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Covering $\mathbb{R}$-trees, $\mathbb{R}$-free groups, and dendrites

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Abstract

We prove that every length space X is the orbit space (with the quotient metric) of an $\mathbb{R}$-tree $\overline{X}$ via a free action of a locally free subgroup $\Gamma(X)$ of isometries of $\overline{X}$. The mapping $\overline{\phi}: \overline{X} \to X$ is a kind of generalized covering map called a URL-map and is universal among URL-maps onto X. $\overline{X}$ is the unique $\mathbb{R}$-tree admitting a URL-map onto X. When X is a complete Riemannian manifold M^n of dimension $n \geqslant 2$, the Menger sponge, the Sierpin'ski carpet or gasket, $\overline{X}$ is isometric to the so-called "universal" $\mathbb{R}$-tree $A_{\mathfrak{c}}$, which has valency $\mathfrak{c} = 2^{\aleph_0}$ at each point. In these cases, and when X is the Hawaiian earring H, the action of $\Gamma(X)$ on $\overline{X}$ gives examples in addition to those of Dunwoody and Zastrow that negatively answer a question of J.W. Morgan about group actions on $\mathbb{R}$-trees. Indeed, for one length metric on H, we obtain precisely Zastrow's example.

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1. Introduction and main results

A metric space such that every pair of its points is joined by a path of length arbitrarily close to the distance between them is called an *inner metric space* or *length space*. A *geodesic space* is a metric space such that every pair of points is joined by a geodesic, i.e. a path whose length

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is equal to the distance between them. Evidently every geodesic space is a length space and it is a classical result that a complete, locally compact length space is a geodesic space. A geodesic space that contains no topological circle is called an $\mathbb{R}$-*tree*. A *submetry* (resp. *weak submetry*) $f: X \to Y$ between metric spaces is a function which maps every closed (respectively, open) ball in X centered at any point $x \in X$ onto the closed (open) ball in Y of the same radius at the point $f(x)$ [6]. Note that a submetry or weak submetry is open, surjective and distance non-increasing, hence 1-Lipschitz and uniformly continuous. A map is *light* if every point pre-image is totally disconnected [3,37]. A function $f: X \to Y$ is a *metric quotient* if $d_Y(x, y)$ is the Hausdorff distance between $f^{-1}(x)$ and $f^{-1}(y)$; clearly a metric quotient is a weak submetry. Recall that a group is *locally free* if each of its finitely generated subgroups is free.

Theorem 1. *Every length space (resp. complete length space) (X, d) is the metric quotient of a (resp. complete) $\mathbb{R}$-tree $(\overline{X}, \overline{d})$ via the free isometric action of a locally free subgroup $\Gamma(X)$ of the isometry group $Isom(\overline{X})$. The quotient mapping $\overline{\phi}: \overline{X} \to X$ is a weak submetry (hence open) and light map, and $\overline{\phi}$ is a submetry if X is geodesic.*

The $\mathbb{R}$-tree $\overline{X}$ is defined as the space of based "non-backtracking" rectifiable paths in X, where the distance between two paths is the sum of their lengths from the first bifurcation point to their endpoints. The group $\Gamma(X) \subset Isom(\overline{X})$ is naturally identified with the subset of loops in $\overline{X}$ with a natural group structure, and the quotient mapping $\overline{\phi}: \overline{X} \to X$ is the endpoint map. We will refer to $\overline{X}$ as the *covering* $\mathbb{R}$-*tree* of X. The term "$\mathbb{R}$-tree" was coined by Morgan and Shalen [33] in 1984 to describe a type of space that was first defined by Tits [36] in 1977. In the last three decades $\mathbb{R}$-trees have played a prominent role in topology, geometry, and geometric group theory (see, for example, [2,9,13,33,22]). They are the most simple of geodesic spaces, and yet Theorem 1 shows that every length space, no matter how complex, is an orbit space of an $\mathbb{R}$-tree.

Unless otherwise stated, "dimension" refers to the covering dimension $\dim(X)$ of X. The small and large inductive dimensions of a metric space X satisfy the Katetov equality $Ind(X) = \dim(X)$ and the inequality $ind(X) \leqslant Ind(X)$ (see [1]). If X is also separable, in particular compact, then $ind(X) = Ind(X) = \dim(X)$ (see [23]). The above theorem and the fact that a (non-trivial) $\mathbb{R}$-tree X is simply connected with $ind(X) = 1$ [2,31,5] give us:

Corollary 2. *Every non-trivial topological space admitting a compatible length metric is the image via a light open mapping of a simply connected space X with $ind(X) = 1$.*

Corollary 2 is broadly applicable because in 1949 Bing and Moise [11,30] independently and positively answered a 1928 question of Menger: whether (in modern terminology) every Peano continuum (continuous image of [0, 1]) admits a compatible geodesic metric [29]. The Bing–Moise Theorem shows that Corollary 2 contributes to a 70-year-old program to construct dimension-raising open mappings, beginning with an example of Kolmogorov in 1937 [25] from a Peano curve (1-dimensional Peano continuum) to a 2-dimensional space. Later examples include [24] and the spectacular theorem partly stated without proof by Anderson [3,4] and proved by Wilson in [37]: every Peano continuum is the image via a light open mapping of the Menger sponge $\mathbb{M}$. Recall that $\mathbb{M}$ is called the "universal curve" because every Peano curve may be embedded in it; the Anderson–Wilson Theorem provides a second sense in which $\mathbb{M}$ is "universal". We will be interested in two more Peano curves: the Sierpin'ski carpet S_c and gasket S_g. As is well known, each of these three *fractal curves* [19] admits a geodesic metric d bi-Lipschitz equivalent to the metric induced by the ambient Euclidean space of which it is a subspace: $d(x, y)$ is simply the shortest Euclidean length of a path joining x and y in the space.

Next, recall that for a point t in an $\mathbb{R}$-tree T, the valency at t is the cardinality of the set of connected components of $T\backslash\{t\}$, and T is said to have valency at most μ if the valency of every point in T is at most μ. A non-trivial complete metrically homogeneous $\mathbb{R}$-tree can be characterized as a complete $\mathbb{R}$-tree A_μ with valency μ at each point for a cardinal number $\mu \geqslant 2$. It is unique up to isometry, and μ-universal in the sense that every $\mathbb{R}$-tree of valency at most μ isometrically embeds in A_μ. The existence of A_μ and the results just mentioned were proved in [28]. Another construction of A_μ was given in [18], where it was shown that $A_{\mathfrak{c}}$ ($\mathfrak{c} = 2^{\aleph_0}$, the cardinality of the continuum) can be isometrically embedded at infinity in a complete simply connected Riemannian manifold of constant negative curvature.

Theorem 3. *If X is a separable length space, then $\overline{X}$ is a subtree of $A_{\mathfrak{c}}$. If in addition X is complete and each point of X is contained in a bi-Lipschitz copy of S_g or S_c, e.g. when X is S_c, S_g, $\mathbb{M}$, or a complete Riemannian manifold M^n of dimension $n \geqslant 2$, then $\overline{X}$ is isometric to $A_{\mathfrak{c}}$.*

Put another way, every separable length space may be obtained by starting with a subtree of $A_{\mathfrak{c}}$ and taking a quotient of that subtree via a free isometric action. Another consequence of this theorem is an explicit construction of $A_{\mathfrak{c}}$ starting with any of the above spaces (see the proof of Theorem 1). Notice that by using different Banach spaces X with their natural geodesic metric, we can in a similar way realize A_μ as $\overline{X}$ for arbitrary $\mu \geqslant \mathfrak{c}$. Our results, combined with the Anderson–Wilson Theorem, show that $A_{\mathfrak{c}}$ is "universal" in a way analogous to the second way in which $\mathbb{M}$ may be regarded as "universal":

Corollary 4. *Every non-trivial Peano continuum is the image of $A_{\mathfrak{c}}$ via a light open mapping.*

The function $\overline{\phi}$ from Theorem 1 is generally not a locally isometric covering map in the traditional sense, but shares important properties with any such map $f : X \to Y$: (I) f preserves the length of rectifiable paths in the sense that $L(c) = L(f \circ c)$ for every path c in X with finite length $L(c)$. (II) If c is any rectifiable path in Y starting at a point p and $f(q) = p$ then there is a unique path c_L starting at q such that $f \circ c_L = c$, and moreover c_L is rectifiable. A function f between length spaces will be called *unique rectifiable lifting (URL)* if it has these two properties. Note that a map between length spaces with condition (I) is known as an *arcwise isometry* [21]; such maps are a distant generalization of isometric immersions from differential geometry. Any URL-map is an arcwise isometric weak submetry (Proposition 29). For Riemannian manifolds, the notion of weak submetry is the same as that of Riemannian submersion [7], which is in some sense dual to isometric immersion. Generally a URL-map may not be locally injective at any point, as Theorems 1, 3, and 5 show.

