 0 \leqslant \delta < 2\pi. \tag{3}$$

In addition, in matter, it is not necessary to consider mass schemes with different mass arrangements $\delta m_{ij}^2 = \pm|\delta m_{ij}{}^2|$. With the full range of parameters, it is enough to include only the “canonical” order of masses $(m_1 < m_2 < m_3)$. All other mass schemes with negative δm_{ij}^2 are equivalent to that with $\delta m_{21}^2 > 0$ and $\delta m_{32}^2 > 0$ and a different region in the parameter space of Eq. (3). Finally we argue that exploring δ in its full range will be important in future experiments and not necessarily in connection with the explicit measurements of CP violation effects.

In Section 2, in a simple analysis we find the domain of parameters for neutrino oscillation in vacuum. Even if the range Eq. (3) is well known from the quark sector, we discuss it as it is a suitable introduction to understand the more complicated case of neutrino oscillations in matter. This is presented in Section 3. Finally, the conclusions are gathered in Section 4.

2. Parameter space for neutrino oscillations in vacuum

The vacuum neutrino flavour oscillation probability for an initially produced ν_α with an energy E converted into detected ν_β after traveling a distance L is given by

$$P_{\nu_\alpha \to \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{a>b} R_{\alpha\beta}^{ab} \sin^2 \Delta_{ab} - Y \sum_{\gamma} \epsilon_{\alpha\beta\gamma}, \tag{4}$$

where

$$R_{\alpha\beta}^{ab} = \mathrm{Re}\left[W_{\alpha\beta}^{ab}\right], \tag{5}$$

$$Y = 8\,\mathrm{Im}\left[W_{e\mu}^{12}\right] \sin\Delta_{21} \sin\Delta_{31} \sin\Delta_{32}, \tag{6}$$

$$\Delta_{ab} = 1.27\, \delta m_{ab}^2 \text{ [eV]} \frac{L \text{ [km]}}{E \text{ [GeV]}}, \tag{7}$$

and

$$\delta m_{ab}^2 = m_a^2 - m_b^2, \qquad W_{\alpha\beta}^{ab} = U_{\alpha a} U_{\beta b} U_{\alpha b}^* U_{\beta a}^*.$$

We can see that the mixing matrix elements $U_{\alpha a}$ enter the oscillation probability by $W_{\alpha\beta}^{ab}$ tensors which are invariant under the phase transformation

$$U_{\gamma c} \to e^{-i\delta_\gamma} U_{\gamma c} e^{i\eta_c}. \tag{8}$$

Freedom of this transformation can be used to show that all mixing angles Θ_{ij} originally belonging to the interval $\langle 0, 2\pi)$ can be mapped onto the first quadrant $\Theta_{ij} \in \langle 0, \frac{\pi}{2}\rangle$. As the real and the imaginary parts of the phase factor $e^{i\delta}$ are allowed to change sign, the appropriate interval for δ is $\langle 0, 2\pi)$.

Now we show in a way which will be useful in the more complicated case of neutrino oscillations in matter, that in fact the domain from Eq. (3) covers the full parameter space of possible neutrino transitions. First of all, from unitarity of the U matrix follows that all $R^{ab}_{\alpha\beta}$ tensors can be expressed by squares of moduli of the U matrix elements

$$R^{ab}_{\alpha\beta} = \frac{1}{2}\big(|U_{\gamma a}|^2|U_{\gamma b}|^2 - |U_{\alpha a}|^2|U_{\alpha b}|^2 - |U_{\beta a}|^2|U_{\beta b}|^2\big), \tag{9}$$

for $\alpha \neq \beta \neq \gamma$, $a \neq b$, and

$$R^{ab}_{\alpha\alpha} = |U_{\alpha a}|^2|U_{\alpha b}|^2, \qquad R^{aa}_{\alpha\beta} = |U_{\alpha a}|^2|U_{\beta a}|^2, \tag{10}$$

otherwise.

