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Real elements in the mapping class group of T^2

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ABSTRACT

We present a complete classification of elements in the mapping class group of the torus which have a representative that can be written as a product of two orientation reversing involutions. Our interest in such decompositions is motivated by features of the monodromy maps of real fibrations. We employ the property that the mapping class group of the torus is identifiable with $SL(2,\mathbb{Z})$ as well as that the quotient group $PSL(2,\mathbb{Z})$ is the symmetry group of the *Farey tessellation* of the Poincaré disk.

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1. Introduction

In this work, we are interested in finding the isotopy classes of orientation preserving diffeomorphisms of the torus which have a representative that decomposes into a product of two orientation reversing involutions. The question arose when we were studying properties of the monodromy maps of real Lefschetz fibrations.

Let us recall that a Lefschetz fibration is a projection of an oriented smooth 4-manifold onto an oriented surface such that apart from finitely many fibers, which have a single node, the fibers are actually smooth oriented surfaces. Intuitively, a real structure can be regarded as a topological generalization of the complex conjugation. We define a *real structure* on an oriented 4-manifold as an orientation preserving involution whose fixed point set, if it is not empty, has dimension 2. Likewise, a *real structure* on an oriented surface is an orientation reversing involution. A smooth manifold together with a real structure is called a *real manifold* and the fixed point set of a real structure is called the *real part*. A *real Lefschetz fibration* is, thus, defined as a Lefschetz fibration of a real 4-manifold over a real surface where the fiber structure is compatible with the real structures.

The fundamental fact about real Lefschetz fibrations is that their monodromy maps along the loops on which the real structure induces an orientation reversing action decompose into a product of two orientation reversing involutions. These involutions are the real structures of the two real fibers over the two real points of the loop. The diffeomorphisms (as well as their classes) with such a feature are called *real*. Let us remark that if, in particular, the base space is a surface with a single boundary component, then the monodromy map along the boundary component satisfies this property.

We can rephrase the decomposition property as follows: the monodromy maps as above of real Lefschetz fibrations are conjugated to their inverses by a real structure. At this point, a weaker property, that of being conjugated to its inverse by an orientation reversing diffeomorphism appears naturally. Diffeomorphisms (as well as their classes) with the latter property are called *weakly real*. We study weakly real and real diffeomorphisms simultaneously.

As is well known, the restriction of a Lefschetz fibration to loops in the complement of the critical set is a usual fibration over a circle with fiber compact connected oriented smooth 2-manifold F. We call such fibrations *F-fibrations* and extend the above definitions to F-fibrations to proceed with them.

The aim of this article is to answer the following questions in the case of $F = T^2$ (elliptic):

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(1) Which classes are real/ weakly real?
(2) Obviously real classes are weakly real. What about the converse: are weakly real classes real?

Employing the identification of the mapping class group of the torus with $SL(2,\mathbb{Z})$ we state the main results as follows:

Theorem 1. *All elliptic and parabolic matrices in $SL(2,\mathbb{Z})$ are real.*
A hyperbolic matrix $A \in SL(2,\mathbb{Z})$ is real if and only if its cutting period-cycle $[a_1 a_2 \dots a_{2n}]_A$ is odd-bipalindromic.
Moreover, a matrix in $SL(2,\mathbb{Z})$ is real if and only if it is weakly real.

(The three statements of the above theorem are indeed presented separately as Theorem 6, Theorem 9 and Theorem 12, respectively.)

Elliptic, parabolic and hyperbolic matrices in $SL(2,\mathbb{Z})$ differ by the nature of their fixed points. In Section 6, we discuss the conjugacy classes of elliptic and parabolic matrices. Conjugacy classes of hyperbolic matrices are explained in Section 7, where we also give the definition of the cutting period-cycle. The last two sections are devoted to the real factorizations of the matrices. We define odd-bipalindromic cutting period-cycle in the last section.

Finally, we would like to note that this work is self-contained and the tools used are explained in detail.

2. Real F-fibrations and their monodromies

Let Y be a compact connected oriented smooth 3-manifold, and $\pi : Y \to S^1$ be a fibration whose fiber is an oriented smooth 2-manifold F. We call π an *F-fibration*. In particular, when $F = T^2$, we call π an *elliptic F-fibration*.

Definition. An F-fibration $\pi : Y \to S^1$ is called *weakly real* if there is an orientation preserving diffeomorphism $H : Y \to Y$ which sends fibers into fibers reversing their orientations. If $H^2 = id$, then H will be called a *real structure* on the F-fibration $Y \to S^1$. An F-fibration equipped with a real structure will be called *real*.

Note that H induces an orientation reversing diffeomorphism $h : S^1 \to S^1$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{H} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ S^1 & \xrightarrow{h} & S^1. \end{array}$$

Here, it is not difficult to see that the set of orientation reversing involutions form a single conjugacy class in the diffeomorphism group of S^1 (the crucial observation is that any such involution has precisely two fixed points). Therefore, any real F-fibration is equivariantly isomorphic to an F-fibration whose involution h is standard. Let it be the complex conjugation $c_{S^1} : S^1 \to S^1$, $z \mapsto \bar{z}$, $z \in S^1 \subset \mathbb{C}$.

In the case of a weakly real F-fibration, h may be not an involution; however, it also has precisely two fixed points and can be changed into an involution by an isotopy. This isotopy can be lifted to an isotopy of H. Thus, by modification of H we can always make h an involution. So it is not restrictive for us to suppose always that $h = c_{S^1}$ for both real and weakly real F-fibrations.

The restrictions of H to the invariant fibers $F_\pm = \pi^{-1}(\pm 1)$ will be denoted $h_\pm : F_\pm \to F_\pm$. In the case of real F-fibrations, we will prefer using the notation c_Y for the involution H, and $c_\pm$ for the involutions $h_\pm$.

In what follows, we choose the point b in the upper semi-circle, S^+. The restriction $Y_+ = \pi^{-1}(S^+) \to S^+$ of π admits a trivialization $\phi_+ : Y_+ \to F \times S^+$ which is identical on the fiber $F = F_b$. In the case of real fibrations, this allows us to consider the pull-back of $c_\pm$ via ϕ, namely, the two involutions $x \mapsto \phi_+(c_\pm(\phi_+^{-1}(x \times \pm 1)))$ on the same fiber F. We will stick to the notation $c_\pm$ for these involutions.

