A Gerschgorin Theorem for Linear Difference Equations and Eigenvalues of Matrix Products

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ABSTRACT

The Gerschgorin circle theorem is used here to give sufficient conditions for the solution space of the difference equation $x(m+1)=A(m+1)x(m)$ to admit a type of exponential dichotomy. The result obtained is then used to establish a result on regions of eigenvalue inclusion for the product of finitely many square matrices. An application to differential equations is also given.

1. INTRODUCTION AND SUMMARY

In [2] A. C. Lazer has given sufficient conditions for the solution space of the linear differential equation

$$x'(t)=A(t)x(t) \tag{1}$$

to admit a type of exponential dichotomy. Here $A(t)=(a_{ij}(t))$ denotes an $n\times n$ matrix valued function whose entries are continuous complex functions defined on the real line; and $x(t)=\mathrm{col}(x_1(t),\dots,x_n(t))$ denotes a complex n-column vector. In order to paraphrase Lazer's result we make the following definitions and conventions.

For $x=\mathrm{col}(x_1,\dots,x_n)$ we set

$$\|x\|=\max_{1\leqslant i\leqslant n}|x_i|.$$

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For any $n \times n$ matrix $A=(a_{ij})$ we set

$$r(A,i)=\sum_{\substack{j=1\\ j\neq 1}}^{n} |a_{ij}|.$$

DEFINITION. Let S be a subset of $\Omega=\{1,2,\ldots,n\}$, and let δ be a positive real number. We say that the matrix $A=(a_{ij})$ is (S,δ) *diagonally dominant* if $\mathrm{Re}(a_{ii})+r(A,i)\leqslant -\delta<0$ for each index $i\in S$ and $\mathrm{Re}(a_{ii})-r(A,i)\geqslant \delta>0$ for each index $i\in\Omega\setminus S$.

DEFINITION. Let X denote the solution space of (1) and let δ be a positive real number. We say that X admits a *δ-comparative exponential dichotomy* if X is the direct sum of two subspaces, X_1 and X_2, such that

$$\|x(t_2)\|\leqslant\|x(t_1)\|\exp\delta(t_1-t_2) \qquad \text{if } t_1\leqslant t_2$$

for every solution x in X_1, and

$$\|x(t_2)\|\geqslant\|x(t_1)\|\exp\delta(t_2-t_1) \qquad \text{if } t_1\leqslant t_2$$

for every solution in X_2.

Lazer's result states that if $A(t)$ in (1) is (S,δ) diagonally dominant for each real t, then the solution space of (1) admits a δ-comparative exponential dichotomy, and the *dimension* of X_1 is the same as the *cardinality* of S. Lazer's result was motivated by a result due to S. A. Gerschgorin [1] (which appears in [3]) which states that if $A=(a_{ij})$ is an $n\times n$ complex matrix, then each eigenvalue of A is contained in one of the closed discs

$$|z-a_{ii}|\leqslant r(A,i), \qquad i=1,\ldots,n.$$

The appeal of Lazer's result is that his hypotheses concern only the entries of the coefficient matrix $A(t)$ in (1).

In Sec. 2 of this paper we use techniques similar to those of Gerschgorin and Lazer to give sufficient conditions for the solution space of the linear difference equation

$$x(m+1)=A(m+1)x(m), \qquad m=0,\pm 1,\pm 2,\ldots \tag{2}$$

to admit a type of exponential dichotomy. As in [2] our hypotheses concern only the location of the Gerschgorin discs for $A(m)$.

In Sec. 3 we use the result of Sec. 2 to prove a purely linear algebra result concerning the eigenvalues of the product of a finite number of square matrices.

Finally, in Sec. 4 we give an application to differential equations by showing that any equation of the type (1) which satisfies Lazer's hypotheses gives rise to an equation of the type (2) which satisfies our hypotheses.