The $|U_{ea}|^2$ for $a = 1, 2, 3$ and $|U_{\alpha 3}|^2$ for $\alpha = \mu, \tau$ depend only on sine and cosine squares of Θ_{ij} and do not feel the transformations among the four quadrants. Only $|U_{\alpha a}|^2$'s for $\alpha = \mu, \tau$ and $a = 1, 2$ depend linearly on sines and cosines of Θ_{ij} angles, namely

$$|U_{\alpha a}|^2 = K_{\alpha a} \pm S, \tag{11}$$

where $K_{\alpha a}$ are still functions of $\cos^2\Theta_{ij}$ and $\sin^2\Theta_{ij}$, but

$$S = \frac{1}{2}F\cos\delta, \tag{12}$$

$$F = \sin 2\Theta_{12}\sin 2\Theta_{23}\sin\Theta_{13}. \tag{13}$$

Exactly the same factor F appears in the Jarlskog invariant (Eq. (6)) [5]

$$J \equiv \mathrm{Im}\big[W^{12}_{e\mu}\big] = \frac{1}{4}F\cos^2\Theta_{13}\sin\delta. \tag{14}$$

Only the F factor is sensitive to the change of sign when the angles Θ_{ij} are mapped from the full domain $\langle 0, 2\pi)$ to the final range $\langle 0, \frac{\pi}{2}\rangle$ ($n_{ij} = 0, \ldots, 3$)

$$F\left(\Theta_{12} + n_{12}\frac{\pi}{2}; \Theta_{13} + n_{13}\frac{\pi}{2}; \Theta_{23} + n_{23}\frac{\pi}{2}\right)\bigg|_{0\leqslant\Theta_{ij}\leqslant\frac{\pi}{2}} = (-)^{n_{12}+n_{23}} f(n_{13}) F(\Theta_{12}; \Theta_{13}; \Theta_{23}), \tag{15}$$

where

$$f(n_{13}) = \begin{cases} +1, & \text{for } n_{13} = 0, 1, \\ -1, & \text{for } n_{13} = 2, 3. \end{cases}$$

In order to compensate the possible change of signs in Eq. (15) other factors in Eqs. (6), (14) and (12) must have the freedom to change sign. The only possible choices are the CP phase δ and the combination Δ in Eq. (6) defined as

$$\Delta = \sin\Delta_{21}\sin\Delta_{31}\sin\Delta_{32}. \tag{16}$$

There are two possibilities.

- If $\delta \in \langle 0, 2\pi)$ then a change of sign by $\sin\delta$ in Eq. (14) and $\cos\delta$ in Eq. (12) compensates the sign in Eq. (15). In this case the order of masses can be kept canonical, $m_1 < m_2 < m_3$.
- If $\delta \in \langle 0, \pi)$ then $\cos\delta$ is able to compensate the sign in Eq. (12), but $\sin\delta > 0$, so Δ must be used in the CP violating Y quantity (Eq. (6)). In such a case, schemes with $\Delta > 0$ ([123], [231], [312]) are distinguishable from schemes with $\Delta < 0$ ([132], [321], [213]) in oscillation appearance experiments (see Fig. 1 for notation).

3 2 1	[123]	$\delta m^2_{21} > 0$ $\delta m^2_{32} > 0$ $\delta m^2_{31} > 0$	$\Delta > 0$
2 1 3	[312]	$\delta m^2_{21} > 0$ $\delta m^2_{32} < 0$ $\delta m^2_{31} < 0$	$\Delta > 0$
1 3 2	[231]	$\delta m^2_{21} < 0$ $\delta m^2_{32} > 0$ $\delta m^2_{31} < 0$	$\Delta > 0$
1 2 3	[321]	$\delta m^2_{21} < 0$ $\delta m^2_{32} < 0$ $\delta m^2_{31} < 0$	$\Delta < 0$
3 1 2	[213]	$\delta m^2_{21} < 0$ $\delta m^2_{32} > 0$ $\delta m^2_{31} > 0$	$\Delta < 0$
2 3 1	[132]	$\delta m^2_{21} > 0$ $\delta m^2_{32} < 0$ $\delta m^2_{31} > 0$	$\Delta < 0$

Fig. 1. Possible configurations of neutrino masses. The first one is called canonical. The first two ([123], [312]) are usually discussed in literature. All schemes are completely equivalent since marking neutrinos with numbers has no physical meaning. It is not important how a neutrino with number "i" couples to the α flavour. It is only meaningful how neutrinos of different masses couple to the α flavour.