Theorem 5. *Under the assumptions and with the notation of Theorem* 1*:*

1. *The map $\overline{\phi} : \overline{X} \to X$ is a URL-map.*
2. *If Z is a length space and $f : Z \to X$ is a URL-map then there is a unique (up to basepoint choice) URL-map $\overline{f} : \overline{X} \to Z$ such that $\overline{\phi} = f \circ \overline{f}$.*
3. *There exists a unique (up to isometry) length space (X_1, d_1) with a map $\phi_1 : X_1 \to X$, having the previous two properties.*

(4) *If there is an* $\mathbb{R}$*-tree* X_1 *with a URL-mapping* $\phi_1 : X_1 \to X$*, then there is an isometry* $\overline{\phi}_1 : X_1 \to \overline{X}$ *such that* $\phi_1 = \overline{\phi} \circ \overline{\phi}_1$.

In the language of category theory, this theorem means that $\overline{\phi}$ is the initial object in the category of URL-mappings over X, i.e. $\overline{X}$ is "universal" in this category. This result, combined with Theorem 3, shows that $A_{\mathfrak{c}}$ is "universal" in yet a third sense.

One can easily deduce from Theorems 1 and 5 the following corollary.

Corollary 6. *Let* $f : (X_1, *) \to (X_2, *)$ *be a basepoint preserving URL-map of length spaces. Then there is a commutative diagram*

$$\begin{array}{ccc} (\overline{X_1}, *) & \xrightarrow{\overline{f}} & (\overline{X_2}, *) \\ \downarrow \overline{\phi_1} & & \downarrow \overline{\phi_2} \\ (X_1, *) & \xrightarrow{f} & (X_2, *) \end{array}$$

of URL-maps preserving basepoints, with unique $\overline{f}$*, where* $\overline{\phi_1}$ *and* $\overline{\phi_2}$ *are the* $\mathbb{R}$*-tree covering maps for* X_1 *and* X_2 *respectively, and* $\overline{f}$ *is an isometry. The identification* $(\overline{X_1}, *)$ *with* $(\overline{X_2}, *)$ *by the isometry* $\overline{f}$ *induces the homomorphic inclusion* $\Gamma(X_1) \subset \Gamma(X_2)$ *of the corresponding isometry groups.*

In particular, Corollary 6 shows that *all* URL-maps, including the traditional universal cover of a length space, are obtained via quotients of the covering $\mathbb{R}$-tree. Notice that the proof of Theorem 1 implies that $\Gamma(X)$ is naturally identified with the group Γ from Proposition 19. There naturally arises:

Question 7. *For a given length space* $(X, *)$*, which subgroups of* $\Gamma(X, *)$ *correspond to URL-maps onto* $(X, *)$*?*

We plan to occupy ourselves with this problem in the future. It may be useful to consider the well-known bijective correspondence between geodesically complete $\mathbb{R}$-trees with basepoints ("rooted $\mathbb{R}$-trees") and ultrametric spaces of diameter 1 with nonempty spheres of radius 1 that come from considering the end space of the $\mathbb{R}$-tree [22]. The ultrametric space corresponding to A_μ is complete, metrically homogeneous, and does not depend up to isometry on the choice of a basepoint in A_μ. Then an approach of Bruce Hughes in [22] associates to each isometry of the $\mathbb{R}$-tree a so-called local similarity equivalence of the corresponding ultrametric space. The preprint [26] may also be useful for this problem.

Remark 8. Generally, for a given length space X, we can find a proper subtree $\tilde{X} \subset \overline{X}$ and a proper subgroup $\tilde{\Gamma}(X) \subset \Gamma(X)$ such that X is the metric quotient of the $\mathbb{R}$-tree $\tilde{X}$ via the free isometric action of the group $\tilde{\Gamma}(X)$ on $\tilde{X}$, and the quotient mapping $\tilde{\phi} : \tilde{X} \to X$ is an arcwise isometry, a weak submetry, and $\tilde{\phi}$ is a submetry if X is geodesic. Under these conditions, $\tilde{X}$ is not necessarily complete, even if X is geodesic and complete. This is shown in Theorem 45 for any Riemannian manifold M^n of dimension $n \geqslant 2$. It follows from Corollary 6 that $\tilde{\phi}$ is not a URL-map.

Previously we discussed three main actors: any length space X, the $\mathbb{R}$-tree $\overline{X}$, and the URL-map $\overline{\phi} : \overline{X} \to X$. Theorem 1 implies that for any pointed length space X, the group $\Gamma(X)$ acts freely by isometries on the covering $\mathbb{R}$-tree $\overline{X}$. So our paper is closely connected with the following general question of J.W. Morgan from [32]:

Question 9. *Which (finitely presented) groups act freely (by isometries) on $\mathbb{R}$-trees?*

This question inspired us to study more closely the structure of the fourth actor, the group $\Gamma(X)$. The answer to Question 9 is known for finitely generated groups [34,20,13]. However, there are examples by Dunwoody [17] and Zastrow [38] of infinitely generated groups that are not free products of fundamental groups of closed surfaces and abelian groups, but which act freely on an $\mathbb{R}$-tree. Zastrow's group G contains one of the two Dunwoody groups as a subgroup. The other group is a Kurosh group. We prove the following theorems.

Theorem 10. *Let X be S_c, $\mathbb{M}$, a complete Riemannian manifold M^n of dimension $n \geqslant 2$, or the Hawaiian earring H with any compatible length metric d. Then $\Gamma(X)$ is an infinitely generated, locally free group that is not free and not a free product of surface groups and abelian groups, but acts freely on the $\mathbb{R}$-tree $\overline{X}$. Moreover, the $\mathbb{R}$-tree $\overline{X}$ is a minimal invariant subtree with respect to this action.*

Theorem 11. *For any two length metrics d_1, d_2 on H (compatible with the usual topology), there is an injective homomorphism of $\Gamma(H, d_1)$ into $\Gamma(H, d_2)$. For a particular choice of $d = d_Z$ on H, $\Gamma(H, d_Z)$ and its action on $\overline{(H, d_Z)}$ coincide with Zastrow's group G and its free action by isometries on Zastrow's $\mathbb{R}$-tree.*

An important role in the proofs is played by classical results about *normal paths* from [15] and *dendrites* from [27], and a more recent characterization of $\mathbb{R}$-trees as Gromov 0-*hyperbolic* geodesic spaces [21]. There is an interesting connection between these topics of different eras: It is not hard to show, using the Bing–Moise Theorem, that a topological space is a dendrite if and only if it is metrizable as a compact $\mathbb{R}$-tree.

This paper is connected with, and was inspired by, our previous paper [8] and a 20-year-old announcement of the first author, cited in [18]. The results from [8] may be used to give an alternative proof of Proposition 18. In an upcoming paper we will use constructions of [8] to obtain additional examples of URL-maps that are not local isometries.

2. The covering $\mathbb{R}$-tree

Definition 12. Let $c : [a, b] \to X$ be a path in a metric space X; c is called normal if there is no non-trivial subsegment $J = [u, v] \subset [a, b]$ such that $c(u) = c(v)$ and $c|_J$ is path homotopic to a constant. Here "path homotopic" means fixed-endpoint homotopic. We define c to be weakly normal if c is normal in its image $c([a, b])$.

Remark 13. An immediate consequence of the above definition is that every normal path is weakly normal.