2. DIAGONALLY DOMINANT DIFFERENCE EQUATIONS

Throughout this paper we denote the set of integers by Z and observe the following conventions:

$$Z_n = \{n, n+1, \dots\}, \qquad Z^n = \{\dots, n-1, n\}.$$

DEFINITION. We say that the coefficient matrix $A(m)$ in (2) is (S,δ) *diagonally dominant with respect to the unit circle* if S is a subset of Ω and δ is a positive real number such that $|a_{ii}(m)| + r(A(m),i) \leqslant 1-\delta < 1$ for each $i \in S$, and $|a_{ii}(m)| - r(A(m),i) \geqslant 1+\delta > 1$ for each $i \in \Omega \setminus S$. Throughout we assume $A(m)$ to be nonsingular on Z.

THEOREM 1. *Let $A(m)$ in (2) be (S,δ) diagonally dominant with respect to the unit circle for each integer m. Let k be the cardinality of S. Then each of the following statements holds.*

(a) *There exist k independent solutions $y^1, \dots, y^k$ of (2) such that if x is any nontrivial solution of (2) of the form*

$$x(m) = \alpha_1 y^1(m) + \cdots + \alpha_k y^k(m),$$

then

$$\|x(m)\| \leqslant \|x(p)\| (1-\delta)^{m-p} \qquad \text{if } p \leqslant m.$$

(b) *There exist $n-k$ independent solutions $z^1, \dots, z^{n-k}$ of (2) such that if x is any nontrivial solution of (2) of the form*

$$x(m) = \alpha_1 z^1(m) + \cdots + \alpha_{n-k} z^{n-k}(m),$$

then

$$\|x(m)\| \geqslant \|x(p)\| (1+\delta)^{m-p} \qquad \text{if } p \leqslant m$$

(c) *The solutions $y^1, \dots, y^k, z^1, \dots, z^{n-k}$ form a basis for the solution space of (2).*

The proof of Theorem 1 uses four preliminary lemmas.

LEMMA 1. *Under the assumptions of Theorem* 1, *if* x *is a nontrivial solution of* (2) *such that* $\|x(m_0)\| = |x_j(m_0)|$ *for some* $j \in \Omega \setminus S$, *then* $\|x(m)\| = |x_h(m)|$ *with* $h \in \Omega \setminus S$ *for every integer* $m > m_0$. *Furthermore,* $\|x(m)\| \geqslant \|x(m_0)\|(1+\delta)^{m-m_0}$ *whenever* $m > m_0$.

Proof. We have by (2) and our hypotheses that

$$|x_j(m_0+1)| \geqslant |a_{jj}(m_0+1)|\cdot|x_j(m_0)| - \sum_{\substack{k=1\\k\neq j}}^{n} |a_{kj}(m_0+1)|\cdot|x_k(m_0)|$$

$$\geqslant \{|a_{jj}(m_0+1)| - r(A(m_0+1),j)\}\|x(m_0)\|.$$

Since $j \in \Omega \setminus S$, the above inequality and our hypotheses show that

$$|x_h(m_0+1)| \geqslant \|x(m_0)\|(1+\delta) \tag{3}$$

for at least one $h \in \Omega \setminus S$.

If $i \in S$, we have by our hypotheses that

$$|x_i(m_0+1)| \leqslant \left(\sum_{k=1}^{n} |a_{ik}(m_0+1)|\right)\|x(m_0)\|$$

$$\leqslant (1-\delta)\|x(m_0)\| \tag{4}$$

Comparing the above inequality with (3) establishes Lemma 1 in the case $m = m_0 + 1$. The cases $m = m_0 + p$, $p > 1$ follow by induction. ∎

LEMMA 2. *Under the assumptions of Theorem* 1, *if* x *is a nontrivial solution of* (2) *such that* $\|x(m_0)\| = |x_j(m_0)|$ *for some* $j \in S$, *then* $\|x(m)\| = |x_h(m)|$ *with* $h \in S$ *for every integer* $m < m_0$. *Furthermore,* $\|x(m)\|(1-\delta)^{m_0-m} \geqslant \|x(m_0)\|$ *for every* $m < m_0$.