We see that the chosen Θ_{ij} and δ angles given in Eq. (3) exhaust the full parameter space. We can bind the CP violating phase to the smaller range

$$(0 \leqslant \delta \leqslant \pi) \tag{17}$$

and in the same time distinguish the neutrino mass schemes with $\Delta > 0$ (cyclic mass permutations from the canonical case) from $\Delta < 0$ cases (noncyclic mass permutations of the canonical scheme). It is impossible to disentangle schemes inside these two groups. Therefore, an approach with the canonical order of masses is clearer from the point of view of neutrino oscillation parametrization: a point (region) in the parameter space of Eq. (3) determines the scheme of masses and the mixture of the weak states in an unambiguously way. We will turn back to the interpretation of δm^2_{ij} signs in the next section.

Presently, as statistical errors are large any subleading effects in neutrino oscillations are neglected and experimental data for neutrino (disappearance) oscillations are fitted by the formula where only sine and/or cosine squares of the Θ_{ij} mixing angles appear. Therefore, we do not have to explore the full parameter space in Eq. (3). However, if the future precision improves and subleading effects are measured then it may be necessary to do it. Now we show an example where taking into account the δ phase is important even if CP violation is not measured. Let us consider atmospheric $\nu_\mu \to \nu_\mu$ disappearance probability in vacuum

$$\begin{aligned} P_{\mu\mu} = 1 - 4\big[&\big(K_{\mu 2}K_{\mu 1} - S^2\big)\sin^2\Delta_{21} + |U_{\mu 3}|^2\big(K_{\mu 1}\sin^2\Delta_{31} - K_{\mu 2}\sin^2\Delta_{32}\big)\big] \\ &- 4S\big\{(K_{\mu 2} - K_{\mu 1})\sin^2\Delta_{21} + |U_{\mu 3}|^2\big(\sin^2\Delta_{31} - \sin^2\Delta_{32}\big)\big\}, \end{aligned} \tag{18}$$

where $K_{\mu i}$ $(i = 1, 2)$ are defined in Eq. (11). We can see that there is a part proportional to S which exists only if $\Delta_{21} \neq 0 \Leftrightarrow \Delta_{31} \neq \Delta_{32}$. In Fig. 2 $P_{\mu\mu}$ as function of L/E is given for $\delta m^2_{21} \equiv \delta m^2_{\rm sol} = 2.5 \times 10^{-4}$ eV^2,

