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A Generalized Firing Squad Problem*

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This article presents a generalization of the Firing Squad Synchronization Problem in which the General may be any soldier. A seventeen-state minimum time solution is given.

INTRODUCTION

The firing squad problem, well known in switching and automata theory (Moore, 1964), concerns a finite (but arbitrarily long) one-dimensional array of finite-state machines (or cells), all of which are alike except possibly those at each end. One of the end cells is the General. For this problem, Waksman (1966) has obtained a minimum time solution which takes $2n - 2$ time units for an n-cell firing squad.

In this article, the authors consider a generalization which allows the General to be located anywhere. Under these conditions, it is shown to be impossible to have a solution which takes less than $2n - 2 - k$ time units to fire (where k is the number of cells on the General's side closest to an end cell). In addition, a minimum time solution that takes exactly $2n - 2 - k$ time units is presented. Thus, when the General is not on the end, the time to fire is less than $2n - 2$.

FIRING SQUAD PROBLEM

Figure 1 shows a finite one-dimensional iterative array of n finite-state cells (soldiers). The letter n denotes the length (number of cells) of the firing squad. Note the cell numbering convention. Each cell (machine) is identical, except perhaps the end cells. The time variable t is discrete, taking on values from the integers. Thus the array operates in the synchronous mode with the next state of a cell being determined by both its

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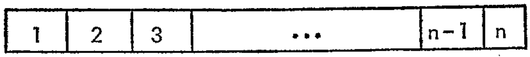

FIG. 1. Firing squad of length n

own present state and the present state of its right and left neighbors. Conceptually, the sequential cell's "inputs" are its neighbor's state variables.

One of the states is the quiescent state, Q. The next state function has the property that the next state of a quiescent cell with quiescent neighbors is again the quiescent state. All soldiers are initially in the quiescent state at time $t = 0$ except cell 1, the General, which is in the "Fire-When-Ready" state. The problem is to specify the structure (states and next-state function) of the cell such that, independent of the number n of cells, all cells enter the "Fire" (terminal) state at exactly the same time. Thus, the same cell structure with a fixed and finite number of states must accomplish this, regardless of the length of the firing squad.

Since signals can propagate no faster than one cell per time unit, the minimum time for all cells to fire is $2n - 2$ time units. This is the time it takes for a signal to propagate from cell 1 to cell n and back to cell 1.

A GENERALIZATION

Consider the same problem in which the General is not necessarily an end cell. Let the letter k be the number of cells between the General and the nearest end (if the General is either cell 3 or cell 8 in a squad of 10 soldiers, $k = 2$). The value of k can never be greater than

$$\left[\frac{n+1}{2}\right] - 1,$$

where $[N]$ equals the largest integer less than N.

The terms "signal," "propagate" and "generate" will be used in the sense of Waksman. For example, a "left propagating signal" may be a state or subset, L, of states with a particular property. A cell assumes a state in the subset L following a time unit (not necessarily the earliest possible) that its right neighbor assumed a state in the subset L. The cell assumes a state not in the subset L only when (1) its left neighbor has entered a state in the subset L, or (2) a neighbor takes on a state calling for the annihilation of L. A "right propagating signal" can be similarly defined. If a cell enters a state in L without its right neighbor having propagated it, the cell is said to "generate" the signal. A left propagating signal of slope 1 is a signal which is propagated at the rate one cell per time unit. Obviously, signals can propagate no faster than this. The use of *slope* in this article corresponds to Waksman's "time unit delay/machine." Conceptually, a slope 3 signal is a state passed from neighbor to neighbor, with each cell keeping the state for 3 time units. In Waksman's solution, signals of slope 3, 7, 15 , $\cdots$, $2^{l+1} - 1$, $\cdots$ play an important role.

THEOREM. *The minimum time in which the firing squad could possibly fire in the general case is no earlier than* $2n - 2 - k$ *time units.* [*When the General is at an end,* k *is equal to zero and the formula reduces to* $2n - 2$, *which is the minimum time for the original problem* (Moore, 1964).