Proof. Assume on the contrary that $\|x(m_0-1)\| = |x_h(m_0-1)|$ for some $h \in \Omega \setminus S$. Then by Lemma 1, $\|x(m_0)\| = |x_l(m_0)|$ for some $l \in \Omega \setminus S$. Applying inequalities (3) and (4) to this situation gives that

$$|x_l(m_0)| \geqslant \|x(m_0-1)\|(1+\delta) > |x_p(m_0)|$$

for every index $p \in S$ which contradicts our hypotheses. Thus $\|x(m_0-1)\| = |x_h(m_0-1)|$ for some $h \in S$.

Now

$$|x_j(m_0)| \leqslant |a_{jj}(m_0)| \cdot |x_j(m_0-1)| + r(A(m_0), j) \|x(m_0-1)\|$$

for every $j \in \Omega$. By the above inequality and our hypotheses we have that

$$\|x(m_0)\| \leqslant (1-\delta)\|x(m_0-1)\|$$

which establishes Lemma 2 in the case $m = m_0 - 1$. The cases $m = m_0 - p$, $p > 1$ follow by induction. ∎

The above two lemmas remain true if we allow $A(m)$ to be singular. However, the next two lemmas require that $A(m)^{-1}$ exist for all $m \in Z$.

In the following K denotes the complex field.

LEMMA 3. *Under the assumptions of Theorem* 1, *for each positive integer q there exists a k-dimensional subspace V_q of K^n such that if x is a nontrivial solution of* (2) *with $x(0) \in V_q$, then $\|x(m)\|$ is strictly increasing on Z_{-q}.*

Proof. Let R be the k-dimensional subspace of K^n defined by

$$R = \{\operatorname{col}(\gamma_1, \dots, \gamma_n) | \gamma_i = 0 \text{ if } i \in S\}. \tag{5}$$

Denote by $Y(j,l)$ the unique $n \times n$ matrix defined for all $(j,l) \in Z \times Z$ such that

$$Y(j+1,l) = A(j+1)Y(j,l), \qquad Y(l,l) = I.$$

Let V_q be the subspace of K^n defined by

$$V_q = \{c \in K^n | c = Y(0,-q)a,\ a \in R\}$$

By the nonsingularity of $Y(0,-q)$ we have $\dim V_q = \dim R = k$. If x is a nontrivial solution of (2) with $x(0) \in V_q$, then

$$x(0) = Y(0,-q)a = Y(0,-q)x(-q),$$

so $x(-q) = a \in R$. From (5) we have that $\|x(-q)\| = |x_j(-q)|$ for some $j \in \Omega \setminus S$. Thus by Lemma 1, $\|x(m)\|$ is strictly increasing on Z_{-q}. ∎

LEMMA 4. *Under the assumptions of Theorem* 1, *for each positive integer q there exists an $(n-k)$-dimensional subspace W_q of K^n such that if x is a nontrivial solution of* (2) *with $x(0) \in W_q$, then $\|x(m)\|$ is strictly decreasing on Z^q.*

Proof. Let R be the $(n-k)$-dimensional subspace of K^n defined by

$$R=\{\operatorname{col}(\gamma_1,\dots,\gamma_n)|\gamma_i=0 \text{ if } i\in\Omega\setminus S\}. \tag{6}$$

Let $Y(m)$ be the unique $n\times n$ matrix function defined for all integers m so that

$$Y(m+1)=A(m+1)Y(m), \qquad Y(0)=I.$$

Define the subspace W_q of K^n by

$$W_q=\{c\in K^n|c=Y^{-1}(q)a,\ a\in R\}.$$

By the nonsingularity of $Y(q)$ we have $\dim W_q=\dim R=n-k$. If x is any nontrivial solution of (2) with $x(0)\in W_q$, then $x(0)=Y^{-1}(q)a$ for some $a\in R$. But since $x(0)=Y^{-1}(q)x(q)$, we have that $x(q)=a\in R$. Thus from (6) we have $\|x(q)\|=|x_h(q)|$ for some $h\in S$, so by Lemma 2 $\|x(m)\|$ is strictly decreasing on Z^q. ■

In the following, if c is a column matrix, c^* will denote the row matrix which is the conjugate transpose of c.

