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A NOTE ON A COLLECTIVELY COMPACT APPROXIMATION FOR WEAKLY SINGULAR INTEGRAL OPERATORS

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Abstract—We prove the collectively compact convergence of the Nyström method when applied on a truncated version of a weakly singular integral operator. As a consequence, we get the quasi compact convergence of finite rank projection operators derived from the Kantorovitch singularity subtraction approximation.

1. INTRODUCTION

Let $\mathcal{C}^0[0,1]$ denote the space of continuous functions $\varphi : [0,1] \to \mathbb{C}$ and $\|\cdot\|_\infty$ the L^∞ norm. $T : \mathcal{C}^0[0,1] \to \mathcal{C}^0[0,1]$ is an integral operator

$$(T\varphi)(t) = \int_0^1 k(t,s)\,\varphi(s)\,ds, \tag{1.1}$$

with weakly singular kernel $k(t,s) = g(|t-s|)\,h(t,s)$, where $h : [0,1]^2 \to \mathbb{C}$ is continuous and $g :]0,1] \to \mathbb{R}$ satisfies (cf. [1]):

$$g \in \mathcal{C}^0(]0,1]) \cap L^1(0,1), \tag{1.2a}$$
$$\exists \delta \in]0,1] \text{ such that } g \geq 0 \text{ and } g \text{ is a nonincreasing function on }]0,\delta]. \tag{1.2b}$$

T is then a compact operator and examples of g are $g(u) = u^{-\alpha}$, $(\alpha \in]0,1[)$, $g(u) = -\ln u$.

We consider the following approximation T_n^N to T (N for Nyström): For $\mu \in]0,\delta]$ we define g_μ by $g_\mu(u) = g(\mu)$ if $u \in [0,\mu]$ and $g_\mu(u) = g(u)$ if $u \in [\mu, 1]$ and we set

$$(T_n^N\varphi)(t) \equiv \sum_{i=1}^{n} \omega_{i,n}\, g_{\mu_n}(|t - t_{i,n}|)\, h(t, t_{i,n})\varphi(t_{i,n}),$$

where the real sequences $t_{i,n}$, $\omega_{i,n}$ and μ_n satisfy:

$$\forall n \in \mathbb{N}, \qquad 0 \leq t_{1,n} < t_{2,n} < \cdots < t_{n,n} \leq 1, \tag{1.3a}$$
$$\forall n \in \mathbb{N}, \qquad \forall i = 1,2,\ldots,n, \quad \omega_{i,n} \geq 0, \tag{1.3b}$$
$$\text{For any continuous function } f, \quad \sum_{i=1}^{n} \omega_{i,n}\, f(t_{i,n}) \to \int_0^1 f(t)\,dt, \text{ as } n \to \infty, \tag{1.3c}$$
$$\exists c > 0 \text{ such that } \forall n \in \mathbb{N} \text{ and any interval } \mathcal{G} \text{ of type } [a,b[\text{ or }]a,b],$$
$$\text{such that } b - a \leq \frac{1}{n}, \quad \sum_{t_{i,n} \in \mathcal{G}} \omega_{i,n} \leq \frac{c}{n}, \tag{1.3d}$$
$$\forall n \in \mathbb{N}, \ \mu_n \in]0,\delta] \quad \text{and} \quad \mu_n \downarrow 0 \text{ as } n \to \infty, \tag{1.3e}$$
$$\exists \rho > 0 \text{ such that } \forall n \in \mathbb{N}, \qquad \mu_n \geq \frac{\rho}{n}. \tag{1.3f}$$

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2. THE MAIN RESULT

It has been said but never used nor proved, that T_n^N is a collectively compact approximation to T. We give here a proof and we show some of its consequences. We begin by a lemma proved in [1]:

LEMMA 2.1. *Under conditions (1.3) any continuous nonincreasing function f satisfies*

$$\sum_{i=1}^{n} \omega_{i,n}\, f(t_{i,n}) \leq \frac{c}{n}\, f(0) + c\int_0^1 f(t)\,dt. \qquad \blacksquare$$

THEOREM 2.2. *Under conditions (1.3) T_n^N is a collectively compact approximation to T.*

PROOF. We prove that $\exists n_0 \in \mathbb{N}$ such that $W \equiv \{T_n^N\varphi : \ \varphi \in \mathcal{C}^0[0,1] : \ \|\varphi\|_\infty \leq 1, \ n > n_0\}$ is relatively compact. By (1.3e) $\forall \xi \in]0,\delta]$, $\exists n_0 \in \mathbb{N}$ such that $n > n_0 \implies \max\{1/n, \rho/n, \mu_n\} < \xi$. Take $\xi \in]0,\delta]$, $n > n_0$, $\varphi \in \mathcal{C}^0[0,1]$ with $\|\varphi\|_\infty \leq 1$ and $t \in [0,1]$. Then

$$\begin{aligned}|(T_n^N\varphi)(t)| &= \left|\sum_{i=1}^{n} \omega_{i,n}\, g_{\mu_n}(|t-t_{i,n}|)\, h(t,t_{i,n})\,\varphi(t_{i,n})\right| \leq \|h\|_\infty \sum_{i=1}^{n} \omega_{i,n}|g_{\mu_n}(|t-t_{i,n}|)| \\ &\leq \|h\|_\infty \Big(\sum_{|t-t_{i,n}|<\xi} \omega_{i,n}\, g_{\mu_n}(|t-t_{i,n}|) + \sum_{|t-t_{i,n}|\geq\xi} \omega_{i,n}|g_{\mu_n}(|t-t_{i,n}|)| \Big).\end{aligned}$$