Proof. Without loss of generality, for a given k, it is assumed that the General is to the left of center (cell $k + 1$ rather than cell $n - k$). Cell 1 is then the "near end cell" with respect to the General—the critical cell.

To secure an absurdity, assume that there is a cell structure S for the general problem

$$n \text{ arbitrary, with } 1 \leqq k \leqq \left[\frac{n+1}{2}\right] - 1$$

for which there is some length n_o and cell number $k_o + 1$ for the General, such that the firing squad fires at time $t = m$ where $m < 2n_o - 2 - k_o$. This means that cell 1 could not have received a signal from cell n_o, because it takes at least $n_o - k_o - 1$ time units for a signal from the General to reach cell n_o, plus at least $n_o - 1$ time units for a signal from cell n_o to reach cell 1. Therefore, cell 1 has entered the "fire" state unaffected by cell n_o. In other words, cell 1 has "fired" independent of cell n_o, and for that matter, anything to the right of cell n_o. Therefore, if another $n_o + 2$ cells were added to the problem (the right end now being cell $2n_o + 2$), cell 1 would still fire at time m. This is because the cell structure is fixed, cell operation is deterministic and nothing has changed as far as cell 1 is concerned. Since $m < 2n_o - 2 - k_o$, cell $2n_o + 2$ is still in the quiescent state at time $t = m$. Therefore cell structure S does not represent a solution, and a contradiction has been obtained. The argument carries over *mutatis mutandis* when the General is more conservative and is located at cell $n_o - k_o$, the critical cell (cell n_o) now being at the right end.

WAKSMAN'S SOLUTION

Waksman discovered a clever way to cause a cell to generate (via auxiliary R-signals) signals of slope 1, 3, 7, 15 , $\cdots$, $2^l - 1$, $\cdots$. Call such a cell a K-cell (corresponding to Waksman's P_0 or P_1 state), and such signals B-signals. Waksman's method of generating right propagating B-signals of slope 3, 7, 15, $\cdots$, $2^l - 1$, $\cdots$ will be explained. (The left propagating B-signals are generated analogously.) Call the states comprising the B-signal A (Waksman's B_0) and B (Waksman's B_1). When the K-cell first goes to state K it emits a right propagating signal I (Waksman's state A) of slope 1. Every second cell in the I state generates a left propagating signal of slope 1. An important observation here is that there will be three time units separation between R-signals going through each cell. The purpose of the R-signal is to shift the B-signals one cell to the right; thus the slope 3 B-signal shifts one cell to the right every third time unit as desired. However, a B-signal in state A allows the R-signal to pass; a B-signal in state B does not. Furthermore, a B state shifts right to the A state, and an A state changes to the B state in shifting right. Thus, every other R-signal intersecting a B-signal is allowed to pass through. The first R-signal generates an A state to the right of the K-cell. The second R-signal shifts it right three time units later as a B-cell. This slope 3 B-signal allows every other R-signal through so the new separation of R-signals will be $(2 \times 3) + 1 = 7$ time units. Similarly, the R-signal separation through the slope 7 B-signal will be $(2 \times 7) + 1 = 15$ time units. The generation of new B-signals continues in this manner.

Consider Fig. 2. At time $t = 0$, cell 1 (the General) becomes a K-cell. At time $t = n - 1$, a B-signal of slope 1 strikes cell n and cell n becomes a K-cell. Let two intersecting B-signals create a K-cell. Then when the right propagating B-signal of slope 3 from cell 1 strikes the left propagating B-signal of slope 1 from cell n, the center cell of the firing squad becomes a K-cell, and begins generating both right and left propagating B-signals. (The B-signals retain odd or even parity information so that if n is even, both cell $n/2$ and cell $n/2 + 1$ become K-cells simultaneously.) Intersecting B-signals continue to subdivide the interval (defining the quarter points, eighth points, etc.) with K-cell such that at the time $t = (2n - 2) - 2$, each non-K-cell has as a neighbor at least one K-cell, and each K-cell has as a neighbor at least one non-K-cell. At $t = (2n - 2) - 1$, all cells are K-cells. Since the next state of a K-cell both of whose neighbors are K-cells is the only way to enter the "fire" state "T", the firing squad soldiers simultaneously "fire" at time $t = 2n - 2$.