Proof of Theorem 1. For each positive integer q, let V_q be the k-dimensional subspace defined in Lemma 3, and let $\{c_q^j|j=1,\dots,k\}$ be an orthonormal basis for V_q, i.e., such that

$$c_q^{i*}c_q^j=0 \quad \text{for } i\neq j \quad \text{and} c_q^{i*}c_q^i=1.$$

By the compactness of the unit sphere in K^n, there exists a sequence of integers $\{q_l\}$ and vectors $c^j\in K^n$, $j=1,\dots,k$, such that

$$\lim_{l\to\infty}|c^j-c_{q_l}^j|=0, \qquad j=1,\dots,k.$$

Clearly $c^{i*}c^j=0$ for $i\neq j$, and $c^{i*}c^i=1$, $i,j=1,\dots,k$.

Now let $y^1,\dots,y^k$ be the solutions of (2) defined by the initial conditions

$$y^j(0)=c^j, \qquad j=1,\dots,k,$$

which, by their independence at zero, are independent. Let x be any nontrivial solution of (2) of the form

$$x(m)=\alpha_1 y^1(m)+\cdots+\alpha_k y^k(m).$$

For each integer $l=1,2,\dots$, let $x_l(m)$ be the solution of (2) satisfying

$$x_l(0)=\gamma_1 c^1_{q_l}+\cdots+\gamma_k c^k_{q_l}.$$

Now $x_l(0)\in V_{q_l}$, so by Lemma 3 we have that $\|x_l(m)\|$ is strictly increasing on Z_{-q}.

Next observe that, for $m>0$,

$$\|x(m)-x_l(m)\|\leqslant\left(\prod_{j=1}^{m}\|A(j)\|\right)\|x(0)-x_l(0)\|, \tag{7}$$

and that for $m<0$,

$$\|x(m)-x_l(m)\|\leqslant\left(\prod_{j=0}^{m-1}\|A^{-1}(-j)\|\right)\|x(0)-x_l(0)\|. \tag{8}$$

Since

$$\|x(0)-x_l(0)\|\leqslant\sum_{j=1}^{k}|\gamma_j|\cdot|c^j-c^j_{q_l}|\to 0 \qquad \text{as } l\to\infty,$$

we have by (7) and (8) that

$$\|x(m)-x_l(m)\|\to 0 \quad \text{as } l\to\infty, \qquad m\in Z.$$

Thus $|\|x(m)\|-\|x_l(m)\||\to 0$ as $l\to\infty$, which shows that $\|x(m)\|$ is nondecreasing on Z. Now let m_1 and m_2 be integers such that $m_1<m_2$. If $\|x(m_1)\|=|x_j(m_1)|$ for some $j\in S$, then by Lemma 2 we would have $\|x(m_1-1)\|>\|x(m_1)\|$, contradicting the fact that $\|x(m)\|$ is nondecreasing. Hence $\|x(m_1)\|=|x_j(m_1)|$ for some $j\in\Omega\setminus S$, so by Lemma 1 we have that $\|x(m_2)\|\geqslant\|x(m_1)\|(1+\delta)^{m_2-m_1}$, which establishes statement (a).

The proof of (b) follows analogously to the proof of (a), using the subspace W_q of Lemma 4; it is omitted.

To establish (c) we note that if the solutions $y^1,\dots,y^k$, $z^1,\dots,z^{n-k}$ were not independent, there would exist a nontrivial solution x of (2) such that $\|x(m)\|$ would be both strictly increasing and strictly decreasing on Z, which is absurd. ■

3. AN APPLICATION TO MATRIX PRODUCTS

Using Theorem 1 we are now able to give estimates for the moduli of the eigenvalues of a product of finitely many $n\times n$ matrices.