But, from Lemma 2.1,

$$\sum_{|t-t_{i,n}|<\xi} \omega_{i,n}\, g_{\mu_n}(|t-t_{i,n}|) \leq \frac{2c}{n}\, g(\mu_n) + 2c\int_0^\xi g(u)\,du \leq 2c\, 1 + \frac{1}{\rho}\int_0^\xi g(u)\,du.$$

Besides

$$\sum_{|t-t_{i,n}|\geq\xi} \omega_{i,n}|g_{\mu_n}(|t-t_{i,n}|)| = \sum_{|t-t_{i,n}|\geq\xi} \omega_{i,n}|g(|t-t_{i,n}|)| \leq c \sup_{u\in[\xi,1]} |g(u)|.$$

Then $|(T_n^N\varphi)(t)| \leq C \equiv c\|h\|_\infty \max\{\sup_{u\in[\xi,1]} |g(u)|, 2(1+1/\rho)\int_0^\xi g(u)\,du\}$ and W is bounded. We now show that W is equicontinuous. Take $t \in [0,1]$ and $\varepsilon > 0$. By (1.2) $\exists \xi \in]0,\delta]$ such that $\int_0^\xi g(u)\,du < \frac{\varepsilon\rho}{16c\|h\|_\infty(3\rho+2)}$. The uniform continuity of h in $[0,1]^2$ and of g in $[\xi,1]$ imply that $\exists \eta(\varepsilon,t) > 0$ such that $\forall t' \in [0,1]$, $\forall s \in [0,1]$, $|t'-t| < \eta \implies |h(t',s)-h(t,s)| < \frac{\varepsilon\|h\|_\infty}{2C}$ and $\forall u', u \in [\xi,1]$, $|u'-u| < \eta \implies |g(u')-g(u)| < \frac{\varepsilon}{12c\|h\|_\infty}$. So that $\forall t' \in [0,1]$ satisfying $|t'-t| < \eta$, we have

$$\begin{aligned}|(T_n^N\varphi)(t') - (T_n^N\varphi)(t)| &= \left|\sum_{i=1}^{n} \omega_{i,n}\, [g_{\mu_n}(|t'-t_{i,n}|)\, h(t',t_{i,n}) - g_{\mu_n}(|t-t_{i,n}|)\, h(t,t_{i,n})]\varphi(t_{i,n})\right| \\ &\leq \left|\sum_{i=1}^{n} \omega_{i,n}\, g_{\mu_n}(|t'-t_{i,n}|)[h(t',t_{i,n}) - h(t,t_{i,n})]\,\varphi(t_{i,n})\right| \\ &\quad + \left|\sum_{i=1}^{n} \omega_{i,n}\, [g_{\mu_n}(|t'-t_{i,n}|) - g_{\mu_n}(|t-t_{i,n}|)]\, h(t,t_{i,n})\,\varphi(t_{i,n})\right| \\ &< \frac{\varepsilon\|h\|_\infty}{2C} \sum_{i=1}^{n} \omega_{i,n}|g_{\mu_n}(|t'-t_{i,n}|)| + \|h\|_\infty(S_1+S_2+S_3+S_4) \\ &< \frac{\varepsilon}{2} + \|h\|_\infty(S_1+S_2+S_3+S_4),\end{aligned}$$