It is well known that any F-fibration $\pi : Y \to S^1$ is isomorphic to the projection $M_f \to S^1$ of a mapping torus $M_f = F \times I/_{(f(x),0)\sim(x,1)}$ of some orientation preserving diffeomorphism $f : F \to F$. More precisely, if we fix a particular fiber $F = F_b = \pi^{-1}(b)$, $b \in S^1$, then an isomorphism $\phi : M_f \to Y$ can be chosen so that $F \times 0$ and $F \times 1$ are identified with the fiber F_b, so that $x \times 0 \mapsto x$ and $x \times 1 \mapsto f(x)$.

An F-fibration π determines an orientation preserving diffeomorphism f up to isotopy and thus, provides a well-defined element in the mapping class group $[f] \in Map^+(F)$ called the *monodromy of π* (relative to the fiber $F = F_b$). A map f representing the class $[f]$ will be also often called *monodromy*, or more precisely, a *monodromy map*.

In some cases, we fix a marking $\rho : \Sigma_g \to F_b$ (an identification of F_b with an abstract genus-g surface Σ_g). Then the diffeomorphism $\rho^{-1} \circ f \circ \rho : \Sigma_g \to \Sigma_g$ (the pull-back of f) as well as its isotopy class $[\rho^{-1} \circ f \circ \rho] \in Map^+(\Sigma_g)$ will be called the *monodromy of π relative to the marking ρ*.

Theorem 2. *Let $\pi : Y \to S^1$ be a weakly real F-fibration with a distinguished fiber $F = F_b$, $b \in S^+$. Then the two product diffeomorphisms of the fiber F, $(h_+)^{-1} \circ h_-$, and $h_+ \circ (h_-)^{-1}$ are isotopic and equal the monodromy of π relative to the fiber F. In particular, if π is a real F-fibration, then the monodromy can be factorized as $c_+ \circ c_-$.*

Proof. Consider a trivialization $Y_- \to F \times S^-$ of the restriction $Y_- = \pi^{-1}(S^-) \to S^-$ of π over the lower semi-circle, S^-, which is the composition of $\phi_+ \circ H : Y_- \to F \times S^+$, with the map $F \times S^+ \to F \times S^-$, $(x, z) \mapsto (x, c_{S^1}(z))$.

If S^1 is split into several arcs and a fibration over S^1 is glued from trivial fibrations over these arc, then the monodromy is clearly the product of the gluing maps of the fibers over the common points of the arcs, ordered in the counter-clockwise direction beginning from a marked point $b \in S^1$. In our case, the arcs are S^+, S^-, their common points follow in the order -1, $+1$, and the corresponding gluing maps, are h_- and h_+^{-1}. This gives monodromy $(h_+)^{-1} \circ h_-$. If we consider another trivialization $Y_- \to F \times S^-$ replacing in its definition H by H^{-1}, then the gluing maps will be h_-^{-1} and h_+, and the monodromy is factorized as $h_+ \circ (h_-)^{-1}$. □

Corollary 3.

(1) *If $f : F \to F$ is a monodromy map of a weakly real F-fibration, then the diffeomorphisms $h^{-1} \circ f \circ h$ as well as $h \circ f \circ h^{-1}$, where h stands either for h_+, or for h_-, are all isotopic to the inverse f^{-1}.*
(2) *If f is a monodromy map of a real F-fibration, then $f^{-1} = c_+ \circ f \circ c_+ = c_- \circ f \circ c_-$.*

Remark. It is obvious that $f = c_+ \circ c_-$ for some real structures c_-, c_+ if and only if $f^{-1} = c_\pm \circ f \circ c_\pm$. It follows from the known cases of the Nielsen realization problem that $[f] = [c_+ \circ c_-]$ if and only if $[f^{-1}] = [c_\pm \circ f \circ c_\pm]$, (cf. [1,2]). In another words, if $[f^{-1}] = [c \circ f \circ c]$, then there is a diffeomorphism g isotopic to f such that $g^{-1} = c \circ g \circ c$. Similarly, if $[f^{-1}] = [h \circ f \circ h^{-1}]$ for some orientation reversing diffeomorphism h, then there are diffeomorphisms g, k such that $[g] = [f]$ and $[k] = [h]$ with $g^{-1} = k \circ g \circ k^{-1}$. (All the class equalities above are considered in the extended mapping class group $Map(T^2)$ of the torus.)

Definition. A diffeomorphism $f : F \to F$ as well as its isotopy class $[f] \in Map^+(F)$ will be called *real* if there is a real structure $c : F \to F$ such that $f^{-1} = c \circ f \circ c$.

We call $f : F \to F$ as well as its isotopy class $[f] \in Map^+(F)$ *weakly real* if $f^{-1} = h \circ f \circ h^{-1}$ for some orientation reversing diffeomorphism h of F.

Proposition 4. *An F-fibration is real* (*respectively weakly real*) *if and only if its monodromy f is real* (*respectively weakly real*).

Proof. We give the proof for real F-fibrations; the proof for weakly real ones is analogous.

Necessity of the condition follows from Corollary 3. As for the converse, consider an F-fibration $\pi : Y \to S^1$ with the monodromy class $[f] \in Map^+(F)$, and f its representative such that $f^{-1} = c \circ f \circ c$ for some real structure c on F. Presenting Y as $F \times [0,1] / _{(f(x),0) \sim (x,1)}$, we obtain a well-defined involution $c_Y : Y \to Y$ induced from the involution $(x, t) \mapsto (c(x), 1-t)$ on $F \times [0,1]$. It preserves the fibration structure and acts as c and $f \circ c$ on the real fibers $F \times \frac{1}{2}$ and $F \times 0 = F \times 1$ respectively. □

3. Homology monodromy factorization of elliptic F-fibrations

It is well known that $Map^+(T^2) = SL(2, \mathbb{Z})$, due to the fact that every diffeomorphism $f : T^2 \to T^2$ is isotopic to a *linear diffeomorphism*. The latter diffeomorphisms by definition are induced on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ defined by a matrix $A \in SL(2, \mathbb{Z})$. Note that we can naturally identify $T^2 = H_1(T^2, \mathbb{R})/H_1(T^2, \mathbb{Z})$ and interpret matrix A as the induced automorphism f_* in $H_1(T^2, \mathbb{Z})$. The latter automorphism is called the *homology monodromy*. Since isotopic diffeomorphisms have the same homology monodromy in $H_1(T^2, \mathbb{Z})$, we obtain well-defined homomorphisms $Map^+(T^2) \to Aut^+(H_1(T^2, \mathbb{Z})) \to SL(2, \mathbb{Z})$ which are in fact isomorphisms (here Aut^+ stand for the orientation preserving automorphisms). Let a denote the simple closed curve on T^2 represented by the equivalence class of the horizontal interval $I \times 0 \subset \mathbb{R}^2$, and b is similarly represented by the vertical interval $0 \times I$. We have $a \circ b = 1$; hence, the homology classes represented by these curves are integral generators of $H_1(T^2, \mathbb{Z})$. The mapping class group $Map^+(T^2)$ of T^2 is generated by the Dehn twists t_a and t_b, which can be characterized by their homology monodromy homomorphism matrices $t_{a*} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $t_{b*} = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$.