Proposition 14. *Consider the following three statements for a path $c : [a, b] \to X$:*

(1) *c is normal;*
(2) *c is weakly normal;*

(3) *there is no non-trivial subsegment* $J = [u, v] \subset [a, b]$ *such that* $c(u) = c(v)$ *and* $c|_J$ *is path homotopic in* $c(J)$ *to a constant.*

All three are equivalent if X *is a separable,* 1*-dimensional metric space. If* X *is an arbitrary metric space and* c *is rectifiable then the second and third are equivalent.*

Proof. Suppose that c is weakly normal and X is separable and one-dimensional. Then for every non-trivial subsegment $J = [u, v] \subset [a, b]$ such that $c(u) = c(v) = y$, $c|_J$ is not path homotopic in $Y := c([a, b])$ to a constant map, i.e. it represents a non-trivial element of $\pi_1(Y, y)$. By Corollary 2.1 in [15], the inclusion map $i : Y \to X$ induces an injective homomorphism $i_* : \pi_1(Y, y) \to \pi_1(X, y)$. So, the path $c|_J$ is not path homotopic in X to constant path. This implies that the path c is normal. The last statement follows from the previous statement and the well-known fact that the image Z of a non-trivial rectifiable path c is one-dimensional. This fact follows from inequalities $\dim(Z) \leqslant \dim_H(Z) \leqslant 1$, where $\dim_H$ is the so-called Hausdorff dimension [19,21], and non-triviality of c. □

Definition 15. Two paths $c_1, c_2 : I = [a, b] \to X$ are called Fréchet equivalent if there exist order-preserving monotone (continuous) maps m_1, m_2 of I onto itself such that $c_1 m_1 = c_2 m_2$.

In Lemma 3.1 and Theorem 3.1 the authors of [15] proved the following results for a 1-*dimensional separable metric space* X.

Lemma 16. *Each path* $f : I \to X$ *is path homotopic to a normal path.*

Theorem 17. *Two normal paths in* X *are path homotopic if and only if they are Fréchet equivalent.*

The definition of normal loop was given in [15] along with the not-quite-standard Definition 15. The statements were proved for loops, but the same arguments work for paths. Evidently, the "if" part of Theorem 17 is valid for any space X. Curtis and Fort proved Lemma 16 in [15] in the following way. Let S be the collection of all subsets G of I, open in $\mathbb{R}$, such that: If (u, v) is a component of G, then $f(u) = f(v)$ and $f|_{[u,v]}$ is path homotopic to a constant. The collection S is partially ordered by inclusion. It is proved that S contains a maximal element G^*. Define g to be the map that agrees with f on $I - G^*$ and is constant on each component of G^*. Then g is path homotopic to f and g is normal.

Proposition 18. *Any rectifiable path* c *in a metric space is path homotopic in its image to a weakly normal path* c_n*. Moreover,* $L(c_n) \leqslant L(c)$ *and the parameterization of* c_n *by arclength is uniquely determined by* c.

Proof. The first statement follows from Proposition 14 and Lemma 16. The third statement is a corollary of the above hint for the proof of Lemma 16, Theorem 17, and the evident statement that two rectifiable Fréchet equivalent paths have equal parameterizations by arclength. □

By a ρ*-path* in a metric space X we mean a weakly normal, rectifiable, arclength parameterized path $c : [0, L] \to X$. Note that the concatenation $c * d$ of a ρ-path c followed by a ρ-path d may not be a ρ-path. To resolve this problem we define the "cancelled concatenation" $c \star d$ to be the unique ρ-path which is the arclength parameterization of the weakly normal path in the path homotopy class of the concatenation $c * d$, in the image of $c * d$ (Proposition 18). From the uniqueness and the last statement in Proposition 14, one can easily see more concretely that $c \star d$ is obtained from $c * d$ by removing the maximal final segment of c that coincides with an initial segment of d with reversed orientation, and removing that initial segment of d as well.

Proposition 19. *The associative law $(a \star b) \star c = a \star (b \star c)$ is satisfied. Moreover, cancelled concatenation on the set Γ of all ρ-loops at a fixed basepoint $*$ of any metric space X is a group operation, where the constant loop is the identity and the inverse of $c : [0, L] \to X$ is the ρ-loop $c^{-1}(t) := c(L - t)$.*

Proof. All these statements follow from the uniqueness of the ρ-path $c \star d$ for any two ρ-paths c and d in the case when $c * d$ makes sense (see the discussion right before this proposition). $\square$

We shall exclude in the future the trivial case when X contains only one point (this is traditional in discussing $\mathbb{R}$-trees). The following are equivalent for a geodesic space X (see [31,12, 5]): (1) X is an $\mathbb{R}$-tree. (2) X is 0-hyperbolic in Gromov's sense. (3) X is $CAT(K)$-space for all $K \leqslant 0$. (4) X is simply connected and $ind(X) = 1$ [5]. See [12] for the definition of $CAT(K)$-space. We will not use this notion in the present paper except to observe the corollary that every geodesic space is the metric quotient of a $CAT(K)$-space.

Proof of Theorem 1. Choose a basepoint $* \in X$ and define the set $\overline{X}$ to be the set of all ρ-paths $c : [0, L] \to X$ starting at $*$. For $c_1, c_2 \in \overline{X}$, let $c_1 \wedge c_2 : [0, b] \to X$ be the restriction of c_1 (and c_2) to the largest interval $[0, b]$ on which c_1 and c_2 coincide, and define

$$\bar{d}(c_1, c_2) := L(c_1) + L(c_2) - 2L(c_1 \wedge c_2) = L\big(c_1^{-1} \star c_2\big). \tag{1}$$

To see that $\overline{X}$ is an $\mathbb{R}$-tree, we will use the characterization (2) above. We will also denote by $*$ the element of $\overline{X}$ that is simply the constant path at $* \in X$. Let $c_1, c_2 \in \overline{X}$, defined on $[0, L_1]$, $[0, L_2]$, respectively. Let

$$s_0 := \max\big\{s\colon\ c_1(t) = c_2(t) \text{ for all } t \in [0, s]\big\}$$

and define $C(s)$ for $s \in [0, L_1 + L_2 - 2s_0]$ as follows. For $s \in [0, L_1 - s_0]$ let $C(s)$ be the restriction of c_1 to $[0, L_1 - s]$. For $s \in [L_1 - s_0, L_1 + L_2 - 2s_0]$ let $C(s)$ be the restriction of c_2 to $[0, s - L_1 + 2s_0]$. Certainly $C(s)$ is a geodesic in $\overline{X}$ joining c_1 and c_2. This implies that $\overline{X}$ is a geodesic space.

We see from formula (1) that the so-called Gromov product

$$(c_1, c_2)_* := \frac{1}{2}\big[\bar{d}(*, c_1) + \bar{d}(*, c_2) - \bar{d}(c_1, c_2)\big]$$

with respect to the point $*$ (see, for example, [12, p. 410]) is equal to $L(c_1 \wedge c_2)$. Also we see immediately that $c_1 \wedge c_2$ contains as a subpath $(c_1 \wedge c_3) \wedge (c_2 \wedge c_3)$ for any c_3. Then it follows from these two statements that

$$(c_1, c_2)_* \geqslant \min\big\{(c_1, c_3)_*, (c_3, c_2)_*\big\} \tag{2}$$

for any $c_1, c_2, c_3 \in \overline{X}$. This means that $\overline{X}$ is 0-hyperbolic "with respect to the point $*$", whereas 0-hyperbolicity itself means that Eq. (2) must be satisfied with respect to any point $c \in \overline{X}$. But by Remark 1.21 (page 410 of [12]), 0-hyperbolic at a single point is sufficient for $\overline{X}$ to be 0-hyperbolic, and hence an $\mathbb{R}$-tree.

By definition, $\overline{\phi} : (\overline{X}, \overline{d}) \to (X, d)$ associates to a path $c \in \overline{X}$ its endpoint in X. Let c_1, c_2 be any elements in $\overline{X}$. Then the path $c_1^{-1} \star c_2$ joins the points $x_1 := \overline{\phi}(c_1)$ and $x_2 := \overline{\phi}(c_s)$. Thus by definition

$$d(x_1, x_2) \leqslant L\big(c_1^{-1} \star c_2\big) = \overline{d}(c_1, c_2). \tag{3}$$

This means that $\overline{\phi}$ does not increase distances. Now let $x_1, x_2 \in X$, $x_1 := \overline{\phi}(c_1)$ and $\varepsilon > 0$ be given. Then there is a rectifiable path c in X joining the points x_1, x_2 so that $L(c) \leqslant d(x_1, x_2) + \varepsilon$. Denote by c_n the ρ-path in X from Proposition 18. Then c_n joins the points x_1, x_2, $L(c_n) \leqslant L(c)$, and the ρ-path $c_2 := c_1 \star c_n$ joins the points $*$ and x_2. Moreover,

$$\overline{d}(c_1, c_2) = L\big(c_1^{-1} \star c_2\big) = L\big(c_1^{-1} \star c_1 \star c_n\big) = L(c_n) \leqslant d(x_1, x_2) + \varepsilon. \tag{4}$$

This together with inequality (3) means that $\overline{\phi}$ is a weak submetry, hence a metric quotient [35]. If (X, d) is a geodesic space, we can take c_n to be a geodesic and instead of the inequality (4), we get

$$\overline{d}(c_1, c_2) = L\big(c_1^{-1} \star c_2\big) = L\big(c_1^{-1} \star c_1 \star c_n\big) = L(c_n) = d(x_1, x_2).$$

This together with inequality (3) means that $\overline{\phi}$ is a submetry.

