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Local-to-global spectral sequences for the cohomology of diagrams

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ABSTRACT

We construct local-to-global spectral sequences for the cohomology of a diagram, which compute the cohomology of the full diagram in terms of smaller pieces. These are motivated by the obstruction theory of D. Blanc et al. [D. Blanc, M.W. Johnson, J.M. Turner, On realizing diagrams of Π-algebras, Algebraic Geom. Topol. 6 (2006) 763–807] for realizing a diagram of Π-algebras, but are valid in quite general algebraic settings.

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0. Introduction

The cohomology of diagrams arises as a natural object of study in several mathematical contexts: in deformation theory (see [23,22,21]), and in classifying diagrams of groups, as in [13]. If I is the one-object category corresponding to a group G, a diagram $X \in \mathcal{C}^I$ is just an object in $\mathcal{C}$ equipped with a G-action, and its cohomology is the equivariant cohomology of [26] (cf. [33, Section 2]). On the other hand, for any discrete or Lie group G, let $I = \mathcal{O}_G$ denote the orbit category of G: if X is a G-space, $\mathcal{X} : \mathcal{O}_G \to \mathcal{T}op$ is the corresponding fixed point diagram $\mathcal{X}(G/H) := X^H$, and $M : \mathcal{O}_G \to \mathcal{A}bGp$, is any system of coefficients, then the corresponding cohomology $H(\mathcal{X}; \mathcal{M})$ is Bredon cohomology (cf. [28, I, Section 4]). Finally, when I consists of a single arrow, and the coefficients are constant, we have the usual cohomology of a pair. See [5,17,20,31,32,6] for further applications.

0.1. Diagrams in homotopy theory

The cohomology of diagrams also plays a major role in the Dwyer–Kan–Smith theory for the rectification of homotopy-commutative diagrams (cf. [19,16,18]). In fact, our interest in the subject was motivated by the related realization problem for diagrams of Π-algebras (graded groups with an action of the primary homotopy operations): as in the case of a single Π-algebra (cf. [8]), the obstructions to realizing a diagram of Π-algebras $\Lambda : I \to \Pi\text{-}\mathcal{A}lg$ lie in appropriate cohomology groups of Λ (see [9, Thm. 6.3]).

Furthermore, given a Π-algebra Γ, all distinct homotopy types realizing Γ may be distinguished by a set of higher homotopy operations associated to a collection $(I^\alpha)_{\alpha\in A}$ of finite indexing categories I^α and homotopy-commutative diagrams $X^\alpha : I^\alpha \to \mathrm{ho}\,\mathcal{T}op$, where all the spaces X^α_i are wedges of spheres (cf. [7]). Since these higher operations are obstructions to the rectification of the diagrams $(X^\alpha)_{\alpha\in A}$ (and thus the associated diagrams $\Lambda^\alpha := \pi_* X^\alpha : I^\alpha \to$

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Π-$\mathcal{A}lg$), they correspond to elements in the cohomology of Γ. Understanding the cohomology groups of such diagrams may therefore be helpful in algebraicizing (and organizing) the "higher Π-algebra" of a space Y, consisting of all higher homotopy operations in $\pi_* Y$.

0.2. Computing diagram cohomology

Even the cohomology of a single map may be hard to calculate (cf. [9, Section 5.16]), so some computational tools are needed. For this purpose we construct "local-to-global" spectral sequences for the cohomology of a diagram, which can be used to compute the cohomology of the full diagram in terms of smaller pieces.

Given a small category I, a model category $\mathcal{C}$ (in the sense of [35]), and an I-diagram $X \in \mathcal{C}^I$, one can define the cohomology of X with coefficients in any abelian group object $Y \in \mathcal{C}^I$. For technical reasons, we shall concentrate on the case where $\mathcal{C} = s\mathcal{A}$ is the category of simplicial objects over some variety of universal algebras $\mathcal{A}$: since the homotopy category of simplicial groups is equivalent to that of (pointed connected) topological spaces, this actually covers all cases of interest above. Some of our results are valid, however, for an arbitrary simplicial model category $\mathcal{C}$.