In a like manner, it can be shown that the extended mapping class group $Map(T^2)$ is isomorphic to the general linear group $GL(2, \mathbb{Z})$. As a consequence, for elliptic F-fibrations, the question of characterization of real monodromy classes $[f] \in Map^+(T^2)$ can be interpreted as the question on the decomposability of their homology monodromy $f_* \in SL(2, \mathbb{Z})$ into a product of two *linear real structures* in the group $GL(2, \mathbb{Z})$. The latter structures by definition are linear orientation reversing maps of order 2 defined by integral (2×2)-matrices. Such decomposability is equivalent to the property that f_* is conjugate to its inverse by a linear real structure. Hence, a necessary condition for a matrix A to be real is that both A and A^{-1} lie in the same conjugacy classes in the group $GL(2, \mathbb{Z})$.

Recall that there are three types of real structures on T^2 distinguished by the number of their real components: 0, 1, or 2. Note that the automorphisms of $H_1(T^2, \mathbb{Z})$ induced by the real structures with 0 or 2 real components are diagonalizable over $\mathbb{Z}$, namely, their matrices are conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in $GL(2, \mathbb{Z})$. So we cannot determine if the number of components 0

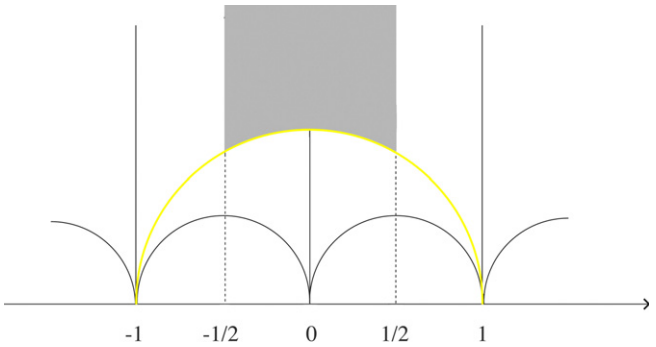

Fig. 1. The upper half plane model of the hyperbolic plane and the standard fundamental domain of $SL(2,\mathbb{Z})$.

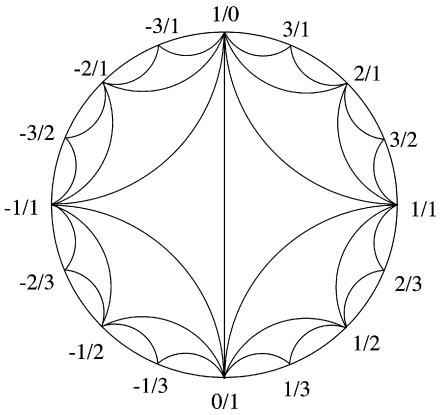

Fig. 2. Tessellation of $\mathbb{D}$.

or 2 knowing only the matrix representing the homology action of the real structure. The homology action of a real structure with 1 real component is presented by a matrix conjugate to $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$.

4. The modular action on the hyperbolic half-plane

Let $\mathbb{C}^2$ be considered as the vector space of 2×1 matrices over $\mathbb{C}$. Then a matrix $A=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ in $GL(2,\mathbb{Z})$ acts on $\mathbb{C}^2$ from the left as matrix multiplication.

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} z_1\\ z_2\end{pmatrix}=\begin{pmatrix} az_1+bz_2\\ cz_1+dz_2\end{pmatrix}.$$

This action can be extended to $\mathbb{C}P^1=\mathbb{C}^2\setminus\{(0,0)\}/_{(z_1,z_2)\sim(\lambda z_1,\lambda z_2)}$. Let us identify $\mathbb{C}P^1\cong\{(z_1,z_2)\in\mathbb{C}^2, z_2\neq 0\}\cup\{\infty\}\cong\mathbb{C}\cup\{\infty\}$ and rewrite the action of $GL(2,\mathbb{Z})$. We obtain a linear fractional transformation $z\to\frac{az+b}{cz+d}$ where $z=\frac{z_1}{z_2}$. In particular, if $A\in SL(2,\mathbb{Z})$, then the transformation preserves the orientation of $\mathbb{C}$ and takes $\mathbb{R}\cup\{\infty\}$ to itself preserving its orientation. Hence, it gives rise to a diffeomorphism of the upper half plane $\mathbb{H}$ which can be seen as a model for the hyperbolic plane where the geodesics are the semi-circles centered at a real point or vertical half-lines which can also be considered as arcs of infinite radius. By identifying the upper half plane with the lower half plane by the complex conjugation, one extends the action of $SL(2,\mathbb{Z})$ to an action of $GL(2,\mathbb{Z})$. The standard fundamental domain of the action is the set $\{z\,|\,|Re(z)|\leqslant\frac{1}{2},|z|\geqslant 1\}$ which is shown in Fig. 1.

5. The Farey tessellation

Let us identify the upper half plane model with the Poincaré disk model $\mathbb{D}$. We will consider the disk $\mathbb{D}$ together with its boundary $\mathbb{R}\cup\infty$ and define a tessellation on $\mathbb{D}$ as follows:

Set ∞ as $\frac{1}{0}$ and consider the two fractions $\frac{0}{1}$ and $\frac{1}{0}$, spot them on $\mathbb{D}$ as the south and the north poles respectively and connect them with a line which will be the vertical diameter. Consider their mediant $\frac{0+1}{1+0}=\frac{1}{1}$ and connect each of them with a geodesic to the mediant. Apply the same to the fractions $\{\frac{0}{1},\frac{1}{1}\}$ and $\{\frac{1}{1},\frac{1}{0}\}$. Iterating this process one obtains a tessellation of the right semi-disk. By taking the symmetry one extends the tessellation to $\mathbb{D}$, see Fig. 2.