For an arbitrary interior point w of a non-trivial geodesic segment $[y, z]$ in an $\mathbb{R}$-tree T, y, z lie in different connected components of $T - \{w\}$. Then any connected subset $C \subset \overline{X}$ containing two different points y, z must include $[y, z]$. Now it follows from the definition that in this case $\overline{\phi}([y, z])$ is the image of a non-trivial path in X. Then the inclusion $C \subset \overline{\phi}^{-1}(x)$ is impossible for any $x \in X$, which shows that $\overline{\phi}$ is a light map.

Now let Γ be the group from Proposition 19. It follows from formula (1) and Proposition 19 that Γ acts freely via isometries on $(\overline{X}, \overline{d})$ if we define $l(c) = l \star c$ for any ρ-loop $l \in \Gamma$ and $c \in \overline{X}$. Obviously, $\overline{\phi}(l(c)) = \overline{\phi}(c)$. Also, if $c_1, c_2 \in \overline{X}$ and $c_1, c_2 \in \overline{\phi}^{-1}(x)$, $x \in X$, then $c_2 = l(c_1)$, where $l = c_2 \star c_1^{-1}$. This means that every pre-image $\overline{\phi}^{-1}(x)$, $x \in X$, is an orbit via the action of Γ. The orbits of this action are precisely the sets $\phi^{-1}(x)$ for $x \in X$, which verifies that X is the metric quotient with respect to the action of Γ.

Assume that X is complete. Suppose that $c_k : [0, L_k] \to X$ is a Cauchy sequence in X. By definition of the metric, $\{L_k\}$ converges to a real number L and so is bounded above by some finite number M. By extending all paths to be constant at their endpoints we may assume that all paths are defined on $[0, M]$ (these extensions generally are not in $\overline{X}$). Now all these paths are 1-Lipschitz maps. That is, the sequence of these paths is uniformly Cauchy and since X is complete, it converges uniformly to some path $c : [0, M] \to X$. It follows from the uniform convergence that $c_{[0,L]} \in \overline{X}$ and $c_k \to c_{[0,L]}$ in $(\overline{X}, \overline{d})$. This proves the completeness of $(\overline{X}, \overline{d})$.

Finally we check that $\Gamma(X)$ is locally free. Let $l_1, \ldots, l_n$ be ρ-loops in X starting at $*$ and Y be the union of their images, which is separable and 1-dimensional. According to [16], $\pi_1(Y)$ is locally free. Evidently the subgroup $\Gamma(l_1, \ldots, l_n)$ of $\Gamma(X)$ generated by $l_1, \ldots, l_n$ is naturally identified with a subgroup of $\Gamma(Y)$. Moreover, it follows from the definition of $\Gamma(X)$, Lemma 16, and Theorem 17 that $\Gamma(Y)$ is naturally isomorphic to a subgroup of the locally free group $\pi_1(Y)$. By the Nielsen–Schreier Theorem, $\Gamma(l_1,\dots,l_n)$ is free. $\square$

Lemma 20. *Let γ_c denote the unique geodesic in $\overline{X}$ parameterized by arclength and joining the points $*$ (constant path at the point $* \in X$) and c. Then*

(1) $\bar{\phi}\circ\gamma_c = c$,
(2) $\gamma_{c_1}^{-1}\star\gamma_{c_2}$ *is the unique geodesic in $\overline{X}$ parameterized by arclength and joining the points c_1 and c_2,*
(3) $\bar{\phi}\circ(\gamma_{c_1}^{-1}\star\gamma_{c_2}) = c_1^{-1}\star c_2$, *and*
(4) $L(\gamma_{c_1}^{-1}\star\gamma_{c_2}) = L(c_1^{-1}\star c_2)$.

Proof. These statements follow from definition of $\bar{d}$, the fact that the path C considered in the proof of Theorem 1 is the unique arclength-parameterized geodesic in $(\overline{X},\bar{d})$ joining the points $c_1, c_2 \in \overline{X}$, and the equations $C = \gamma_{c_1}^{-1}\star\gamma_{c_2}$, $\bar{\phi}\circ C = c_1^{-1}\star c_2$. $\square$

In fact, the construction of $\overline{X}$ and $\Gamma(X)$ in the proof of Theorem 1 depends on the choice of the basepoint $* \in X$, so, strictly speaking, we must write $\overline{(X,*)}$ and $\Gamma(X,*)$ instead of $\overline{X}$ and $\Gamma(X)$. Proposition 22 below shows that this dependence is not so essential. Recall the following definition.

Definition 21. An action of a group Γ_1 on a metric space X_1 via isometries is said to be equivalent to an action of a group Γ_2 on a metric space X_2 via isometries if there are an isometry $f: X_1 \to X_2$ and an isomorphism $\phi: \Gamma_1 \to \Gamma_2$ such that $f(g(x)) = \phi(g)(f(x))$ for any point $x \in X$ and any element $g \in \Gamma_1$.

Proposition 22. *Let X be a length space and $*, \star \in X$ be two of its points. Then the action of the group $\Gamma(X,*)$ on the $\mathbb{R}$-tree $\overline{(X,*)}$ is equivalent to the action of the group $\Gamma(X,\star)$ on the $\mathbb{R}$-tree $\overline{(X,\star)}$.*

Proof. By Proposition 18, there is a ρ-path k in X which starts at $\star$ and ends at $*$. Define maps $f: \overline{(X,*)} \to \overline{(X,\star)}$ and $\phi: \Gamma(X,*) \to \Gamma(X,\star)$ by the formulas $f(c) := k\star c$ and $\phi(\gamma) = k\star\gamma\star k^{-1}$. By Proposition 19 and formula (1),

$$\begin{aligned}\bar{d}\big(f(c_1), f(c_2)\big) &= \bar{d}(k\star c_1, k\star c_2) = L\big((k\star c_1)^{-1}\star(k\star c_2)\big)\\ &= L\big(c_1^{-1}\star k^{-1}\star k\star c_2\big) = L\big(c_1^{-1}\star c_2\big) = \bar{d}(c_1, c_2)\end{aligned}$$

for any two ρ-paths $c_1, c_2 \in \overline{(X,*)}$; for any element $c' \in \overline{(X,\star)}$, $k^{-1}\star c' := c \in \overline{(X,*)}$ and $f(c) = k\star k^{-1}\star c' = c'$. Thus f is an isometry. Now by Proposition 19,

$$\phi(\gamma_1\star\gamma_2) = k\star(\gamma_1\star\gamma_2)\star k^{-1} = \big(k\star\gamma_1\star k^{-1}\big)\star\big(k\star\gamma_2\star k^{-1}\big) = \phi(\gamma_1)\star\phi(\gamma_2)$$

for any $\gamma_1, \gamma_2 \in \Gamma(X,*)$, and for any element $\gamma' \in \Gamma(X,\star)$,

$$\gamma'^{-1}\star\gamma'^{-1} = \phi\big(k^{-1}\star\gamma'\star k\big), \quad \text{where } k^{-1}\star\gamma'\star k \in \Gamma(X,*).$$

So ϕ is an isomorphism. Finally,

$$f\big(\gamma(c)\big) = k \star (\gamma \star c) = \big(k \star \gamma \star k^{-1}\big) \star (k \star c) = \phi(\gamma)\big(f(c)\big)$$

for any elements $c \in \overline{(X, *)}$ and $\gamma \in \Gamma(X, *)$. $\square$

3. Continua, fractals, and manifolds

For the next proposition, let X be a length space, $p \in X$. Define ρ-paths $\alpha : [0, a] \to X$ and $\beta : [0, b] \to X$ starting at p to be equivalent if α, β coincide on $[0, \varepsilon)$ for some $\varepsilon > 0$. We denote the cardinality of the set of the resulting equivalence classes by κ_p.