THEOREM 2. *Let $A(1), A(2), \ldots, A(p)$ be $n\times n$ complex nonsingular matrices and let $A(k)=(a_{ij}(k))$ for $1\leqslant k\leqslant p$. Let*

$$\alpha_i = \min_{1<k<m} \{|a_{ii}(k)| - r(A(k),i)\}$$

and

$$\beta_i = \max_{1<k<m} \{|a_{ii}(k)| + r(A(k),i)\}.$$

Let $[c_l, d_l]$, $l=1,\ldots,s$, denote the components of $\cup_{i=1}^{n}[\alpha_i,\beta_i]$. Let n_j be the number of indices i such that $[\alpha_i,\beta_i]\subseteq[c_j,d_j]$ for $1\leqslant j\leqslant s$, and let $P=A(1)A(2)\cdot\cdots\cdot A(p)$. Then P has exactly n_j eigenvalues λ (counting multiplicities) such that

$$c_j \leqslant |\lambda|^{1/p} \leqslant d_j, \qquad 1\leqslant j\leqslant s. \tag{9}$$

We remark that this result can be considered as an extension of Gerschgorin's theorem, since it follows from Gerschgorin's theorem when all of the matrices are equal.

The proof of Theorem 2 uses one preliminary lemma.

LEMMA 5. *Let S denote a subset of $\Omega=\{1,2,\ldots,n\}$, and let k be the cardinality of S. Let $B(1), B(2),\ldots,B(p)$ be nonsingular $n\times n$ complex matrices each of which is (S,δ) diagonally dominant with respect to the unit circle. Let $P=B(1)B(2)\cdot\cdots\cdot B(p)$. Then P has exactly k eigenvaues λ such that $|\lambda|\leqslant(1-\delta)^p$ and exactly $n-k$ eigenvalues λ such that $|\lambda|\geqslant(1+\delta)^p$ (counting multiplicities).*

Proof. Define matrices $C(m)$, $m\in Z$, by

$$C(rp+j)=B(p+1-j) \qquad \text{for } 1\leqslant j\leqslant p \text{ and } r\in Z$$

and consider the difference equation

$$x(m+1)=C(m+1)x(m), \qquad m\in Z. \tag{10}$$

Note that the difference equation (10) satisfies the hypotheses of Theorem 1. Thus if V denotes the solution space of (10), it follows from Theorem 1 that

$$V=V_1\oplus V_2,$$

where

$$\|x(m_2)\| \leqslant (1-\delta)^{m_2-m_1}\|x(m_1)\|, \qquad m_1\leqslant m_2 \tag{11}$$

for any solution $x \in V_1$, and that

$$\|x(m_2)\| \geqslant (1+\delta)^{m_2-m_1}\|x(m_1)\|, \qquad m_1 \leqslant m_2 \tag{12}$$

for any solution $x \in V_2$. Furthermore, $\dim V_1 = k$. Now note that if $x(m)$ is any solution of (10), then

$$x(r_2 p) = P^{r_2-r_1}x(r_1 p), \qquad r_1, r_2 \in Z. \tag{13}$$

Thus it follows from (11), (12), and (13) that if W denotes the solution space of the difference equation

$$y(m+1) = Py(m)$$

then $W = W_1 \oplus W_2$ where

$$\|y(m_2)\| \leqslant \|y(m_1)\|(1-\delta)^{p(m_2-m_1)}, \qquad m_1 \leqslant m_2$$

for any solution $y \in W_1$, and

$$\|y(m_2)\| \geqslant \|y(m_1)\|(1+\delta)^{p(m_2-m_1)}, \qquad m_1 \leqslant m_2$$

for any solution $y \in W_2$. Furthermore, $\dim W_1 = k$. The lemma now follows from these observations. ■

Proof of Theorem 2. Without loss of generality, assume that

$$c_1 < d_1 < c_2 < d_2 < \cdots < c_s < d_s.$$

First let j be any index such that $1 < j \leqslant s$. Let

$$\gamma = \tfrac{1}{2}(d_{j-1} + c_j), \qquad \delta = \tfrac{1}{2}(c_j - d_{j-1}). \tag{14}$$

Let $B(m) = (1/\gamma)A(m)$, $1 \leqslant m \leqslant p$, and let $B(m) = (b_{ik}(m))$. Note that if $[\alpha_i, \beta_i] \subseteq (-\infty, d_{j-1}]$, then by (14),

$$\max_{1<k<p}\{|b_{ii}(k)| + r(B(k),i)\} = \frac{\beta_i}{\gamma} \leqslant \frac{d_{j-1}}{\gamma} = 1 - \delta\gamma^{-1}. \tag{15}$$