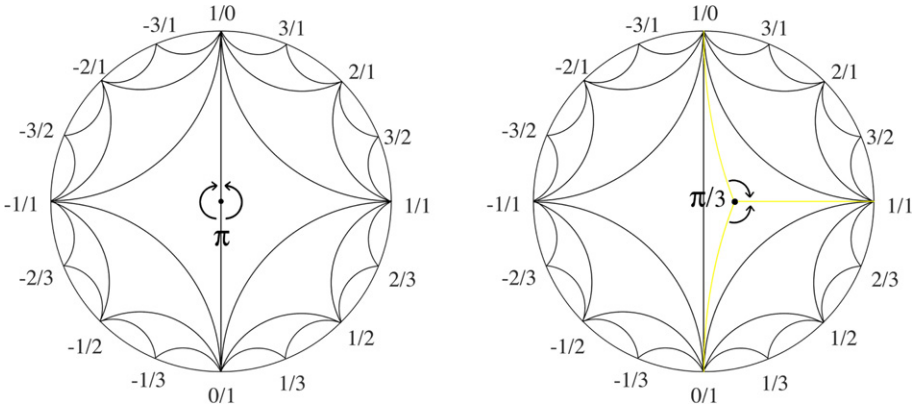

Fig. 3. Modular actions of elliptic matrices, $E_{\pm\pi}$, $E_{\pm\frac{2\pi}{3}}$.

In the literature this tessellation is called the *Farey tessellation*. Let us denote the disk together with the Farey tessellation by $\mathbb{D}_F$. Note that the Farey tessellation is a tessellation of $\mathbb{D}$ by ideal triangles (i.e. triangles with vertices on the boundary $\mathbb{D}_F$). In fact, the set of vertices of the triangles is exactly $\mathbb{Q}\cup\{\infty\}$. Moreover, two fractions $\frac{m_1}{n_1}, \frac{m_2}{n_2}$ are connected by a *line* if and only if $m_1n_2 - m_2n_1 = \pm 1$. Hence, the action of $GL(2,\mathbb{Z})$ on $\mathbb{D}$ induces an action on $\mathbb{D}_F$ which is transitive on the geodesics of $\mathbb{D}_F$. Only $\pm I$ acts as the identity; hence, the modular group $PGL(2,\mathbb{Z}) = GL(2,\mathbb{Z})/\pm I$ is the symmetry group of $\mathbb{D}_F$ where the subgroup $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\pm I$ gives the orientation preserving symmetries. In what follows we denote by Γ the triangle with vertices $\{0, 1, \infty\}$. Note that Γ splits in 3 copies of the fundamental region.

The fixed points of the modular action of a matrix $A \in PSL(2,\mathbb{Z})$, $A \neq I$, in $\mathbb{D}_F$ are solutions of $z = \frac{az+b}{cz+d}$. This gives a quadratic equation with the discriminant $tr(A)^2 - 4$. If the trace $|tr(A)| < 2$, then the discriminant is negative and the modular action is a rotation around an imaginary point (an interior point of $\mathbb{D}_F$). Such matrices are called *elliptic*. If $|tr(A)| = 2$, then the discriminant vanishes, and A acts as a translation with one fixed rational point, $\frac{d-a}{2}$ (on the boundary of $\mathbb{D}_F$). Such matrices are called *parabolic*. The *hyperbolic matrices* have $|tr(A)| > 2$ and define a translation of $\mathbb{D}_F$ with two fixed quadratically irrational real points on the boundary of $\mathbb{D}_F$.

6. Conjugacy classes of elliptic and parabolic matrices

Elliptic matrices: As mentioned above an elliptic matrix $A \in PSL(2,\mathbb{Z})$ acts on $\mathbb{D}_F$ as a rotation around a point in the interior of $\mathbb{D}_F$. The center of the rotation belongs to one of the triangles of the tessellation. Without loss of generality, let us assume that the fixed point belongs to the triangle Γ. If the fixed point belongs to an edge of Γ, then A rotates Γ by angles $\pm\pi$. Let us denote such rotations by $E_{\pm\pi}$. The other possibility is the rotations $E_{\pm\frac{2\pi}{3}}$ by angles $\pm\frac{2\pi}{3}$ around the center of Γ, see Fig. 3. It is not hard to see that the matrices $E_{\pm\pi}$ (respectively $E_{\pm\frac{2\pi}{3}}$) are conjugate to each other via an orientation reversing matrix in $PGL(2,\mathbb{Z})$.

Since $PGL(2,\mathbb{Z})$ acts transitively on the triangles of the tessellation, $E_\pi = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $E_{\frac{2\pi}{3}} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ represent the conjugacy classes in $PGL(2,\mathbb{Z})$ of elliptic matrices in $PSL(2,\mathbb{Z})$.

Each matrix A in $PSL(2,\mathbb{Z})$ defines two matrices $\pm A$ in $SL(2,\mathbb{Z})$. It is not hard to see that the matrices $\pm E_\pi$ are conjugate to each other via reflection with respect to the edge containing the fixed point while $\pm E_{\frac{2\pi}{3}}$ are not, simply by the fact that they have different traces. Hence, there are three conjugacy classes (represented by $E_\pi, \pm E_{\frac{2\pi}{3}}$) in $GL(2,\mathbb{Z})$ of elliptic matrices in $SL(2,\mathbb{Z})$.

Parabolic matrices: The fixed point of the action of a parabolic matrix in $PSL(2,\mathbb{Z})$ is rational; thus, it is a common vertex of an infinite set of the triangles of $\mathbb{D}_F$. Since $PGL(2,\mathbb{Z})$ acts transitively on the rational points, it is not restrictive to assume that the fixed point of the translation is 0.

Hence, a parabolic element can shift the triangle Γ by an arbitrary number n of triangles to the right or to the left (see Fig. 4) fixing the point 0. The left shift is conjugated to the right shift by the reflection with respect to the vertical line. Hence, the equivalence classes in $PGL(2,\mathbb{Z})$ are determined by the number n of shifts. Such a shift can be represented by the matrix $P_n = \left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$, $n \in \mathbb{N}$.

The matrix $P_n \in PSL(2,\mathbb{Z})$ corresponds to two matrices $\pm P_n$ in $SL(2,\mathbb{Z})$. Having traces with opposite signs, they belong to two different conjugacy classes which are, thus, determined by the integer $\pm n$. Representatives of the conjugacy classes can be chosen as $\pm\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$, $n \in \mathbb{N}$.