Proposition 23. *The valency of $\overline{X}$ at any point $\overline{p} \in \overline{\phi}^{-1}(p)$ is equal to κ_p. If X is separable then $\kappa_p \leqslant \mathfrak{c} = 2^{\aleph_0}$.*

Proof. The construction of $\overline{X}$ immediately implies the first statement. If X is separable, then X itself has cardinality $\mathfrak{c}$ (unless it is a point). Since every path is determined by its value at rational numbers in its domain, the cardinality of κ_p is at most $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. $\square$

Example 24. Consider the space B consisting of countably many circles $\{C_i\}_{i \in \mathbb{N}}$ each of length $s_i > 0$, all attached at a common basepoint $*$, and given the induced geodesic metric. We are interested in two cases: (1) s_i is a constant s; (2) s_i is a strictly decreasing sequence converging to zero. In the first case B is not compact. The valency of points in $\overline{B}$ is either $\aleph_0$ or 2 depending on whether they are in $\overline{\phi}^{-1}(p)$ or not. In the second case B is homeomorphic to the Hawaiian earring H, considered below.

An early result concerning H is Theorem 2.1 from [15] which states: If a one-dimensional separable locally connected continuum is not locally simply connected, then it contains a subspace which has the homotopy type of H.

Proposition 25. *For the Hawaiian earring H, supplied with any length metric of the second kind from Example* 24*, we have $\kappa_p = \mathfrak{c}$ for the point $p = *$ and $\kappa_p = 2$ for any point $p \neq *$.*

Proof. The last statement is evident. It is easy to find $\mathfrak{c}$ unit weakly normal loops starting (and ending) at $*$ such that no two coincide on any interval $[0, \varepsilon)$. In fact, let us take first an increasing integer sequence $\{i(n),\ n \in \mathbb{N}\}$ so that $s_{i(n)} < \frac{1}{2^n}$. One can define a path that wraps one of two ways around $C_{i(1)}$, then one of two ways around $C_{i(2)}$, and so on. Then reverse the parameterization, so that $C_{i(1)}$ is wrapped around last. It is clear that any such path is weakly normal and is encoded as a sequence $\{x_n\}_{n \in \mathbb{N}}$ with values in the set $\{-1, 1\}$. Two such arclength parameterized paths are equivalent if and only if they wrap the same way around $C_{i(n)}$ for all sufficiently large n, or in other words, define equivalent sequences $\{x_n\}$ and $\{y_n\}$—we mean here that x is equivalent to y if there is a number $m \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geqslant m$. The following lemma finishes the proof. $\square$

Lemma 26. *We have $\mathfrak{c}$ different equivalence classes of the above type of sequences.*

Proof. Let us take first the sequence $x_n \equiv 1$. Consider now an arbitrary sequence $z = \{z_n\}$ of natural numbers and define subsequently two sequences $s(z)_n := \sum_{i=1}^n z_i$ and $\sigma(z)_n := \sum_{i=1}^n 2^{s(z)_i}$. It is clear that if for another such sequence $w = \{w_n\}$, $w_m \neq z_m$, then $\sigma(w)_n \neq \sigma(z)_n$ for all $n \geqslant m$. Then for every sequence z above, define another sequence $x(z)_n$ with values in $\{-1, 1\}$ by the equations $x(z)_m = -1$ if $m = \sigma(z)_k$ for some $k \in \mathbb{N}$, and $x(z)_m = 1$ for all other $m \in \mathbb{N}$. Then it follows from the statement above that $x(z)_n$ is not equivalent to x and is not equivalent to $x(w)$ if $w \neq z$. So we get $\aleph_0^{\aleph_0} = \mathfrak{c}$ pairwise non-equivalent sequences of the above sort. □

Proof of Theorem 3. The first part of the theorem is an immediate consequence of Proposition 23 and the theorem from [28] that every $\mathbb{R}$-tree of valency at most $\mathfrak{c}$ isometrically embeds in $A_{\mathfrak{c}}$. Next, for any point $p \in S_c$ there is clearly a bi-Lipschitz embedding $h : H \to S_c$ such that $h(*) = p$. Therefore $\kappa_p \geqslant \mathfrak{c}$. The case of S_g is more tricky. There are countably many rectifiable loops C_i in S_g starting at any fixed point p such that every C_i is a topologically embedded circle and $L(C_{i+1}) < \frac{1}{3}L(C_i)$ for all natural numbers i. Then one can prove with a little more detail than in Proposition 25, that $\kappa_p \geqslant \mathfrak{c}$. A similar argument now finishes the proof of the theorem. □

4. URL-maps

The following statement easily follows from definitions.

Proposition 27. *The collection of pointed length spaces with URLs as morphisms is a category.*

Proposition 28. *A map $f : X \to Y$ between length spaces is a URL-map if and only if f is 1-Lipschitz and for some choice of basepoints, f is basepoint preserving, each arclength parameterized rectifiable path p in Y starting at the basepoint has unique lift p_L starting at the basepoint, and $L(p) = L(p_L)$.*

Proof. The necessity of these conditions easily follows from the definition of URL-map. Let us prove sufficiency. Assume first that c is an arclength parameterized rectifiable path starting at $y \in Y$ and $x \in X$ satisfies $f(x) = y$. Let k be a ρ-path from the basepoint in X to x. Then $d := f \circ k$ is rectifiable and since each of its initial segments has, by assumption, a unique lift of the same length, d is also arclength parameterized and has the same length as k. Then $d * c$ is rectifiable and arclength parameterized, so has a unique lift $(d * c)_L$ to the basepoint in X. We may write $(d * c)_L = k * k'$ for some path k'. Then k' is the desired lift of c; k' must be unique since if it were not then $d * c$ would not have a unique lift. Now suppose that $c : [0, a] \to Y$ is a rectifiable path starting at y, and $f(x) = y$. Let $\mathcal{C}$ be the collection of maximal (closed) intervals on which c is constant. As is well known, there are a non-decreasing continuous function $h : [0, a] \to [0, L(c)]$ and an arclength parameterized rectifiable path $c_1 : [0, L(c)] \to Y$ such that $c = c_1 \circ h$ and h is strictly increasing everywhere except on the intervals in $\mathcal{C}$. Let d_1 be the unique lift of c_1 at x. Define $c_L : [0, a] \to X$ by $c_L(t) = d_1 \circ h(t)$. Then c_L has the same length as d_1, hence as c_1 and c. Since f is 1-Lipschitz, it now follows from the lifting property proved above that if c is rectifiable in X then $f \circ c$ has the same length as c. If c is not rectifiable then $f \circ c$ cannot be rectifiable either, for if it were, $f \circ c$ would have a rectifiable lift and a non-rectifiable lift. □

Proposition 29. *Every URL-map $f : X \to Y$ is a weak submetry. If Y is a geodesic space then f is a submetry.*

Proof. Since Y is a length space, for any $\varepsilon > 0$ and $x, y \in Y$ we may join x, y by a rectifiable path c with the length less than $d(x, y) + \varepsilon$. By definition, c has a lift c_L of the same length with endpoints w, z such that $f(w) = x$ and $f(z) = y$. Since X is a length space, $d(w, z) \leqslant L(c_L) = L(c) \leqslant d(x, y) + \varepsilon$ and the proof of the first part is complete. If Y happens to be a geodesic space then we may take c to be a geodesic with corresponding inequality $d(w, z) \leqslant L(c_L) = L(c) = d(x, y)$. But from the first part we have that f is 1-Lipschitz, so $d(x, y) \leqslant d(z, w)$ and $d(x, y) = d(z, w)$. This implies that f is a submetry. $\square$