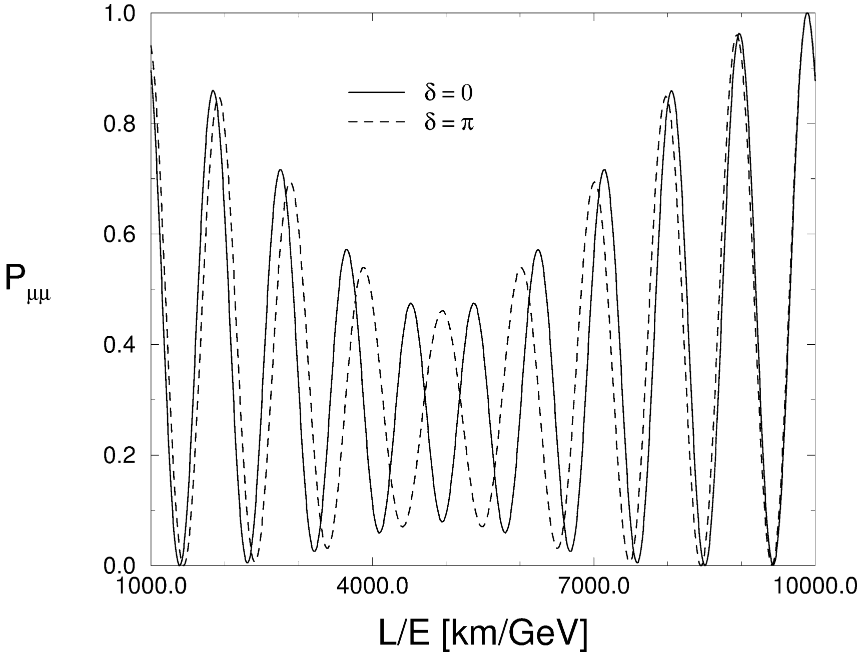

Fig. 2. The effect of the CP-violating phase in the disappearance $\nu_\mu \to \nu_\mu$ transition (Eq. (18)).

$\delta m^2_{31} \equiv \delta m^2_{\mathrm{atm}} = 2.5 \times 10^{-3}\ \mathrm{eV}^2$, $\Theta_{23} = \Theta_{12} = \pi/2$ and $\Theta_{13} = 0.2$. δ is taken to be 0 and π. The difference between $\delta = 0$ and $\delta = \pi$ cases can be easily seen. This difference diminishes with decreasing $\delta m^2_{\mathrm{sol}}$. Taking into account some new results where exploration of large values of $\delta m^2_{\mathrm{sol}}$ (even up to a scale of $\delta m^2_{\mathrm{atm}}$) is discussed seriously [6–9] this effect should be keep in mind when a precise, global analysis of oscillation data is undertaken, especially with incoming neutrino factory physics. Let us note, that only $\delta = 0$ is used presently. Usually it is assumed, that in the CP conservation case it is allowed to take $\delta = 0$ and $0 \leqslant \Theta_{ij} \leqslant \pi/2$. Surprisingly it is not true. If CP is conserved, then the discussion given above implies that mapping the full range $0 \leqslant \Theta_{ij} < 2\pi$ onto $0 \leqslant \Theta_{ij} \leqslant \pi/2$ requires the term $\cos\delta$ in Eq. (12) to have two discrete values ± 1.

3. Parameter space for neutrino oscillations in matter

The probability $P^m_{\nu_\alpha \to \nu_\beta}$ of neutrino oscillations in matter of density N_e is given by the vacuum formula (Eq. (4)) with modified $U_{\alpha a}$, Δ_{ab} and J [10]

$$P^m_{\nu_\alpha \to \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{a>b} \mathrm{Re}\big[W^{(m)ab}_{\alpha\beta}\big] \sin^2 \Delta^m_{ab} - Y^m \sum_{\gamma} \epsilon_{\alpha\beta\gamma}, \tag{19}$$

where

$$W^{(m)ab}_{\alpha\beta} = U^m_{\alpha a} U^m_{\beta b} U^{m*}_{\alpha b} U^{m*}_{\beta a}, \tag{20}$$

$$U^{\mathrm{m}}_{\alpha a} = \frac{N_a}{D_a} U_{\alpha a} + \frac{A}{D_a} U_{ea}\big[\big(\lambda^2_a - m^2_b\big) U^*_{ec} U_{\alpha c} + \big(\lambda^2_a - m^2_c\big) U^*_{eb} U_{\alpha b}\big], \quad \text{with } a \neq b \neq c, \tag{21}$$

$$\Delta^m_{ab} = 1.27 \frac{(\lambda^2_a - \lambda^2_b)\ [\mathrm{eV}]^2 L\ [\mathrm{km}]}{E\ [\mathrm{GeV}]}, \tag{22}$$