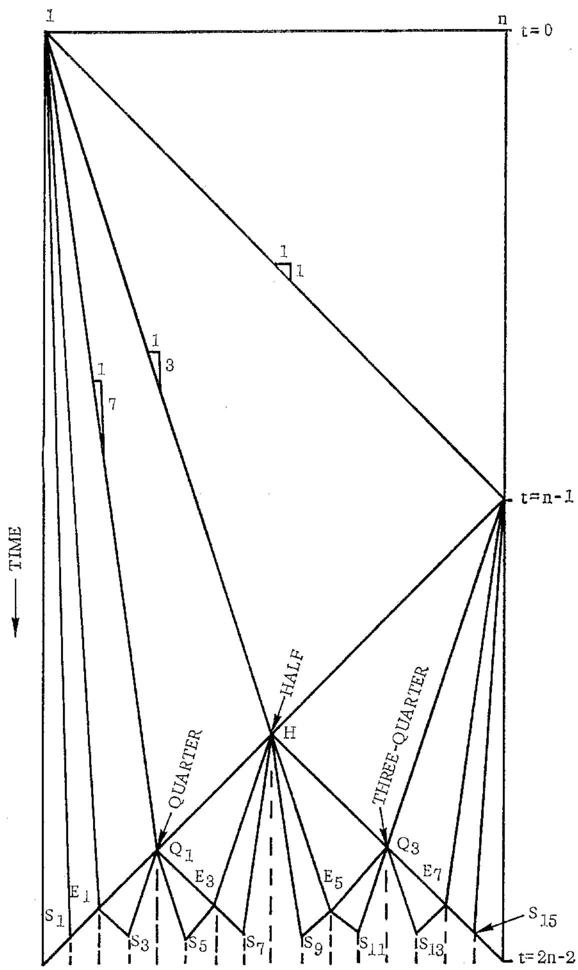

FIG. 2. Geometry for Waksman's solution

SOLUTION FOR $t = 2n - 2 - k$

In the light of the proof that the solution to the general problem cannot have the soldiers firing earlier than the time required to propagate a signal from the General to the far end and back to the near end, the following solution in which the soldiers fire at exactly time $t = 2n - 2 - k$ is presented. Contrary to what might be expected, when the General is not on the end, the minimum time is less than $2n - 2$.