Proof of Theorem 5. (1) By Theorem 1, $\overline{\phi}$ is a weak submetry, so is a 1-Lipschitz map. Now, by Proposition 28, we need only to prove condition (II) for points $p = * \in X$ and $q = * \in \overline{X}$ and rectifiable arclength parameterized paths starting at $* \in X$. To do so, let $c : [0, L] \to \overline{X}$ be any rectifiable path in (X, d) starting at $*$. By Proposition 18 we have, for every $0 \leqslant s \leqslant L$, a unique up to Fréchet equivalence weakly normal path $p_s : [0, s] \to c([0, s])$ such that p_s is path homotopic to $c_s := c|_{[0,s]}$ in $c([0, s])$ and $L(p_s) \leqslant L(c_s)$. Define a ρ-path in (X, d) (and so an element in $\overline{X}$) $\bar{c}(s)$ to be the arclength parameterization of p_s. Proposition 18 implies that $\bar{c}(s)$ is uniquely defined by c and $s \in [0, L]$. If $0 \leqslant s_1 < s_2 \leqslant L$ then by definition of $\overline{d}$,

$$\overline{d}\big(\bar{c}(s_1), \bar{c}(s_2)\big) = L\big(\bar{c}(s_1)^{-1} \star \bar{c}(s_2)\big).$$

It is clear that the ρ-path $\bar{c}(s_1)^{-1} \star \bar{c}(s_2)$ is similarly defined by the path $c_{s_1}^{-1} * c_{s_2} = c_{s_1,s_2} := c|_{[s_1,s_2]}$. It follows from Proposition 18 and the argument above that $\overline{d}(\bar{c}(s_1), \bar{c}(s_2)) \leqslant L(c_{s_1,s_2})$. This implies that the path $\bar{c}(s)$, $s \in [a, b]$, is continuous in $(\overline{X}, \overline{d})$, $\overline{\phi} \circ \bar{c} = c$, and $L(\bar{c}) \leqslant L(c)$. Finally, since $\overline{\phi}$ is a 1-Lipschitz map we have $L(\bar{c}) = L(c)$.

To finish the proof of the condition (II), we need to prove that if $\bar{c}' : [0, L] \to \overline{X}$ is any path such that $\bar{c}'(0) = *$ and $\overline{\phi} \circ \bar{c}' = c$ then $\bar{c}' = \bar{c}$. Since $\overline{X}$ is an $\mathbb{R}$-tree by Theorem 1, for any $s \in [0, L]$, $C_s = \bar{c}'([0, s])$ is a Peano continuum that contains no topological circle. By the Hahn–Mazurkiewicz–Sierpin'ski Theorem, C_s is locally connected, hence a dendrite (see Section 51, VI of [27]). Then C_s is contractible by Corollary 7 in Section 54, VII of [27]. Hence there is a path homotopy $h_s : [a, s] \times [0, 1] \to C_s$ such that $h(\cdot, 0) = \bar{c}'|_{[0,s]}$, and $h(\cdot, 1) := \bar{c}'_s$ is a topological embedding whose image is the unique arc a_s in C_s, joining $\bar{c}'(0) = *$ and $\bar{c}'(s)$ (this arc exists by Corollary 7, Section 51, VI in [27]). Since $\overline{X}$ is an $\mathbb{R}$-tree, the arc a_s is a geodesic segment in $(\overline{X}, \overline{d})$. By Lemma 20, $\overline{\phi} \circ \bar{c}'_s = \overline{\phi} \circ \gamma_{\bar{c}'(s)} = \bar{c}'(s)$. It is clear that $\overline{\phi} \circ h_s$ is a path homotopy in $c([0, s])$ from $c_s := c|_{[0,s]}$ to the path $\bar{c}'(s)$. So the last path coincides with the ρ-path $\bar{c}(s)$ considered above, and we have proved the required equality $\bar{c}' = \bar{c}$.

(2) Given a URL-map $f : Z \to X$ with some choice of basepoints define $\overline{f}(c)$ to be the endpoint of the unique lift of c starting at the basepoint in Z. Obviously $f \circ \overline{f} = \overline{\phi}$. For $c, k \in \overline{X}$, the lift of $c \star k^{-1}$ is a path joining $f(c)$ and $f(k)$ having the same length as $c \star k^{-1} = d(c, k)$, and therefore $\overline{f}$ is 1-Lipschitz. Now let γ be a rectifiable path starting at the basepoint in Z. Then $f \circ \gamma$ is rectifiable in X, so has a lift $(f \circ \gamma)_L$ at the basepoint in $\overline{X}$. Now $\overline{f} \circ (f \circ \gamma)_L$ is a lift of $f \circ \gamma$ starting at the basepoint in Z and so must be equal to γ. That is, $(f \circ \gamma)_L$ is a lift of γ starting at the basepoint in $\overline{X}$ having the same length as γ. Suppose k is any lift of γ starting at the basepoint in $\overline{X}$. Then γ is also a lift of $f \circ \gamma$ to $\overline{X}$ and therefore $k = (f \circ \gamma)_L$. We have checked the conditions of Proposition 28 to show that $\overline{f}$ is a URL-map. Finally, suppose we have a URL-map h that preserves the basepoints with $f \circ h = \overline{\phi}$. For any $c \in \overline{X}$, $h \circ \gamma_c$ is a rectifiable path from the basepoint to $h(c)$, which is also a lift of $\overline{\phi} \circ \gamma_c = c$ starting at the basepoint in Z. Since this lift is unique, $h(c) = \overline{f}(c)$.

(3) The uniqueness of $\overline{X}$ follows from Proposition 29 and the second part of the theorem.

(4) Let (X_1, d_1) be an $\mathbb{R}$-tree and $\phi_1 : (X_1, d_1) \to (X, d)$ be a URL-map that preserves some basepoints $*$. By property (2) there is a unique basepoint preserving URL-map $\overline{\phi_1} : \overline{X} \to X_1$ such that $\overline{\phi} = \phi_1 \circ \overline{\phi_1}$. Since X_1 contains no topological circle, using the same arguments as in the proof of condition (II) above, we get that any ρ-path c in X_1 is injective (otherwise we could prove that c is not weakly normal), hence a topological embedding and a geodesic in X_1. This together with the construction of $\overline{X_1}$ implies that we can take $\overline{X_1} = X_1$ and $\overline{\phi}_1 = id_{X_1}$. Again applying property (2), we find a URL-map $\overline{\overline{\phi_1}} : X_1 \to \overline{X}$ such that $\overline{\phi_1} \circ \overline{\overline{\phi_1}} = id_{X_1}$. Since all three of these maps are weak submetries by Proposition 29, they are all isometries. $\square$

Using the same argument as in the proof for part (4) of Theorem 5, we get

Proposition 30. *If $f : X \to Y$ is URL-map between length spaces, and Y is an $\mathbb{R}$-tree, then f is an isometry.*

5. $\mathbb{R}$-free groups

The primary reference for the discussion that follows is [13]. A group acting freely on an $\mathbb{R}$-tree is usually called $\mathbb{R}$-*free*.

Theorem 31. *(See [34].) The fundamental group of a closed surface is $\mathbb{R}$-free, except for the non-orientable surfaces of genus* 1, 2, *and* 3 *(the connected sum of* 1, 2, *or* 3 *real projective planes).*

The non-orientable surfaces of genus 1, 2, and 3 are called *exceptional* and their fundamental groups *exceptional surface groups*. The torus has fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ and is embeddable in $(\mathbb{R}, +)$ [13]. Any subgroup of $(\mathbb{R}, +)$ acts freely by isometries on $\mathbb{R}$, so is $\mathbb{R}$-free.

Question 32. *(See [31].) It follows easily that any free product of non-exceptional surface groups and subgroups of* $(\mathbb{R}, +)$ *is $\mathbb{R}$-free. The question is whether the converse statement is true.*

The positive answer in the case of finitely generated groups is given by the following *Rips' Theorem*:

Theorem 33. *(See [20,10,13].) Any finitely generated $\mathbb{R}$-free group G can be written as a free product $G = G_1 * \cdots * G_n$ for some integer $n \geqslant 1$, where each G_i is either a finitely generated free abelian group or a non-exceptional surface group.*

As was pointed out in the Introduction, the answer to Question 32 is negative in general. All spaces below are length spaces with basepoints and maps are basepoint-preserving.