Another reason for our interest in the "local-to-global" approach to diagram cohomology is that in order for the higher homotopy operation corresponding to a homotopy-commutative diagram $X : I \to \operatorname{ho}\mathcal{T}op$ to be *defined*, all lower order operations (corresponding to subdiagrams of I) must vanish *coherently*. Thus an essential step in a cohomological description of higher order operations is the ability to piece together local data to obtain global information.

Remark 0.3. We should point out that our methods work (almost exclusively) for a *directed* indexing category I (i.e., with only identities as endomorphisms), which is a significant restriction. However, such diagrams certainly suffice for the description of higher homotopy operations, as above: even the linear case – when I consists of a single composable sequence of arrows – is of interest, since the realizability of such a diagram is essentially equivalent to calculating higher Toda brackets. Furthermore, diagrams arising in deformation theory (indexed by the nerve of a covering) are of this form. Our methods, suitably modified (cf. Remark 1.7), also apply to diagrams indexed by the orbit category $\mathcal{O}_G$ of a group G.

0.4. The spectral sequences

Let $\mathcal{C}$ be a simplicial model category and I a directed index category, and assume given diagrams $Z : I \to \mathcal{C}$, and $X, Y \in \mathcal{C}^I/Z$, with Y an abelian group object in $\mathcal{C}^I/Z$. Our main results may be summarized as follows:

Theorem A. There is a first quadrant spectral sequence with:

$$E^2_{s,t} = \prod_{j \in \widetilde{J}_s} H^{t+s}(X_j/Z_j, \hat{\phi}_j) \Longrightarrow H^{s+t}(X/Z; Y).$$

This is constructed by taking increasing truncations of the coefficient diagram Y (cf. Theorem 3.5). Here $H^*(X/Z, \phi)$ denotes relative cohomology for a map of the coefficients (see Definition 3.1).

Theorem B. There is a first quadrant spectral sequence with:

$$E^2_{s,t} = H^{s+t}(\eta_s; Y) \Longrightarrow H^{s+t}(X/Z; Y).$$

This spectral sequence is constructed dually to the previous one, by taking increasing truncations of the *source* diagram X (see Theorem 3.7). Here $H^*(\eta, Y)$ denotes the usual cohomology of a map (or pair).

Theorem C. If I is countable, then for any ordering $(c_s)_{s=1}^{\infty}$ of the objects of I, there is a first quadrant spectral sequence with $E^2_{s,t} = H^{t+s}_{c_s}(X/Z; Y) \Longrightarrow H^{s+t}(X/Z; Y)$.

This is constructed by successively omitting the objects c_s from I (see Theorem 7.7). Here $H^*_c(X/Z, Y)$ denote the local cohomology groups at an object $c \in I$ (see Definition 7.4).

There are versions of all three spectral sequences defined for any suitable cover $\mathcal{J}$ of I (Definition 1.1). In particular, the spectral sequences always converge if $\mathcal{J}$ is finite, hence if I itself is finite.

0.5. Other variants

Other spectral sequences for the cohomology of a diagram have appeared in the literature. One should mention the universal coefficient spectral sequence of Piacenza (see [34, Section 1]), the the p-chain spectral sequence of Davis and Lück (see [15]), the equivariant Serre spectral sequence of Moerdijk and Svensson (see [30], and the local-to-global spectral sequences of Jibladze and Pirashvili (see [27]) and Robinson (see [37]) — though the last three use a different definition of cohomology, based on the Baues–Wirsching and Hochschild–Mitchell cohomologies of categories (cf. [3,29]). The spectral sequences for the cohomology of a homotopy (co)limit of a diagram (cf. [40] and [2,11]) may also be related to ours in special cases.

0.6. *Organization*

Section 1 provides background material on diagrams, their covers, and the model category of diagrams. In Section 2 we determine when the "restriction tower" associated to a cover of the indexing category I is a tower of fibrations, and in Section 3 we use this to set up the first two spectral sequences.

The second half of the paper is devoted to the (somewhat more technical) approach based on "localizing at an object": Section 4 provides the setting, and explains the method. In Section 5 we describe an auxiliary construction associated to the tower of certain covers of I, and in Section 6 show that this auxiliary tower is a tower of fibrations. Finally, in Section 7 we identify the fibers of the new tower, and obtain the third spectral sequence.