Let S_j be the subset of $\Omega = \{1,\dots,n\}$ for which (15) holds. Then $i \in \Omega\setminus S_j$ implies $[\alpha_i, \beta_i] \subseteq [c_j, \infty)$, and using (14) we compute that

$$\min_{1<k<p}\{|b_{ii}(k)| - r(B(k),i)\} = \frac{\alpha_i}{\gamma} \geqslant \frac{c_j}{\gamma} = 1 + \delta\gamma^{-1}. \tag{16}$$

Now let q_j be the number of indices in S_j. Then by (15), (16), and Lemma 5 we conclude that the matrix $T = B(1)B(2)\cdot\cdots\cdot B(p)$ has exactly q_j eigenvalues λ such that $|\lambda| \leqslant (1-\gamma^{-1}\delta)^p$ and exactly $n-q_j$ eigenvalues λ such that $|\lambda| \geqslant (1+\gamma^{-1}\delta)^p$ (counting multiplicities). Now λ is an eigenvalue of T iff $\lambda\gamma^p$ is an eigenvalue of P. Thus P has q_j eigenvalues λ such that $|\lambda| < [\gamma(1-\gamma^{-1}\delta)]^p$, and $n-q_j$ eigenvalues λ such that $|\lambda| > [\gamma(1+\gamma^{-1}\delta)]^p$. But from (14) we have that

$$\gamma(1-\gamma^{-1}\delta) = d_{j-1} \quad \text{and} \quad \gamma(1+\gamma^{-1}\delta) = c_j.$$

We have therefore established that if $1 < j \leqslant s$, there are exactly $q_j = n_1 + n_2 + \cdots + n_{j-1}$ eigenvalues λ of P such that $|\lambda| \leqslant d_{j-1}^p$ and exactly $n - q_j = n_j + \cdots + n_s$ eigenvalues λ such that $|\lambda| > c_j^p$. To complete the proof we must show that there are no eigenvalues λ of P such that either $|\lambda| > d_s^p$ or $|\lambda| < c_1$ (in the case $c_1 > 0$).

Now assume that $|\lambda_0| > d_s^p$ for some eigenvalue λ_0 of P. Let ξ be a real number such that

$$\xi d_s < 1 < \xi |\lambda_0|^{1/p}, \tag{17}$$

and let $C(k) = \xi A(k)$, $1 \leqslant k \leqslant p$. Then for $C(k) = \big(c_{i,j}(k)\big)$, we have

$$|c_{ii}(k)| + r\big(C(k), i\big) \leqslant \xi d_s < 1, \qquad 1 \leqslant k \leqslant p,\ 1 \leqslant i \leqslant n.$$

By Lemma 5 we conclude that $|\lambda| < 1$ for every eigenvalue λ of $U = C(1)C(2)\cdot\cdots\cdot C(p)$. But λ is an eigenvalue of P iff $\xi^p\lambda$ is an eigenvalue of U. Thus $|\lambda| < 1/\xi^p$ for every eigenvalue λ of P. Using (17) we have that

$$|\lambda_0| < \frac{1}{\xi^p} < d_s^p,$$

contradicting our assumption. Thus $|\lambda| \leqslant d_s^p$ for each eigenvalue λ of P.

If we assume the existence of an eigenvalue λ of P such that $|\lambda| < c_1$ (where $c_1 \geqslant 0$), a similar argument produces the desired contradiction. ■

4. AN APPLICATION TO DIFFERENTIAL EQUATIONS

Let x denote any nontrivial solution of the differential equation (1). For $v > 0$ we define the function $y(m)$ on Z by $y(m) = x(mv)$. Let $X(t)$ denote the fundamental matrix for (1) such that $X(0) = I =$ identity. Then $y(m) = X(mv)x(0)$, so $y(m+1) = X(mv+v)X^{-1}(mv)y(m)$ for each $m \in$ Z. Defining $C^v(m+1)=X(mv+v)X^{-1}(mv)$, we have that the differential equation (1) gives rise to the difference equation

$$y(m+1)=C^v(m+1)y(m). \tag{18}$$

The following result shows that, from the viewpoint of diagonal dominance, (18) is a generalization of (1).