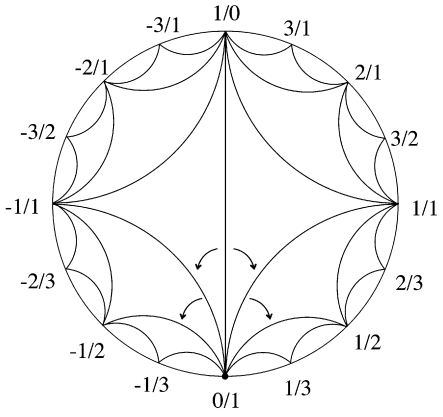

Fig. 4. Modular actions of parabolic matrices P_n.

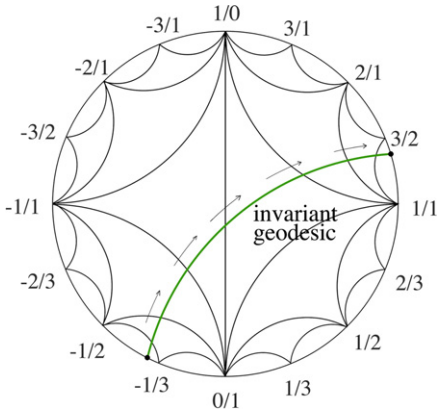

Fig. 5. Modular action of a hyperbolic matrix.

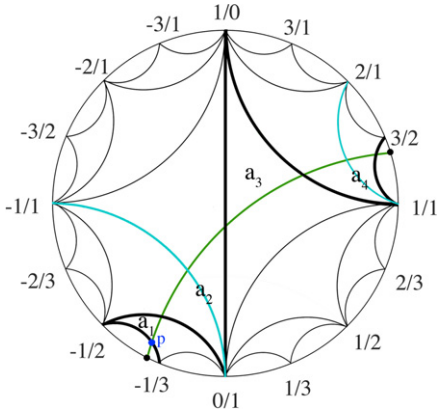

Fig. 6. Periodic pattern of the truncated triangles of the Farey tessellation.

7. Conjugacy classes of hyperbolic matrices

A hyperbolic matrix $A \in PSL(2,\mathbb{Z})$ acts on $\mathbb{D}_F$ as translation fixing two irrational points. The geodesic (a semi-circle), l_A, connecting these fixed points, oriented in the direction of translation, remains invariant under the translation, so A preserves also the set of the triangles of $\mathbb{D}_F$ which are cut by l_A, see Fig. 5.

With respect to the orientation of l_A (orientation is defined by the action of A), such triangles are situated in two different ways: a set of triangles with a common vertex lying on the left of l_A followed by a set of triangles with common vertex lying on the right of l_A, see Fig. 6.

Let us label right and left triangles by R and L, respectively. Then we encode the arrangement of left and right triangles with respect to l_A as an infinite word, $\dots LL\dots LRR\dots RLL\dots L\dots$, of 2 letters. This word is called the *cutting word* of l_A. Let us fix a point p at the intersection of l_A with an edge of a triangle. Relative to this point, we obtain a sequence, $(a_1, a_2, a_3, \dots)_p$, from the cutting word where a_{2i-1} stands for the number of consecutive triangles of one type while a_{2i} is the number of consecutive triangles of the other type. For example, if the cutting word with respect to p reduced to the word $\underbrace{LL\dots L}_{a_1}\underbrace{RR\dots R}_{a_2}\underbrace{LL\dots L}_{a_3}\dots = L^{a_1}R^{a_2}L^{a_3}\dots$, then we obtain $(a_1, a_2, \dots)_p$. This sequence is called the *cutting sequence* relative to the point p.

Left and right triangles form a periodic pattern, and the action of A is a shift by a period. Since the choice of p is arbitrary the cutting sequence relative to p is periodic after possibly some finite terms. Moreover, its period is of even length. Note that the choice of the point p is not canonical; hence, we can encode the period only as a cycle, $[a_1a_2\dots a_{2n-1}a_{2n}]_A$, which we call the *cutting period-cycle associated to the matrix A*.

Because of the fact that $PGL(2,\mathbb{Z})$ is the full symmetry group of $\mathbb{D}_F$, the cutting period-cycle of a hyperbolic matrix $A \in PSL(2,\mathbb{Z})$ gives the complete invariant of the conjugacy class in $PGL(2,\mathbb{Z})$ of A. In other words, two matrices $A, B \in PSL(2,\mathbb{Z})$ are in the same conjugacy class in $PGL(2,\mathbb{Z})$ if and only if $[a_1a_2\dots a_{2n}]_A = [a_{\sigma(1)}a_{\sigma(2)}\dots a_{\sigma(2n)}]_B$ for a cyclic permutation σ. Hence, we will denote the conjugacy classes in $PGL(2,\mathbb{Z})$ of hyperbolic matrices of $PSL(2,\mathbb{Z})$ by the cycle $[a_1a_2\dots a_{2n}]$ (defined up to cyclic ordering).

It can be seen geometrically that with respect to the triangle Γ a matrix representing a translation corresponding to the cutting period-cycle $[a_1a_2\dots a_n]$ can be chosen as the following product of parabolic matrices.

$$\begin{pmatrix}1 & a_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ a_2 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & a_{2n-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ a_{2n} & 1\end{pmatrix}.$$

For the sake of simplicity, let us denote $U = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$. Then the above product is written as $U^{a_1}V^{a_2}\dots V^{a_{2n}}$. Note that U is conjugate to V in $PGL(2,\mathbb{Z})$ but not in $PSL(2,\mathbb{Z})$.

Let us note that in certain cases, namely if l_A intersects the vertical line of $\mathbb{D}_F$ (since the action of $PGL(2,\mathbb{Z})$ is transitive on the geodesics of $\mathbb{D}_F$, up to conjugation this property is always satisfied), the cutting sequence of l_A with respect to the point of intersection of l_A with the vertical line is related to the continued fraction expansion of the fixed point ξ which is the "end point" of l_A with respect to the orientation. The corresponding theorem is due to C. Series [3,4].

Theorem 5. *([3,4]) Let $x > 1$, and let l be any geodesic ray joining some point p on the vertical line of $\mathbb{D}_F$ to x, oriented from p to x. Suppose that the cutting word of l with respect to p is $L^{a_1}R^{a_2}L^{a_3}\dots$. Then $x = a_1 + \frac{1}{a_2+\frac{1}{a_3+\cdots}}$.*

Note that if $0 < x < 1$, then the sequence starts with R and $x = \frac{1}{a_1+\frac{1}{a_2+\frac{1}{a_3+\cdots}}}$.