Theorem 34. *Let $L(g)$ denote the length of an element $g \in G = \Gamma(X, *)$, or in other words, $L(g) = \overline{d}(g, *)$ if we consider g as an element of $\overline{X}$. Then:*

(1) *$L(g) \geqslant 0$, and $L(g) = 0$ if and only if $g = 1 = *$.*
(2) *For all $g \in G$, $L(g) = L(g^{-1})$.*
(3) *For all $g, h, k \in G$, $c(g, h) \geqslant \min\{c(g, k), c(h, k)\}$, where $c(g, h)$ is defined to be $\frac{1}{2}(L(g) + L(h) - L(g^{-1}h))$.*

Proof. The first two properties are evident. The third statement is simply the 0-hyperbolicity property of $(\overline{X}, \overline{d})$ for elements g, h, k, since we earlier referred to $c(g, h)$ as the Gromov product. □

This theorem implies that L is a *Lyndon length function* on G, since by definition such a function satisfies precisely these properties except that in condition (1) only the "if" part of the second statement is required. We will refer to a Lyndon function satisfying the stronger condition (1) as *definite*.

Now let L be any definite Lyndon function on a group G. To obtain an $\mathbb{R}$-tree $T(G, L)$, first join any two elements $g, h \in G$ by an edge $[g, h]$ of length $L(g^{-1}h)$; then for any three elements $g, h, k \in G$ isometrically glue the initial segments of length $c(k^{-1}g, k^{-1}h)$ of the edges $[k, g]$ and $[k, h]$ starting at k. By construction, any point $x \in T(G, L)$ lies in some edge $[g, h]$ for $g, h \in G$. Since any two elements of G are joined by a segment, any two points $x, y \in T(G, L)$ are joined by some segment (of a finite length). Really this segment $[x, y]$ is unique, and we can define $\rho(x, y)$ as the length of the segment $[x, y]$. Now the action of G on itself by left multiplication, defined by the formula $l(g)(h) = gh$, has a well-defined extension to $T(G, L)$ by the requirement that any segment $[h, k]$ maps isometrically onto the segment $[gh, gk]$. This is possible because $L((gh)^{-1}(gk)) = L(h^{-1}k)$. Then G acts freely on $T(G, L)$ by isometries. It is clear that $(T(G, L), \rho)$ and the action of G on $T(G, L)$ are uniquely defined by the function L and the described construction. The equality $L(g) = \rho(1, g)$ returns us to the initial function L. Of course we have omitted the explanations of some natural questions arising in the process of this construction, but these details may be found in the literature on the subject.

Suppose now that G acts freely by isometries on an $\mathbb{R}$-tree (T, ρ). Choose any point $x \in T$ and define $L_x(g) = \rho(x, g(x))$. Then L_x is a Lyndon function on G that in general depends on x. We shall identify an element $g \in G$ with the point $g(x) \in T$. The $\mathbb{R}$-tree T is isometric to $T(G, L_x)$ and the action of G on T is equivalent to the action of G on $T(G, L_x)$ if and only if T is a *minimal* $\mathbb{R}$-tree in T containing all the elements of G.

Proposition 35. *Zastrow's group $G = G_Z$ coincides with $\Gamma(B)$ for the space B from Example* 24 *corresponding to the sequence $\{s_i = \frac{1}{i}\}$, and its Lyndon function coincides with the length L of ρ-loops from $\Gamma(B)$, measured in B. Moreover, there is an isometry of the $\mathbb{R}$-tree $T(G, L)$ onto $\overline{B}$, which establishes the equivalence of the action of G on $T(G, L)$ to the action of $G = \Gamma(B)$ on $\overline{B}$.*

Proof. Zastrow's group was defined by him as a subgroup of $\pi_1(H) \subset F$, where F is an inverse limit of a sequence F_n, $n \in \mathbb{N}$, of free groups of rank n, using a complicated combinatorial description of $\pi_1(H)$ and F. But using the fact that by Lemma 16, any element of $\pi_1(H)$ is represented by a normal loops based at $*$, we see that the first statement follows from the definitions of G in terms of the Lyndon function L on G (see [13, p. 231]), B, and $\Gamma(B)$. Also $L = L_{*=1}$, by our definition, $\Gamma(B) \subset \overline{B}$, and $\Gamma(B)$ is the orbit of the point $*$ relative to the isometric action of $\Gamma(B)$ on $\overline{B}$. It is clear that any ρ-path c in B starting at $*$ is an initial part of a ρ-loop in B. This implies that the $\mathbb{R}$-tree $\overline{X}$ is *minimal* in the above sense. By the above discussion, the proof is finished. □

Theorem 36. *If $f : X \to Y$ is an injective map such that for any rectifiable path c in X, the path $f \circ c$ in Y is rectifiable, and f topologically embeds the image of c into Y, then the natural induced map $f_* : \Gamma(X) \to \Gamma(Y)$ is an injective homomorphism. If f is a bijection such that*

f^{-1} has the same properties as f above, then the groups $\Gamma(X)$ and $\Gamma(Y)$ are isomorphic. In particular, the last statement is true for any bi-Lipschitz map $f: X \to Y$.

This theorem is an immediate corollary of the definitions. It gives some sufficient but most likely not necessary conditions for the next open question:

Question 37. *When are the groups $\Gamma(X)$ and $\Gamma(Y)$ isomorphic, or, more specifically, when is f_* an isomorphism?*

The following lemma is a corollary of Proposition 19 and the discussion prior to it.

Lemma 38. *If c is a non-trivial ρ-loop in X starting at $*$, then there is a unique maximal (by inclusion) ρ-path α in X starting at $*$ so that for some non-trivial ρ-loop β in X starting at the end of the path α, $c = \alpha * \beta * \alpha^{-1}$. Then β is also unique. In this situation, for any nonzero integer n, $c^n = \alpha * \beta^n * \alpha^{-1}$, and $L(c^n) = 2L(\alpha) + |n|L(\beta)$ if we consider c^n as an element of $\Gamma(X)$.*

Proposition 39. *Let any ρ-path in X starting at $*$ be an initial part of a ρ-loop in X and suppose there is a topological embedding $f: B \to X$, where B is the same as in Proposition 35, such that for any rectifiable path c in B, the path $f \circ c$ in X is rectifiable. Then $\Gamma(X)$ is an infinitely generated locally free group that is not free and not a free product of surface groups and abelian groups, but acts freely on the $\mathbb{R}$-tree $\overline{X}$. Moreover, the $\mathbb{R}$-tree $\overline{X}$ is a minimal invariant subtree with respect to this action.*

Proof. Lemma 38 implies that for $1 \neq g \in \Gamma(X)$, there is a natural number N such that if $g = h^n$, where n is an integer and $h \in \Gamma(X)$, then $|n| \leqslant N$. The group $\Gamma(X)$ is locally free by Theorem 1. These two statements mean that all the statements of Lemma 5.3.1 in [13] are true for the group $\Gamma(X)$. The special conditions for the map f and Theorem 36 imply that $\Gamma(X)$ contains a group isomorphic to Zastrow's group G. In Lemma 5.3.3 from [13] it is proved that G is not free. Then the Nielsen–Schreier Theorem implies that $\Gamma(X)$ is not free. In Lemma 5.3.4 from [13], which requires only the statements in Lemmas 5.3.1 and 5.3.3, it is proved that G is not a free product of surface groups and abelian groups. Applying the same proof, we get that $\Gamma(X)$ is not a free product of surface groups and abelian groups. We proved in Theorem 1 that $\Gamma(X)$ acts freely by isometries on the $\mathbb{R}$-tree $\overline{X}$. The yet unused assumption implies, as in the proof of Proposition 35, the last statement. □

Proof of Theorem 11. The second statement is proved in Proposition 35. Suppose we are given two length metrics d_s, d_t on H, defined by sequences $\{s_i\}$, $\{t_i\}$, $i \in \mathbb{N}$. Then there is an increasing integer sequence $k(i)$ such that $t_{k(i)} \leqslant s_i$ for all $i \in \mathbb{N}$. We can define a 1-Lipschitz map $f: (H, d_s) \to (H, d_t)$, which is also a topological embedding, by the requirement that $f(*) = *$ and $f|C_i : (C_i, d_s) \to (C_{k(i)}, d_t)$ is a bijective $(t_{k(i)}/s_i)$-Lipschitz map. By Theorem 36, this map induces an injective homomorphism $f_* : \Gamma(H, d_s) \to \Gamma(H, d_t)$. This proves the first statement. □

Proof of Theorem 10. The proof of Theorem 3, Theorem 11, and Proposition 35 imply that all these spaces satisfy the conditions of Proposition 39. An application of this proposition finishes the proof. □

Definition 40. A length space X is a local $\mathbb{R}$-tree at a point $x \in X$ if there is a number $r > 0$ so that the closed ball $B(x, r)$ is an $\mathbb{R}$-tree. The space X is said to be a local $\mathbb{R}$-tree if it is local $\mathbb{R}$-tree at all of its points.