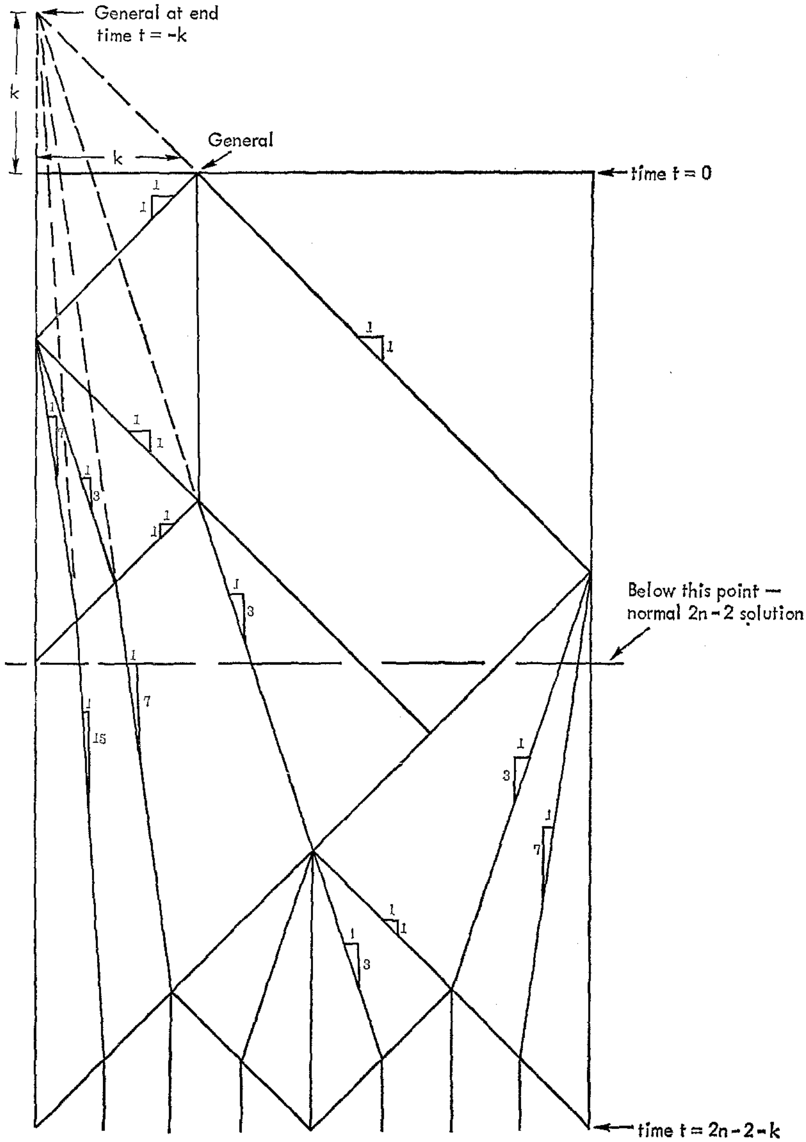

FIG. 3. Geometry for liberal General

The main idea behind the solution is to "reconstruct" Waksman's "*B*" and "*R*" signals (Waksman, 1968) as if the General had been at the near end and had entered the "Fire-When-Ready" state at time $t = -k$.

Figure 3 shows the geometry for a liberal General. Dotted lines show the signals which were reconstructed. A typical solution for $n = 26$, $k = 8$ is shown in Fig. 4. The state tables shown in Fig. 5 (one for each state) should be interpreted as follows: The upper left hand corner denotes the "present state," the column heading is the right neighbor's state, and row headings denote the left neighbor's state. The entry is the cell's next state. If we are searching for rule *ABC*, where