If $x < 0$ everything applies with x replaced by $-x$ and with R and L interchanged.

A matrix $A \in PSL(2,\mathbb{Z})$ corresponds to $\pm A$ in $SL(2,\mathbb{Z})$. Since $\pm A$ have traces with opposite signs, the cutting period-cycle $[a_1a_2\dots a_{2n}]$, together with the sign of the trace determine the conjugacy classes of $\pm A$ in $GL(2,\mathbb{Z})$. Representatives of them can be chosen as $\pm U^{a_1}V^{a_2}\dots V^{a_{2n}}$.

8. Real factorization of elliptic and parabolic matrices

The modular action of the linear real structures $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ on the hyperbolic plane $\mathbb{D}_F$ is $z \mapsto -\bar{z}$ and $z \mapsto \frac{1}{\bar{z}}$, respectively. Geometrically, these are the reflections with respect to the vertical and respectively the horizontal lines, see Fig. 7. In particular, the first reflection takes the basic triangle Γ to the triangle with vertices $\{0, -1, \infty\}$, and the second one takes Γ to itself.

Theorem 6. *All elliptic and parabolic matrices in $SL(2,\mathbb{Z})$ are products of two linear real structures.*

Proof. The explicit real decompositions for the representatives of the conjugacy classes of the elliptic matrices are given below:

$$E_{\frac{2\pi}{3}} = \begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$-E_{\frac{2\pi}{3}} \cong \begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$E_{\pi} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

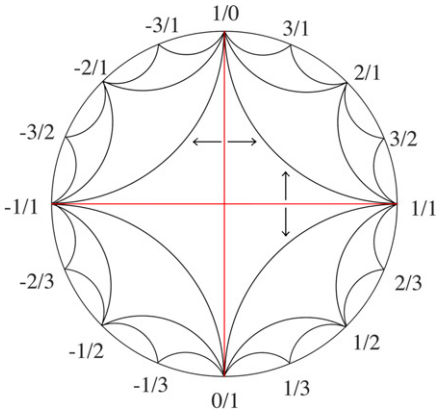

Fig. 7. Modular actions of the linear real structures.

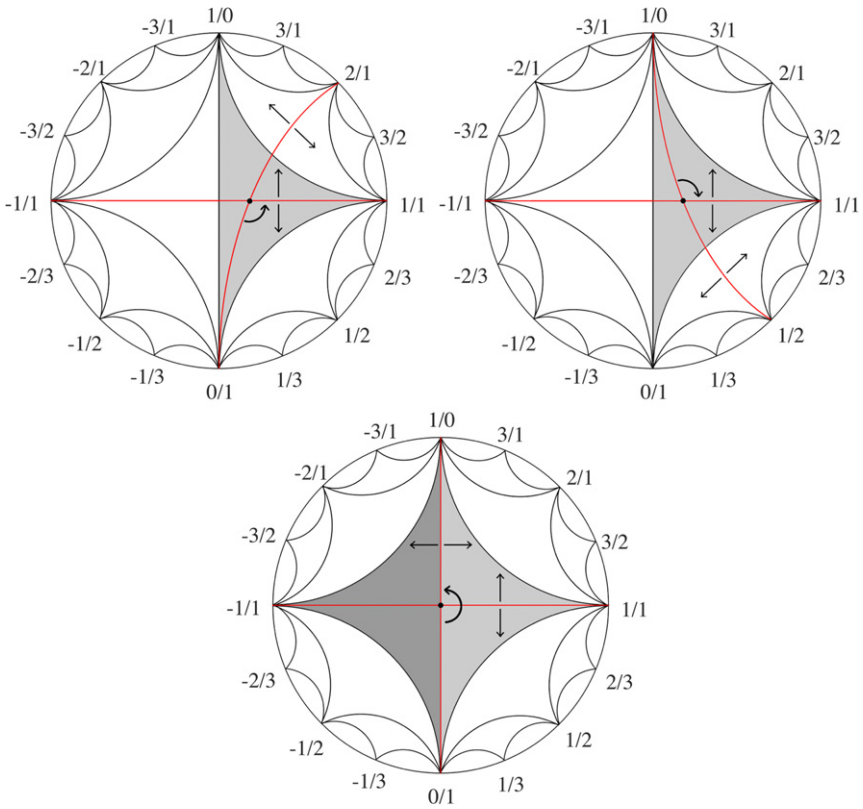

Fig. 8. Decompositions of the modular actions of the elliptic matrices.

Fig. 8 illustrates geometrically the above decompositions in terms of the corresponding modular action of the matrices. Real decompositions for the representatives of the conjugacy classes of the parabolic matrices are as follows:

$$P_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$-P_n = \begin{pmatrix} -1 & 0 \\ -n & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \square$$

Fig. 9 shows the real decompositions of the modular action of matrices $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$, $n = 1, 2$.

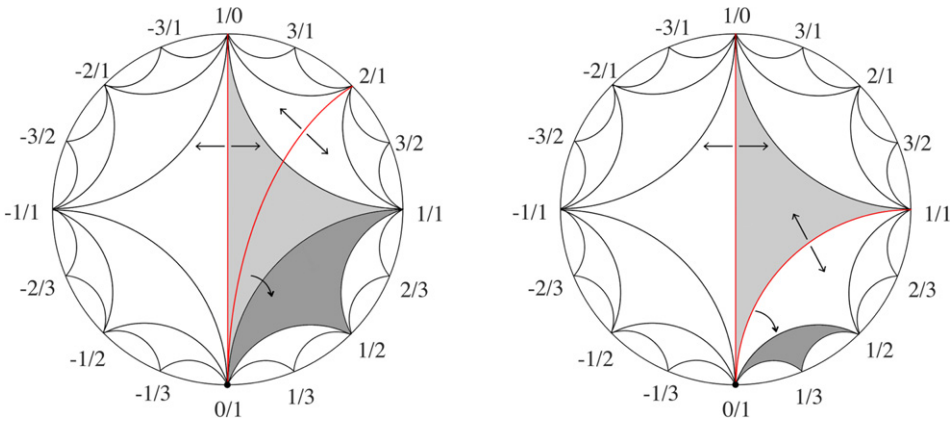

Fig. 9. Decompositions of the modular actions of the parabolic matrices P_1, P_2.