$$Y^m = 8J^m \sin\Delta_{21}^m \sin\Delta_{31}^m \sin\Delta_{32}^m, \tag{23}$$

$$\begin{aligned} J^m = J\,&\frac{(\lambda_1^2-m_2^2)(\lambda_1^2-m_3^2)(\lambda_2^2-m_1^2)(\lambda_2^2-m_2^2)}{D_1^2 D_2^2} \\ &\times\Big\{N_1N_2 - N_2A\big(\lambda_1^2-m_2^2\big)|U_{e1}|^2 - N_1A\big(\lambda_2^2-m_1^2\big)|U_{e2}|^2 \\ &\quad + A^2|U_{e1}|^2|U_{e2}|^2\big[\big(\lambda_2^2-m_1^2\big)\big(\lambda_1^2-m_3^2\big)+\big(\lambda_2^2-m_3^2\big)\big(\lambda_1^2-m_2^2\big)-\big(\lambda_2^2-m_3^2\big)\big(\lambda_1^2-m_3^2\big)\big]\Big\}, \end{aligned} \tag{24}$$

$$N_a = \big(\lambda_a^2-m_b^2\big)\big(\lambda_a^2-m_c^2\big) - A\big[\big(\lambda_a^2-m_b^2\big)|U_{ec}|^2+\big(\lambda_a^2-m_c^2\big)|U_{eb}|^2\big], \tag{25}$$

$$D_a^2 = N_a^2 + A^2|U_{ea}|^2\big[\big(\lambda_a^2-m_b^2\big)^2|U_{ec}|^2+\big(\lambda_a^2-m_c^2\big)^2|U_{eb}|^2\big]. \tag{26}$$

λ_a^2 denote the effective mass squares of neutrinos in matter and follow from diagonalization of an effective Hamiltonian

$$\mathcal{H}_\nu = \frac{1}{2E}\left[\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} + U^\dagger\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}U\right]. \tag{27}$$

m_a ($a = 1, 2, 3$) are neutrino masses and $A = 2\sqrt{2}\,EG_FN_e$. Using the Cardano formula we get

$$\begin{aligned} &\lambda_1^2 = -\frac{a_2}{3} - \frac{1}{3}p\big(\cos\phi+\sqrt{3}\sin\phi\big), \qquad \lambda_2^2 = -\frac{a_2}{3} - \frac{1}{3}p\big(\cos\phi-\sqrt{3}\sin\phi\big), \\ &\lambda_3^2 = -\frac{a_2}{3} + \frac{2}{3}p\cos\phi, \end{aligned} \tag{28}$$

where

$$p = \sqrt{a_2^2 - 3a_1}, \qquad \phi = \frac{1}{3}\arccos\left[-\frac{1}{p^3}\left(a_2^3 - \frac{9}{2}a_1a_2 + \frac{27}{2}a_0\right)\right], \tag{29}$$

and

$$\begin{aligned} a_0 &= -m_1^2m_2^2m_3^2 - A\big[m_1^2m_3^2|U_{e2}|^2 + m_1^2m_2^2|U_{e3}|^2 + m_2^2m_3^2|U_{e1}|^2\big], \\ a_1 &= m_2^2m_3^2 + m_1^2m_2^2 + m_1^2m_3^2 + A\big[m_1^2\big(1-|U_{e1}|^2\big) + m_2^2\big(1-|U_{e2}|^2\big) + m_3^2\big(1-|U_{e3}|^2\big)\big], \\ a_2 &= -\big(m_1^2+m_2^2+m_3^2+A\big). \end{aligned} \tag{30}$$