where

$$
\begin{aligned}
S_1 &\equiv \sum_{\substack{|t'-t_{i,n}|<\xi \\ |t-t_{i,n}|<\xi}} \omega_{i,n} |g_{\mu_n}(|t'-t_{i,n}|) - g_{\mu_n}(|t-t_{i,n}|)| \\
&\leq \sum_{|t'-t_{i,n}|<\xi} \omega_{i,n}\, g_{\mu_n}(|t'-t_{i,n}|) + \sum_{|t-t_{i,n}|<\xi} \omega_{i,n}\, g_{\mu_n}(|t-t_{i,n}|) \\
&\leq 4c\; 1 + \frac{1}{\rho} \int_0^\xi g(u)\, du, \\
S_2 &\equiv \sum_{\substack{|t'-t_{i,n}|\geq\xi \\ |t-t_{i,n}|\geq\xi}} \omega_{i,n} |g_{\mu_n}(|t'-t_{i,n}|) - g_{\mu_n}(|t-t_{i,n}|)| \\
&= \sum_{\substack{|t'-t_{i,n}|\geq\xi \\ |t-t_{i,n}|\geq\xi}} \omega_{i,n} |g(|t'-t_{i,n}|) - g(|t-t_{i,n}|)| < \frac{\varepsilon}{12\|h\|_\infty}, \\
S_3 &\equiv \sum_{\substack{|t'-t_{i,n}|<\xi \\ |t-t_{i,n}|\geq\xi}} \omega_{i,n} |g_{\mu_n}(|t'-t_{i,n}|) - g_{\mu_n}(|t-t_{i,n}|)| \\
&\leq \sum_{|t'-t_{i,n}|<\xi} \omega_{i,n}\, g_{\mu_n}(|t'-t_{i,n}|) + g(\xi) \sum_{|t'-t_{i,n}|<\xi} \omega_{i,n} \\
&\quad + \sum_{\substack{|t'-t_{i,n}|<\xi \\ |t-t_{i,n}|\geq\xi}} \omega_{i,n} |g(\xi) - g(|t-t_{i,n}|)| \\
&< 2c\; 2 + \frac{1}{\rho} \int_0^\xi g(u)\, du + \frac{\varepsilon}{12\|h\|_\infty}, \\
S_4 &\equiv \sum_{\substack{|t'-t_{i,n}|\geq\xi \\ |t-t_{i,n}|<\xi}} \omega_{i,n} |g_{\mu_n}(|t'-t_{i,n}|) - g(|t-t_{i,n}|)| \\
&< 2c\; 2 + \frac{1}{\rho} \int_0^\xi g(u)\, du + \frac{\varepsilon}{12\|h\|_\infty}, \\
S_1 + S_2 &+ S_3 + S_4 < \frac{\varepsilon}{2\|h\|_\infty}.
\end{aligned}
$$

Thus, $\forall t \in [0,1]$, $\forall \varepsilon > 0$, $\exists \eta(\varepsilon, t) > 0$ such that if $\|\varphi\|_\infty \leq 1$, then $\forall n \in \mathbb{N}$, $\forall t' \in [0,1]$, $|t'-t| < \eta \implies |(T_n^N\varphi)(t') - (T_n^N\varphi)(t)| < \varepsilon$. Since $[0,1]$ is compact the equicontinuity of W follows from the Arzela-Ascoli theorem. The pointwise convergence of T_n^N to T is proved in [1]. ∎

3. APPLICATIONS TO SINGULARITY SUBTRACTION

We write T in the form

$$(T\varphi)(t) = \int_0^1 k(t,s)\, \varphi(s)\, ds = \int_0^1 k(t,s)[\varphi(s) - \varphi(t)]\, ds + \varphi(t) \int_0^1 k(t,s)\, ds$$

and Kantorovitch's singularity subtraction approximation T_n^K is motivated:

$$(T_n^K\varphi)(t) \equiv \sum_{i=1}^n \omega_{i,n}\, g_{\mu_n}(|t-t_{i,n}|)\, h(t, t_{i,n})[\varphi(t_{i,n}) - \varphi(t)] + \varphi(t) \int_0^1 k(t,s)\, ds.$$

Let π_n be a sequence of finite rank projections pointwise convergent to the identity. We consider three finite rank operators: $T_n^{KP} \equiv \pi_n T_n^K$, $T_n^{KS} \equiv T_n^K \pi_n$ and $T_n^{KG} \equiv \pi_n T_n^K \pi_n$, where P stands for projection, S for Sloan and G for Galerkin. The quasi compact convergence, as defined in [2], follows:

THEOREM 3.1. *If T_n is any of the approximations T_n^K, T_n^{KP}, T_n^{KS} or T_n^{KG} then T_n is pointwise convergent to T and $(T_n - T)T_n$ converges to zero in the induced operator norm.*

PROOF. Since T_n^N is a collectively compact approximation to T, $(T_n^N - T)T_n^N$ converges in norm to zero (cf. [3]). But $T_n^K = T_n^N + \Delta_n$ where Δ_n converges in norm to zero. Hence $(T_n^K - T)T_n^K$ converges in norm to zero. Since π_n is pointwise convergent to the identity, $\pi_n T_n^N$, $T_n^N \pi_n$ and $\pi_n T_n^N \pi_n$ are collectively compact approximations to T (cf [3]). The Banach-Steinhaus theorem implies that π_n is uniformly bounded. Hence $\pi_n \Delta_n$, $\Delta_n \pi_n$ and $\pi_n \Delta_n \pi_n$ converge in norm to zero. So that $(T_n - T)T_n$ converges in norm to zero for $T_n = T_n^{KP}$, T_n^{KS}, T_n^{KG}. The pointwise convergence of T_n^{KP}, T_n^{KS} and T_n^{KG} follows immediately. Hence T_n^{KG}, T_n^{KP} and T_n^{KS} are quasi compact approximations to T. ∎

We remark that if (1.3d) changes into

$$\exists c > 0 \text{ such that } \forall n \in \mathbb{N}, \quad \forall j = 1, 2, \ldots, n \quad \max\{\omega_{j-1,n}, \omega_{j,n}\} \le c(t_{j,n} - t_{j-1,n}) \tag{1.3d'}$$

then all the conclusions hold provided that (1.3f) changes into

$$\exists \rho > 0 \text{ such that } \mu_n \ge \rho \max_{i=1,2,\ldots,n} \omega_{i,n}. \tag{1.3f'}$$

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