9. The criterion of factorizability for hyperbolic matrices

Lemma 7. *If the cutting period-cycle of a hyperbolic matrix A is $[a_1a_2\dots a_{2n}]$, then the cutting period-cycle of A^{-1} is $[a_{2n}a_{2n-1}\dots a_1]$.*

Proof. Note that $l_A = l_{A^{-1}}$ with the opposite orientation. So the cutting word of A^{-1} can be obtained from the cutting word of A by taking the mirror image of the word and interchanging L with R. Interchanging L and R does not effect the cutting period-cycle; hence, the cutting period-cycle of A^{-1} is the reverse $[a_{2n}a_{2n-1}\dots a_1]$ of the cutting period-cycle $[a_1a_2\dots a_{2n}]$ of A. □

Definition. A finite sequence $(a_1a_2\dots a_k)$ is called *palindromic* if it is equal to the reversed sequence $(a_ka_{k-1}\dots a_1)$. We call k the *length of the sequence*.

A finite sequence is called *bipalindromic* if it can be subdivided into two palindromic sequences.

Definition. A cutting period-cycle is called *bipalindromic* if there is a cyclic permutation such that the permuted period is bipalindromic.

In particular, if the cutting period-cycle is subdivided into two palindromic sequences of odd length (respectively even length) we call it *odd-bipalindromic* (respectively *even-palindromic*).

To illustrate, the period [1213] is odd-bipalindromic, while the period [1122] is even-bipalindromic.

Lemma 8. *If $A^{-1} = Q^{-1}AQ$ for some $Q \in PGL(2,\mathbb{Z})$, then the cutting period-cycle $[a_1a_2\dots a_{2n}]_A$ is bipalindromic.*

Proof. If A and A^{-1} are in the same conjugacy class in $PGL(2,\mathbb{Z})$, then they have the same cutting period-cycles up to cyclic permutation. By Lemma 7 we know that the cutting period-cycle of A^{-1} is $[a_{2n}a_{2n-1}\dots a_1]$ while the cutting period-cycle of A is $[a_1a_2\dots a_{2n}]$. Hence, $[a_{\sigma(1)}a_{\sigma(2)}\dots a_{\sigma(2n)}] = [a_{2n}a_{2n-1}\dots a_1]$ for some cyclic permutation σ. Without loss of generality, let us assume that σ is a shift by $k = 2n - l$, so we have $(a_{l+1}a_{l+2}\dots a_{2n}a_1a_2\dots a_l) = (a_{2n}a_{2n-1}\dots a_{l+1}a_la_{l-1}\dots a_1)$ as finite sequences. Thus, $(a_{l+1}a_{l+1}\dots a_{2n}) = (a_{2n}a_{2n-1}\dots a_{l+1})$ and $(a_1a_2\dots a_l) = (a_la_{l-1}\dots a_1)$ which implies that the cutting period-cycle is bipalindromic. □

Note that if the cutting period-cycle is odd-bipalindromic, then the symmetry of the palindromic pieces lifts to a symmetry of the left and the right triangles corresponding to the cutting period-cycle. However, this is not true for even-bipalindromic periods. In the case of [1213], we have $121 \sim LR^2L = LRRL$ and $3 \sim R^3 = RRR$, whereas of [1122], we have $11 \sim LR$ and $22 \sim L^2R^2 = LLRR$.

Theorem 9. *A hyperbolic matrix A is a product of two linear real structures if and only if its cutting period-cycle $[a_1a_2\dots a_{2n}]_A$ is odd-bipalindromic.*

Lemma 10. *Let $A \in PSL(2,\mathbb{Z})$ such that $A^{-1} = Q^{-1}AQ$ for some $Q \in PGL(2,\mathbb{Z})$, and let l_A be the geodesic invariant under the action of A. Then $Q(l_A) = l_A$.*

Proof. Clearly, if $A(l_A) = l_A$, then $A^{-1}(l_A) = l_A$. Hence,

$$A^{-1}(l_A) = Q^{-1}AQ(l_A) \quad \Leftrightarrow \quad Q(l_A) = A\big(Q(l_A)\big).$$

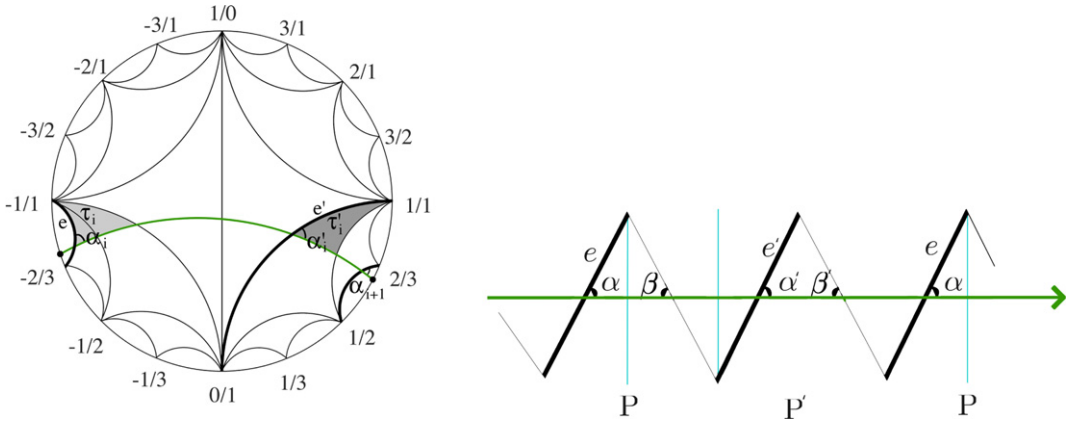

Fig. 10. Triangles τ_i and τ_i' and the interior angles α_i, α_i', β_i, β_i'.

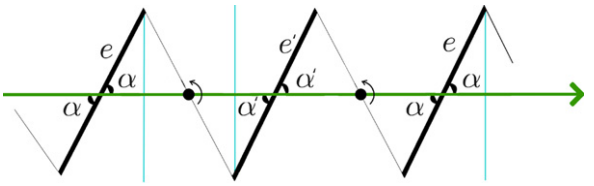

Fig. 11. Elliptic rotations of even-bipalindromic cutting period-cycles.