Now we can proceed as in the vacuum case and consider $P^m_{\nu_\alpha\to\nu_\beta}(\Theta_{ij}, \delta, \delta m_{ij}^2)$ in the full range of parameters. Similarly to the vacuum case the real parts of $W_{\alpha\beta}^{(m)ab}$ depend on $|U_{\delta c}^m|^2$. These subsequently depend on vacuum mixing matrix elements

$$\begin{aligned} \big|U_{\alpha a}^m\big|^2 = \frac{N_a^2}{D_a^2}|U_{\alpha a}|^2 &+ 2\frac{AN_a}{D_a^2}\big\{\big(\lambda_a^2-m_b^2\big)R_{e\alpha}^{ac} + \big(\lambda_a^2-m_c^2\big)R_{e\alpha}^{ab}\big\} \\ &+ \frac{A^2}{D_a^2}|U_{e\alpha}|^2\Big\{\big(\lambda_a^2-m_b^2\big)^2|U_{ec}|^2|U_{\alpha c}|^2 + \big(\lambda_a^2-m_c^2\big)^2|U_{eb}|^2|U_{\alpha b}|^2 \\ &\qquad + 2\big(\lambda_a^2-m_b^2\big)\big(\lambda_a^2-m_b^2\big)R_{e\alpha}^{bc}\Big\}. \end{aligned} \tag{31}$$

We can see that the mixing angles appear in the squared moduli $|U_{\gamma c}|^2$ and inside the R tensors (Eqs. (9), (10)) which also depend on $|U_{\gamma c}|^2$. So, as in the vacuum case, when the full domain $\langle 0, 2\pi)$ is mapped onto $\langle 0, \pi/2\rangle$, nontrivial signs appear only in the F factor (Eq. (13)). Thus again, the change of signs can be compensated by $\cos\delta$ in Eq. (12).

In the CP violating part (Eq. (23)) the vacuum mixing angles are found in the squared moduli of $|U_{ei}|^2$ and J (see Eq. (24)). Again, only the F factor in J (Eqs. (13), (14)) changes sign if the Θ_{ij} angles are reduced to the first quadrant. If $\delta \in \langle 0, 2\pi\rangle$ then $\sin\delta$ term in Eq. (14) is able to compensate the change of sign in F. The mixing angles are also present in the effective neutrino masses λ_a (Eq. (28)). However, only $|U_{ei}|^2$ elements appear (Eq. (30)) and angles can be reduced to the first quadrant without changing λ_a. In this way we have proved that the domains of parameters for neutrino oscillations in matter and vacuum are the same (Eq. (3)).

Now we would like to answer the question, whether introducing permutations of masses to the canonical scheme [123] (see Fig. 1) is able to reduce the parameter space both for Θ_{ij} and δ in Eq. (3). Such an approach to the Θ_{ij} angles was common before the "dark side" era [11]. Statements have also appeared that $\delta \in \langle 0, 2\pi\rangle$ can be shrunk to half of this region when negative signs of δm^2_{ij} are taken into account.

Let us start our considerations from the Θ_{ij} angles. In the case of two flavour neutrino oscillations in vacuum the transition probability depends only on $\sin^2 2\Theta \sin^2\left[\delta m^2 \frac{L}{4E}\right]$. Then it is possible to limit the range of mixing angles to the first octant. The transition probability in matter depends on the combination [12]

$$\left(\frac{A}{\delta m^2} - \cos 2\Theta\right)^2. \tag{32}$$

The relative sign between δm^2 and $\cos 2\Theta$ is important, so two possibilities are considered

$$\delta m^2 > 0 \quad \text{and} \quad 0 < \Theta < \frac{\pi}{2} \qquad \text{or} \qquad \delta m^2 = \pm\left|\delta m^2\right| \quad \text{and} \quad 0 < \Theta < \frac{\pi}{4}.$$

In the case of three flavour neutrino oscillations it is impossible to limit the range of Θ_{ij} angles in this way, even in vacuum. Transition probabilities (Eq. (4)) depend not only on the product $\sin^2\Theta_{ij}\cos^2\Theta_{ij}$ but also on $\sin^2\Theta_{ij}$ and $\cos^2\Theta_{ij}$ separately. [1] Since it is impossible to shrink the range of the mixing angles Θ_{ij} in the case of vacuum oscillations, the same will hold true for the matter case. In spite of that, various schemes (Fig. 1) are considered. Let us show that in this way the same angles are used twice when $0 < \Theta_{ij} < \frac{\pi}{2}$.