THEOREM 3. *Let $A(t)$ in (1) be uniformly continuous and bounded on $[0,\infty)$. If $A(t)$ in (1) is (S,δ) diagonally dominant for each $t\in[0,\infty)$ there exist positive numbers β and η such that $C^v(m)$ in (18) is (S,β) diagonally dominant with respect to the unit circle for all $0<v<\eta$.*

The proof of Theorem 3 uses one preliminary lemma.

LEMMA 6. *Let $A(t)$ be as in Theorem 2, and let $X(t)$ denote the fundamental matrix for (1) such that $X(0)=I$. Then*

$$\left\|\frac{1}{\mu}\left[X(t+\mu)X^{-1}(t)-I\right]-A(t)\right\|\to 0$$

uniformly on $[0,\infty)$ as $\mu\downarrow 0$.

Proof. Let $A(t)\leqslant M$, $t\in[0,\infty)$. Since

$$X(t+\mu)X^{-1}(t)=I+\int_0^{\mu}A(t+v)X(t+v)X^{-1}(t)\,dv,$$

we have that, for each $\mu\in[0,\infty)$,

$$\|X(t+\mu)X^{-1}(t)\|\leqslant 1+\int_0^{\mu}\|A(t+v)\|\cdot\|X(t+v)X^{-1}(t)\|\,dv,$$

which, by application of Gronwall's lemma, gives that

$$\|X(t+\mu)X^{-1}(t)\|\leqslant \exp\int_0^{\mu}\|A(t+v)\|\,dv\leqslant e^{\mu M},$$

which in turn shows that

$$\|X(t+\mu)X^{-1}(t)-I\|\leqslant e^{\mu M}-1\to 0 \qquad \text{uniformly as } \mu\downarrow 0. \tag{19}$$

Since

$$\frac{1}{\mu}[X(t+\mu)X^{-1}(t)-I]-A(t)=\frac{1}{\mu}\int_0^{\mu}[A(t+v)-A(t)]dv+\frac{1}{\mu}\int_0^{\mu}A(t+v)$$
$$\times[X(t+v)X^{-1}(t)-I]dv,$$

the result follows by (19) and the uniform continuity of A. ■

Proof of Theorem 3. For $A(t)=(a_{ij}(t))$ write $a_{ii}(t)=\lambda(t)+i\mu(t)$, where λ and μ are real functions. For real v note that

$$|1+va_{ii}(t)|=\sqrt{1+2v\lambda(t)+v^2[\mu^2(t)+\lambda^2(t)]}\ . \tag{20}$$

Let $a(v,t)=2v\lambda(t)+v^2[\mu^2(t)+\lambda^2(t)]$. Then we have that

$$\sqrt{1+a(v,t)}=1+\tfrac{1}{2}a(v,t)+a^2(v,t)\cdot\epsilon(a(v,t)) \tag{21}$$

where $\epsilon(a(v,t))$ is $O(|a(v,t)|)$.

Since $|a(v,t)|\leqslant 2|v|M+2v^2M^2$, we get that

$$\frac{\epsilon(a(v,t))}{|a(v,t)|}\to 0 \qquad \text{uniformly on } [0,\infty) \text{ as } v\to 0.$$

From (20) and (21) we write that

$$|1+va_{ii}(t)|=1+v\lambda(t)+v^2\left[\frac{\mu^2(t)+v^2(t)}{2}+\left(\frac{a(v,t)}{v}\right)^2\epsilon(a(v,t))\right] \tag{22}$$