By the uniqueness of the invariant geodesic we obtain $Q(l_A) = l_A$. □

Lemma 11. *Let A, Q, l_A be as above. If the cutting period-cycle $[a_1 a_2 \dots a_{2n}]_A$ is even-bipalindromic, then Q is orientation preserving.*

Proof. By means of Lemma 10, we have $Q(l_A) = l_A$; hence, Q preserves triangles meeting l_A. The action of Q on $\mathbb{D}_F$ is a linear fractional transformation; thus, it preserves the angles. An analysis on the angles at the meeting points of l_A and the edges of the triangles will forbid the existence of the orientation reversing map when the cutting period-cycle is even-bipalindromic.

Let us assume that the cutting period-cycle has the form:

$$\big[\underbrace{a_1 a_2 \dots a_k a_k \dots a_2 a_1}_{P}\,\underbrace{a_1' a_2' \dots a_s' a_s' \dots a_2 a_1'}_{P'}\big]$$

where $s + k = n$ and P and P' are two palindromic pieces. Substituting the pieces P, P' to the cutting sequence, we obtain a sequence of P and P' of the form $\dots PP'PP'\dots$. Clearly, the action of the matrix A corresponds to a shift by two: it takes P to P, P' to P'. Let us consider the edges which separate the triangles corresponding to P from the triangles corresponding to P'. They are of two types: with respect to the orientation of l_A, we encounter the edges where we move from P to P', and inversely, the edges where we pass from P' to P. We denote such edges by e_i and e_i' respectively, see Fig. 10. (We enumerate the edges with respect to an auxiliary point p fixed on l_A.)

Each triangle of $\mathbb{D}_\mathbb{F}$ which is cut by l_A splits into two pieces, one of which is a triangle. Let τ_i (respectively τ_i') denote the triangle having one edge e_i (respectively e_i') and obtained as the union of triangle-pieces of the triangles of $\mathbb{D}_F$ with a common vertex on one side of l_A (respectively on the other side of l_A), see Fig. 10. Let α_i (respectively α_i') be the interior angles of τ_i (respectively τ_i') between the edges e_i (respectively e_i') and l_A. In addition, let β_i (respectively β_i') be the other interior angle of τ_i (respectively τ_i') on l_A.

Note that since A shifts triangles by the period PP', A takes α_i to α_{i+1} (respectively α_i' to α_{i+1}'). Hence, all α_i (respectively all α_i') are equal. Let $\alpha = \alpha_i$ for all i (respectively $\alpha' = \alpha_i'$ for all i).

The crucial observation is that if the cutting period cycle is even-bipalindromic, then there is an elliptic matrix in the conjugacy classes of E_π which fixes the point of intersection of l_A with the middle edge of P or P', see Fig. 11. Such matrix interchanges the edges e_i to e_i'. Hence, $\alpha = \alpha'$. (In the same way, we obtain $\beta = \beta_i = \beta_i'$ for all i.)

Let us assume that $\alpha < \frac{\pi}{2}$ (otherwise, we can replace α with β). We choose an orientation of $\mathbb{D}_F$ by specifying (v_1, v_2) where v_1 is a tangent vector of l_A and v_2 is the tangent vector of e_i (or e_i') such that the angle α between v_1 and v_2 is $\alpha < \frac{\pi}{2}$. The matrix Q takes (v_1, v_2) to itself since it preserves l_A and the set of edges e_i, e_i'; hence, it preserves the angles between the two. However, an orientation reversing map can not preserve both the angle $\alpha < \frac{\pi}{2}$ between the vectors (v_1, v_2) and the vectors v_1, v_2 at the same time. Thus, Q is orientation preserving. □

Proof of Theorem 9. If A is a product of two linear real structures. Then the cutting period-cycle is odd-bipalindromic by Lemma 11.

If the cutting period-cycle is odd-bipalindromic, then up to cyclic ordering, it has two palindromic pieces of odd length. Let us assume that the cutting period-cycle is of the form

$$\pm\left[a_1a_2\dots a_ka_{k+1}a_k\dots a_2a_1a_1'a_2'\dots a_s'a_{s+1}'a_s'\dots a_2'a_1'\right]$$

where $(2k+1)+(2s+1)=2n$. Then for some $R\in PGL(2,\mathbb{Z})$, we have $B=R^{-1}AR$ such that $B=\pm U^{a_1}V^{a_2}\dots U^{a_2}V^{a_1}\times U^{a_1'}V^{a_2'}\dots U^{a_2'}V^{a_1'}$. Matrices U^{a_i} and V^{a_i} have the following real decompositions:

$$U^{a_i}=\begin{pmatrix}1&-a_i\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad\text{and}\quad V^{a_i}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\-a_i&-1\end{pmatrix}.$$

Hence, the product $U^{a_1}V^{a_2}\dots U^{a_2}V^{a_1}$ can be rewritten as follows:

$$\begin{pmatrix}1&-a_1\\0&-1\end{pmatrix}\cdots\begin{pmatrix}1&0\\-a_k&-1\end{pmatrix}\begin{pmatrix}1&-a_{k+1}\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\-a_k&-1\end{pmatrix}\cdots\begin{pmatrix}1&-a_1\\0&-1\end{pmatrix}.$$

This gives a linear real structure, since it is a conjugate of $\left(\begin{smallmatrix}1&-a_{k+1}\\0&-1\end{smallmatrix}\right)$. Similarly, the product $U^{a_1'}V^{a_2'}\dots U^{a_2'}V^{a_1'}$ gives a linear real structure conjugate to $\left(\begin{smallmatrix}1&-a_{s+1}\\0&-1\end{smallmatrix}\right)$. □

Theorem 12. *A matrix $A\in SL(2,\mathbb{Z})$ is a product of two linear real structures if and only if there is a matrix Q with* $\det Q=-1$ *such that $A^{-1}=Q^{-1}AQ$.*

Proof. Necessity of the condition is trivial. As for the converse, we only need to consider the case of hyperbolic matrices. Let A be a hyperbolic matrix such that $A^{-1}=Q^{-1}AQ$, for some Q with $\det Q=-1$. Then the cutting period-cycle $[a_1a_2\dots a_{2n-1}a_{2n}]_A$ is odd-bipalindromic by Lemma 11. So A is real by Theorem 9. □

Theorem 12 together with Proposition 4 lead to the following corollary.

Corollary 13. *An elliptic F-fibration is real if and only if it is weakly real.*

Acknowledgements

This work is a part of my PhD thesis where we introduced real Lefschetz fibrations and gave a classification of real elliptic Lefschetz fibrations over S^2 with only real critical values. I would like to express my gratitude to my supervisors Sergey Finashin and Viatcheslav Kharlamov for their guidance and endless support throughout my research.

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