Neutrino oscillation formulae (Eqs. (28)–(30)) are symmetric under permutation of neutrinos. Traditionally some scalar matrix ($\mathbf{1}\cdot\text{const}$) is removed from the effective Hamiltonian (Eq. (27)) giving the same physical predictions. For example, if H_ν is written in the form

$$M^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_1^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m^2_{21} & 0 \\ 0 & 0 & \delta m^2_3 \end{pmatrix}, \tag{33}$$

then the matrix $\mathbf{1}\cdot m_1^2$ can be absorbed giving a common phase factor for all three neutrino flavours. In such a case we diagonalize the hamiltonian H_ν where

$$m_1^2 \to 0, \qquad m_2^2 \to \delta m^2_{21}, \qquad m_3^2 \to \delta m^2_{31}. \tag{34}$$

The new a_i parameters derived from Eq. (30) are not symmetric under permutations of the masses anymore, they depend on δm^2_{ij}'s, namely,

$$\begin{aligned} a_0 &= -A\delta m^2_{21}\delta m^2_{31}|U_{e1}|^2, \\ a_1 &= \delta m^2_{21}\delta m^2_{31} + A\left[\delta m^2_{21}\left(1 - |U_{e2}|^2\right) + \delta m^2_{31}\left(1 - |U_{e3}|^2\right)\right], \\ a_2 &= -\delta m^2_{21} - \delta m^2_{31} - A. \end{aligned} \tag{35}$$

[1] This statement is general. In the approximation with one dominating δm^2 scale some transition probabilities depend only on $\sin^2 2\Theta_{ij}$. For example, the short-baseline reactor disappearance probability $P_{\nu_e\to\nu_e} = 1 - \sin^2\Theta_{13}\sin^2\Delta_{23}$. However, this approximation seems to be questionable, even for present neutrino data [6] and quite probably a full theoretical framework without neglecting some δm^2 should be used in future.

Let us now calculate the eigenvalues for the case of negative δm^2_{3i}, $i=1,2$, i.e., $\delta m^2_{3i} = -|\delta m^2_{3i}|$. We have

$$
\begin{aligned}
&a_0\big(-\big|\delta m^2_{3i}\big|\big) = A\big|\delta m^2_{21}\big|\big|\delta m^2_{31}\big||U_{e1}|^2,\\
&a_1\big(-\big|\delta m^2_{3i}\big|\big) = -\big|\delta m^2_{21}\big|\big|\delta m^2_{31}\big| + A\big[\big|\delta m^2_{21}\big|\big(1-|U_{e2}|^2\big) - \big|\delta m^2_{31}\big|\big(1-|U_{e3}|^2\big)\big],\\
&a_2\big(-\big|\delta m^2_{3i}\big|\big) = -\big(\big|\delta m^2_{21}\big| - \big|\delta m^2_{31}\big| + A\big).
\end{aligned}
\tag{36}
$$

Using these new parameters $a_i(-|\delta m^2_{3i}|)$ different λ_i^2 eigenvalues are obtained. Are these new $\lambda_i(-|\delta m^2_{3i}|)$ eigenvalues equal to the "canonical" λ_i calculated at some other point of the parameter space of Eq. (3)? To show that they are, let us take the scheme [312]. This scheme (as any other in Fig. 1) is completely equivalent to the canonical one [123]. We have only to change the names of particles $2\to3$, $1\to2$, $3\to1$ or more precisely replace $U_{e1}\to U_{e2}$, $U_{e2}\to U_{e3}$, $U_{e3}\to U_{e1}$, $\delta m^2_{23}\to\delta m^2_{31}$, $\delta m^2_{21}\to\delta m^2_{32}$, $\delta m^2_{13}\to\delta m^2_{21}$.