$$A = \text{left neighbor's state,}$$

$$B = \text{present state,}$$

$$C = \text{right neighbor's state,}$$

```
Q Q Q Q Q Q Q Q P Q Q Q Q Q Q Q Q Q Q Q Q Q Q Q Q Q  - t=0
Q Q Q Q Q Q Q L D S Q Q Q Q Q Q Q Q Q Q Q Q Q Q Q Q
Q Q Q Q Q Q L Q D Q S Q Q Q Q Q Q Q Q Q Q Q Q Q Q Q
Q Q Q Q Q L Q Q D Q Q S Q Q Q Q Q Q Q Q Q Q Q Q Q Q
Q Q Q Q L Q Q Q D Q Q Q S Q Q Q Q Q Q Q Q Q Q Q Q Q
Q Q Q L Q Q Q Q D Q Q Q Q S Q Q Q Q Q Q Q Q Q Q Q Q
Q Q L Q Q Q Q Q D Q Q Q Q Q S Q Q Q Q Q Q Q Q Q Q Q
Q L Q Q Q Q Q Q D Q Q Q Q Q Q S Q Q Q Q Q Q Q Q Q Q
K Q Q Q Q Q Q Q D Q Q Q Q Q Q Q S Q Q Q Q Q Q Q Q Q
K I Q Q Q Q Q Q D Q Q Q Q Q Q Q Q S Q Q Q Q Q Q Q Q
K R I Q Q Q Q Q D Q Q Q Q Q Q Q Q Q S Q Q Q Q Q Q Q
K A Q I Q Q Q Q D Q Q Q Q Q Q Q Q Q Q S Q Q Q Q Q Q
K A Q R I Q Q Q D Q Q Q Q Q Q Q Q Q Q Q S Q Q Q Q Q
K A R Q Q I Q Q D Q Q Q Q Q Q Q Q Q Q Q Q S Q Q Q Q
K R B Q Q R I Q D Q Q Q Q Q Q Q Q Q Q Q Q Q S Q Q Q
K A B Q R Q Q I D Q Q Q Q Q Q Q Q Q Q Q Q Q Q S Q Q
K A B R Q Q Q R X Q Q Q Q Q Q Q Q Q Q Q Q Q Q Q S Q
K A Q A Q Q R Q B W Q Q Q Q Q Q Q Q Q Q Q Q Q Q Q K
K A Q A Q R Q Q B R W Q Q Q Q Q Q Q Q Q Q Q Q Q G K
K A Q A R Q Q Q Q A Q W Q Q Q Q Q Q Q Q Q Q Q G H K
K A Q R B Q Q Q Q A Q R W Q Q Q Q Q Q Q Q Q G Q A K
K A R Q B Q Q Q Q A R Q Q W Q Q Q Q Q Q Q G H Q A K
K R B Q B Q Q Q Q R B Q Q R W Q Q Q Q Q G Q Q H A K
K A B Q B Q Q Q R Q B Q R Q Q W Q Q Q G H Q Q B H K
K A B Q B Q Q R Q Q B R Q Q Q R W Q G Q Q H Q B A K
K A B Q B Q R Q Q Q Q A Q Q R Q Q G H Q Q Q H B A K
K A B Q B R Q Q Q Q Q A Q R Q Q G Q Q H Q Q A Q A K
K A B Q Q A Q Q Q Q Q A R Q Q G H Q Q Q H Q A Q A K
K A B Q Q A Q Q Q Q Q R B Q G Q Q H Q Q Q H A Q A K
K A B Q Q A Q Q Q Q R Q B G H Q Q Q H Q Q B H Q A K
K A B Q Q A Q Q Q R Q Q K K Q H Q Q Q H Q B Q H A K
K A B Q Q A Q Q R Q Q G K K I Q H Q Q Q H B Q B H K
K A B Q Q A Q R Q Q G H K K R I Q H Q Q A Q Q B A K
K A B Q Q A R Q Q G Q A K K A Q I Q H Q A Q Q B A K
K A B Q Q R B Q G H Q A K K A Q R I Q H A Q Q B A K
K A B Q R Q B G Q Q H A K K A R Q Q I B H Q Q B A K
K A B R Q Q K H Q Q B H K K R B Q Q R K Q H Q B A K
K A Q A Q G K I H Q B A K K A B Q R G K I Q H B A K
K A Q A G H K R I H B A K K A B R G H K R I A Q A K
K A Q K Q A K A Q K Q A K K A Q K Q A K A Q K Q A K
K A G K I A K A G K I A K K A G K I A K A G K I A K
K K K K K K K K K K K K K K K K K K K K K K K K K K
T T T T T T T T T T T T T T T T T T T T T T T T T T  - t=42 = 2(26) - 2 - 8
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FIG. 4. Solution for $n = 26$, $k = 8$

STATE Q

Q	Q	K	P	L	S	R	G	Y	Z	Ø	–
Q	Q	G	L							Q	
K		K			X				I		I
P	S									K	
L		Z						Z			
S										K	S
A		G									
B		G									
R		G									
H											H
I									I		I
X	W				W						
W											W
Ø	Q		K	K							
–		G		L		R	G		Z		Q

STATE H

H	Q	K	A	B	–
Q		A			
K	I	K			
B		A			
G		A			
I			K	K	I
Y		A			
–			B	A	Q

STATE B

B	Q	A	R	H	G	–
Q					K	
A					K	
R					K	Q
H						
I	K	K		K		
–			Q			B

STATE G

G	Q	K	H
Q		H	
D		H	
A	K	K	
B	H	H	K
R		H	
–	H		Q

STATE W

W	Q	–
Q		R
A	R	
B	R	
R		Q

STATE A

A	Q	K	R	G	–
Q				K	
K		K		K	
H					H
I	K	K			
–			R		A

STATE R

R	K	B	G	I	X	–
Q	G					
K	K	A		A	A	
A			K			B
B			K			A
–			G			Q

STATE D

D	Q	K	S	G	–
Q		Y		Y	
K	X	K	X		
L		Y			
I	X			K	
–					D

STATE X

X	Q
Q	A
K	A
R	B

STATE I

I	Q	D	A	B	H	–
Q			K			R
K	R	R	K	R	R	