An example of a local $\mathbb{R}$-tree is given by the first case of Example 24. Any traditional graph (see [13,14]) with some compatible length metric is also a local $\mathbb{R}$-tree.

Proposition 41. *If X is a length space which is a local $\mathbb{R}$-tree, then $\Gamma(X) = \pi_1(X)$ and $\pi_1(X)$ is a locally free group.*

Proof. Since small inductive dimension is a local notion, X is a local $\mathbb{R}$-tree, and every $\mathbb{R}$-tree has small inductive dimension 1 by [5], then $ind(X) = 1$. Then the image of any path homotopy in X is one-dimensional, and Proposition 14 implies that a path in X is normal if and only if it is weakly normal. By Lemma 16, any path in X is homotopic to a normal path. Since X is a local $\mathbb{R}$-tree, every normal path in X is a rectifiable, so it is a ρ-path. It follows from the above argument that $\Gamma(X) = \pi_1(X)$. Theorem 1 finishes the proof. $\square$

The following two propositions can be easily deduced from definitions.

Proposition 42. *Let $(X, *)$ be the wedge product of length spaces $(X_1, *)$ and $(X_2, *)$ supplied with the natural length metric. Then the group $\Gamma(X, *)$ is isomorphic to the free product $\Gamma(X_1, *) * \Gamma(X_2, *)$ if at least one of X_1 or X_2 is a local $\mathbb{R}$-tree at $*$.*

Proposition 43. *Let (X, d) be a length (respectively, geodesic) space which is a local $\mathbb{R}$-tree at a point $* \in X$. Then the closure in X of any connected component of $X - \{*\}$ with the subspace metric d is a length (respectively, geodesic) space, $(X, *)$ is the wedge product of the family $\{(X_\alpha, *),\ \alpha \in A\}$ of all such closures, and the group $\Gamma(X, *)$ is isomorphic to the free product $\prod^*_{\alpha \in A} \Gamma(X_\alpha, *)$ of the groups $\Gamma(X_\alpha, *)$, $\alpha \in A$.*

Proposition 44. *For any family $\{X_\alpha,\ \alpha \in A\}$ of length (respectively, geodesic) spaces there exists a length (respectively, geodesic) space X such that the group $\Gamma(X)$ is isomorphic to the free product $\prod^*_{\alpha \in A} \Gamma(X_\alpha)$ of the groups $\Gamma(X_\alpha)$, $\alpha \in A$.*

Proof. For every $\alpha \in A$, choose an arbitrary point $\star \in X_\alpha$ and let $(X'_\alpha, *)$ be X_α together with a segment σ_α of fixed length $l > 0$ with endpoints $\star$ and $*$, attached to X_α in such a way that σ_α has the point $\star$ in common with X_α. By Proposition 43, $\Gamma(X'_\alpha, \star)$ is isomorphic to the free product $\Gamma(X_\alpha, \star) * \Gamma(\sigma_\alpha, \star) = \Gamma(X_\alpha, \star)$ because $\Gamma(\sigma_\alpha, \star)$ is the trivial group. By Proposition 22, the groups $\Gamma(X'_\alpha, \star)$ and $\Gamma(X'_\alpha, *)$ are isomorphic. Then define $(X, *)$ as the wedge product of the spaces $(X'_\alpha, *)$. It is clear that $(X, *)$ is a length (respectively, geodesic) space if all X_α are length (respectively, geodesic) spaces, and $(X, *)$ is local $\mathbb{R}$-tree at $*$. Furthermore, the closures of the connected components of $(X, *) - \{*\}$ are exactly the spaces X'_α, $\alpha \in A$, and we can apply Proposition 43. $\square$

6. Piecewise continuously differentiable paths

Let (X, d) be any (connected) Riemannian manifold M^n of dimension $n \geqslant 2$ with its length metric. Let $\tilde{X}$ consist of all ρ-paths in (X, d) starting at $* \in X$ that are piecewise continuously differentiable, and $\tilde{\Gamma}(X) \subset \tilde{X}$ be the corresponding group of loops at $*$. Denote by $\star$ the restriction of the operation $\star$ to $\tilde{\Gamma}(X)$.

Theorem 45. *The $\mathbb{R}$-tree $\tilde{X}$ is a subtree of the $\mathbb{R}$-tree $(\overline{X}, \overline{d})$ with the induced metric and $(\tilde{\Gamma}(X), \star)$ is a (locally free) subgroup of $(\Gamma(X), \star)$. The $\mathbb{R}$-tree $\tilde{X}$ has valency $\mathfrak{c}$ at each point but is never complete. The length space (X, d) is the metric quotient of $(\tilde{X}, \tilde{d})$ via the free isometric action of the group $\tilde{\Gamma}(X)$ on $\tilde{X}$. The quotient mapping $\tilde{\phi} : \tilde{X} \to X$ is an arcwise isometry, a weak submetry (hence open) and light map, and $\tilde{\phi}$ is a submetry if X is geodesic. Moreover, $\tilde{X}$ is the minimal invariant subtree relative to the action of $\tilde{\Gamma}(X)$.*

Proof. The first statement is evident, and this implies that the group $(\tilde{\Gamma}(X), \star)$ acts freely via isometries on $\tilde{X}$. Since $\tilde{X}$ is a subtree of the $\mathbb{R}$-tree $\overline{X}$, which is isometric to $A_{\mathfrak{c}}$ by Theorem 3, $\overline{X}$ has valency at each point no more than $\mathfrak{c}$. On the other hand, we can extend any path $c \in \tilde{X}$ with endpoint $c(L)$ by a geodesic segment starting at $x = c(L)$ which has arbitrary tangent unit vector v at the point x. Since we have $\mathfrak{c}$ such vectors, Proposition 23 implies that $\overline{X}$ has valency $\mathfrak{c}$ at each point. It is easy to construct a rectifiable map $c : [0, L) \to X$ starting at $*$ such that for any number L_k, $0 < L_k < L$, the restriction $c_k = c|[0, L_k]$ is an arclength parameterized piecewise continuously differentiable path, but c either cannot extend continuously to some $c' : [O, L] \to X$ (if X is not complete), or such an extension c' exists but is not a piecewise continuously differentiable path. If we assume now that $L_k \nearrow L$, then c_k is a Cauchy sequence in $\tilde{X}$ which has no limit in $\tilde{X}$, and so $\tilde{X}$ is not complete. As a corollary of Theorem 1, $\overline{\phi}$ is 1-Lipschitz. Then its restriction $\tilde{\phi}$ is also 1-Lipschitz. As in the proof of Theorem 1, using piecewise continuously differentiable paths instead of more general rectifiable paths in X, we get that the quotient mapping $\tilde{\phi} : \tilde{X} \to X$ is an arcwise isometry, a weak submetry (hence open) and light map, and $\tilde{\phi}$ is a submetry if X is geodesic. As in the proof of Theorem 1, this, together with the previously proved statements, implies the third statement. The group $\tilde{\Gamma}(X)$ is locally free as a subgroup of the locally free group $\Gamma(X)$. Considering only piecewise continuous ρ-paths in X, we get from Proposition 39 that $\tilde{X}$ is the minimal invariant subtree relative to the action of $\tilde{\Gamma}(X)$. □

Remark 46. Notice that for $\tilde{\Gamma}(X)$ in the above theorem we may take also the subset of $\overline{X}$ consisting of broken geodesics in X, where X is M^n, $n \geqslant 2$; $\mathbb{M}$, S_c, S_g, or H. Since $\tilde{\Gamma}(X)$ is locally free and satisfies condition (1) from Lemma 5.3.1 in [13], it cannot include a subgroup that is isomorphic to the fundamental group of a surface or a non-cyclic subgroup of $(\mathbb{R}, +)$. So Question 32 for the group $\tilde{\Gamma}(X)$ is equivalent to the question of whether $\tilde{\Gamma}(X)$ is a free group. We don't have an answer to this question.

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