In the scheme [312], as previously we subtract m_1^2 mass from the M^2 matrix. As now m_3 is the lightest mass, we have to diagonalized H_ν with the following replacements

$$
m_1^2\to 0,\qquad m_2^2\to\big|\delta m^2_{21}\big|,\qquad m_3^2\to-\big|\delta m^2_{31}\big|. \tag{37}
$$

The parameters a_i which we get are exactly the same as given by Eq. (36). Similarly we can check that any replacement $\delta m^2_{ij}\to-\delta m^2_{ij}$ in the canonical parameters a_i ([123]) is equivalent to the others given by one of the six schemes in Fig. 1. In this way we have proved that changing the signs of δm^2_{ij} in the canonical [123] eigenvalues is equivalent to evaluating λ_i^2's at some other point of the parameter space Eq. (3), schematically

$$
\lambda_i^2\big(-\big|\delta m^2_{ij}\big|\big)\sim\lambda_i^2\big([ijk]\big). \tag{38}
$$

We can see that using schemes with various permutations of masses does not confine the domain of the parameter space Θ_{ij} and causes double counting only. However, we can find a practical reason for introducing $\pm\delta m^2$'s. We have just shown that using various schemes is equivalent to using the [123] scheme with different values of Θ_{ij} angles in the parameter space. That is why we can reverse the situation by fixing angles to the same physical situation, i.e., Θ_{12} can be connected with $\pm\delta m^2_{12}$ (oscillation of solar neutrinos), Θ_{23} with $\pm\delta m^2_{23}$ (oscillation of atmospheric neutrinos) and Θ_{13} with reactor neutrino oscillations.

Finally, let us consider the δ phase in the case of matter neutrino oscillations. Can we bound it to the smaller range (Eq. (17)) as in the vacuum case? There is a very elegant relationship between the universal CP-violating parameters J^m and J in matter and in vacuum [13]

$$
J^m\big(\lambda_2^2-\lambda_1^2\big)\big(\lambda_3^2-\lambda_1^2\big)\big(\lambda_3^2-\lambda_2^2\big) = J\big(m_2^2-m_1^2\big)\big(m_3^2-m_1^2\big)\big(m_3^2-m_2^2\big). \tag{39}
$$

From this relation follows that the signs of δm^2_{ij} and $\delta\lambda^2_{ij}$ are correlated. If δm^2_{ij} changes sign, the same happens to $\delta\lambda^2_{ij}$. We conclude that for neutrino oscillations in matter we have exactly the same situation as in the vacuum case. The basic domain of δ is $\langle 0,2\pi)$ and it can be restricted to $\langle 0,\pi\rangle$ and then the schemes with $\Delta>0$ and $\Delta<0$ are distinguishable.

4. Conclusions

We have proved in an analytical way that the ranges of the mixing angles Θ_{ij} and the CP-violating phase δ are the same for three flavour neutrino oscillations in vacuum and in matter: $\Theta_{ij}\in\langle 0,\pi/2\rangle$, $\delta\in\langle 0,2\pi)$. It means that probabilities for three flavour neutrino oscillations can be described by points (more reliably by regions) in this parameters' domain without using the signs of δm^2_{ij} ($\delta m^2_{ij}>0$, $i>j$). Contrary to the case of two neutrino oscillations in matter, the possibility of two signs for each δm^2_{ij} does not restrict further the domain of the Θ_{ij} angles. Even though the signs of δm^2_{ij}'s are not needed, they are useful. Taking into account the signs of δm^2_{ij} we can fix angles Θ_{ij} to a given scale of δm_{ij}^2. The range of the δ CP phase can be confined to $\delta \in \langle 0, \pi)$ but then, only sets of schemes with cyclic ($\Delta > 0$) and odd ($\Delta > 0$) neutrino mass permutations are distinguishable to each other. A simple example has been given that δ can be important even for disappearance neutrino oscillation experiments.

Acknowledgement

We would like to thank M. Czakon for careful reading the manuscript and useful discussion. This work was supported by Polish Committee for Scientific Research under Grants Nos. 2P03B04919 and 2P03B05418.

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