A	R					
R				K		Q

STATE K

K	K	Ø	–
K	T	T	
Ø	T		
–			K

STATE Z

Z	Q	A	B	H
Q		H	H	
–	H			Q

STATE P

P	Q	Ø
Q	D	K
Ø	K	T

STATE Y

Y	Q	K	H
Q	A	A	B

STATE L

L	–
–	Q

STATE S

S	–
–	Q

STATE T

T	–
–	T

Upper left-hand corner denotes present state.
Column heading denotes right neighbor's state.
Row heading denotes left neighbor's state.

FIG. 5. State tables

first search for ABC, then search for $-BC$, then search for $AB-$, then search for $-B-$. The "–" is not really a "don't-care" condition because a specific entry in the state table takes precedence over an entry determined by "–" entries. Actually the "–" entries are merely short hand notation for the rules which would be derived by substituting all possible states not previously mentioned by specific rules.

Note also that ϕ indicates the end condition on the left of soldier 1 and the right of soldier n.

HOW THE SOLUTION WORKS

As noted in Fig. 3, there is a point below which the solution only uses states of the normal firing squad (with the General on the end). To this end the nine states (Q, K, A, B, G, H, I, R, T) constitute a trivial modification of Waksman's solution which eliminates the redundant parity information incorporated in his 16-state solution. The remaining states are used to reconstruct the slope, 3, 7, 15 , $\cdots$ (B-signal) lines which would have been propagated in an end-General solution which started at $t = -k$.

When any soldier is promoted to General and given the Fire-When-Ready signal (state "P"), signals are propagated to either end ("L" and "S"). It becomes necessary to determine which end is the nearest the General. On reaching the end, these signals generate the necessary signals on the assumption that that end is the far end (states "I," "R," "A" and "B" on the left and "G," "H," "A" and "B" on the right). In addition, the newly-promoted General holds its place with a divider state ("D") and tells the first state of the pair I, G to reach it that that side really was the near side. At that point a slope 3 line is generated which, since it decides whether to pass the signals R and H, changes the slope of all the preceding lines. A quick glance at Fig. 3 shows that all line slopes change to the next higher one (i.e., $2^l - 1$ becomes $2^{l+1} - 1$).

Seventeen states were used in this solution, since it was desired to illustrate the geometry rather than to achieve a minimum number of states. It is, of course, possible to eliminate a few of the states but the authors leave this as a parlor game for interested readers.

DISCOVERY AND TESTING FOR VALIDITY OF THE SOLUTION

The 17-state realization of the solution presented here was discovered by use of an on-line program which allowed the authors to interact with the Rome Air Development Center Iterative Array Computer. The solution was verified up to length 50 for all positions of the general. But since running time increased as the fourth power of the length, it was estimated that to verify to length 100 for all general positions would take 180 hours. Therefore only random length and general positions were tested for more than 50 soldiers. The 9-state solution for the general at the end was tested to 100 soldiers.

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