Now let $b(v,t)=|1+va_{ii}(t)|-[1+v\lambda(t)]$. For $|v|<1$ sufficiently small that $|\epsilon(a(v,t))|<|a(v,t)|$, we have from (20), (21), and (22) that

$$|b(v,t)|\leqslant M^2+(2M+2|v|M^2)|a(v,t)|^2\leqslant M^2+(2M+2M^2)^3.$$

Thus

$$|1+va_{ii}(t)|=1+v\lambda(t)+v^2b(v,t), \tag{23}$$

where $b(v,t)$ is bounded on $[0,\infty)$ for sufficiently small v. Now by hypotheses we have that, for $i\in\Omega\setminus S$,

$$\lambda(t)>r(A(t),i)+\delta, \qquad t\in[0,\infty).$$

Using (23) and the above, we have, for $v>0$, that

$$|1+va_{ii}(t)|-|v|r(A(t),i)\geqslant|1+va_{ii}(t)|-v[\lambda(t)-\delta],$$

so

$$|1+va_{ii}(t)|-|v|r(A(t),i)\geqslant 1+v\delta-v^2b(v,t),\qquad i\in\Omega\setminus S. \tag{24}$$

For $i\in S$ we have by hypothesis that

$$\lambda(t)\leqslant -r(A(t),i)-\delta,\qquad t\in[0,\infty),$$

which the reader, using (23), can verify gives that

$$|1+va_{ii}(t)|+|v|r(A(t),i)\leqslant 1-v\delta+v^2b(v,t),\qquad i\in S. \tag{25}$$

We now define the $n\times n$ matrix-valued function $C^v(t)=c_{ij}^v(t)$ by $C^v(t)=X(t+v)X^{-1}(t)$. By Lemma 6 we have that

$$C^v(t)=I+vA(t)+v\epsilon(t,v), \tag{26}$$

where $\|\epsilon(t,v)\|\to 0$ uniformly as $v\to 0$. From (26) it follows that

$$\big||c_{ii}^v(t)|-|1+va_{ii}(t)|\big|\leqslant v\|\epsilon(t,v)\|,\qquad i\in\Omega\setminus S,$$

so we have, for $i\in\Omega\setminus S$, that

$$|1+va_{ii}(t)|-v\|\epsilon(t,v)\|\leqslant|c_{ii}^v(t)|\leqslant|1+va_{ii}(t)|+v\|\epsilon(t,v)\|. \tag{27}$$

Similarly, (26) gives that

$$\big|r(C^v(t),i)-|v|r(A(t),i)\big|\leqslant v\|\epsilon(t,v)\|,\qquad i\in\Omega\setminus S,$$

so, for $i\in\Omega\setminus S$,

$$|v|r(A(t),i)-v\|\epsilon(t,v)\|\leqslant r(C^v(t),i)\leqslant|v|r(A(t),i)+v\|\epsilon(t,s)\|. \tag{28}$$

Now using (23) and (27) we conclude, for $i\in\Omega\setminus S$, that

$$|c_{ii}^v(t)|-r(C^v(t),i)\geqslant 1+v\delta-v^2\delta b(v,t)-2v\|\epsilon(t,v)\|. \tag{29}$$

Since $b(v,t)$ is bounded for sufficiently small v, $vb(v,t)\to 0$ uniformly as $v\to 0$. Thus we may choose η so small that

$$\delta-[\eta b(\eta,t)+2\|\epsilon(t,\eta)\|]>0. \tag{30}$$

Letting $\beta = v\{\delta - [\eta b(\eta,t) + 2\|\epsilon(t,\eta)\|]\}$, we have that $\beta > 0$ and, from (29) and (30), that

$$|c_{ii}^{v}(t)| - r(C^{v}(t),i) \geqslant 1+\beta, \qquad t\in[0,\infty),\ i\in\Omega\setminus S,\ 0<v<\eta.$$

Similarly, using (25) and (28), we have, for $i \in S$, that

$$|c_{ii}^{v}(t)| + r(C^{v}(t),i) \leqslant 1 - v[\delta - vb(v,t)] + 2v\|\epsilon(t,v)\|,$$

and so by (30), for η and β as above, we have that

$$|c_{ii}^{v}(t)| + r(C^{v}(t),i) \leqslant 1-\beta, \qquad t\in[0,\infty),\ i\in S,\ 0<v<\eta.$$

■

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