1. The category of diagrams

Our object of study will be the category $\mathcal{C}^I$ of diagrams — i.e., functors from a fixed small (often finite) indexing category I into a model category $\mathcal{C}$. The maps are natural transformations. In this section we define some concepts and introduce notation related to I and $\mathcal{C}^I$:

Definition 1.1. Let I be any small category. By an N-indexed *cover* of I we mean some collection $\mathcal{J} = \{J_\nu\}_{\nu \in N}$ of subcategories of I, such that each arrow in I belongs to at least one J_ν.

A cover $\mathcal{J} = \{J_\nu\}_{\nu \in N}$ for I will be called *orderable* if the relation:

$$\nu_1 \prec \nu_2 \overset{\text{Def}}{\Longleftrightarrow} \exists\, i_1 \in J_{\nu_1}, i_2 \in J_{\nu_2} \exists\, \phi : i_2 \to i_1 \quad \text{in } I \text{ with } i_1 \notin J_{\nu_2} \text{ or } i_2 \notin J_{\nu_1}$$

defines a partial order on N, and the partially ordered set $(N, \prec)$ can be embedded as a (possibly infinite) segment of $(\mathbb{Z}, \leq)$. Choosing such an embedding $N \subseteq \mathbb{Z}$, we may think of $\mathcal{J}$ as being indexed by integers, and we can then filter I by setting $J[n] := \bigcup_{i \leq n} J_i$. If N is bounded below in $\mathbb{Z}$ we say that $\mathcal{J}$ is *right-orderable*, and if it is bounded above we say it is *left-orderable*.

Remark 1.2. Note that the linear ordering of $\mathcal{J}$ (indicated by the indices) is not generally uniquely determined by the partial order $\prec$: there may be elements of $\mathcal{J}$ which are not comparable under $\prec$. This happens when all maps out of J_n actually land in $J[k]$ for $k < n - 1$. In this case the linear ordering of J_n and J_{n-1}, for example, may be switched with impunity.

1.3. *Directed indexing categories*

A *directed indexing category* is a small category I equipped with a map $\deg : \mathrm{Obj}(I) \to \mathbb{Z}$, such that for every non-identity map $\phi : j \to i$ in I, $\deg(j) > \deg(i)$. Then I is filtered by the full subcategories $I_n = J[n]$ whose objects have degree at most n.

An orderable cover $\mathcal{J} = \{J_n\}_{n \in N}$ for such an I will be called *compatible* (with the choice of deg) if there is a strictly increasing sequence of integers $(k_n)_{n \in N}$ such that $\mathrm{Obj}(J_n) = \deg^{-1}([k_{n-1}, k_n])$.

Example 1.4. The *fine cover* for a directed indexing category I is defined by letting J_n be the subcategory obtained from the "difference categories" $\widetilde{J}_n := I_n \setminus I_{n-1}$ (discrete, by assumption) by adding all the maps from any of these objects into I_{n-1}.

For instance, if $I = [\mathbf{n}]$ is the *linear* category of n composable maps (with degrees as labels):

$$n \xrightarrow{\phi_n} n-1 \xrightarrow{\phi_{n-1}} \cdots 2 \xrightarrow{\phi_2} \cdots 1 \xrightarrow{\phi_1} 0,$$

then I_k consists of the k arrows on the right, $\widetilde{J}_k = \{k\}$, and the fine cover thus is $J_k := \{\phi_k\}$.

Example 1.5. If I is the commutative square diagram

$$\begin{tikzcd} 4 \arrow[r, "d"] \arrow[d, "c"'] & 3 \arrow[d, "b"] \\ 2 \arrow[r, "a"'] & 1 \end{tikzcd} \tag{1.6}$$

then $\widetilde{J}_k$ contains only k, while $J_2 = \{a : 2 \to 1\}$, $J_3 = \{b : 3 \to 1\}$, and J_4 contains both $c : 4 \to 2$ and $d : 4 \to 3$ (since I_3 contains both 2 and 3).

Remark 1.7. As noted in the introduction, a group (or monoid) G may be thought of as a category with a single object. If we start with a directed indexing category I', and for $i \in I'$, we add maps $g : i \to i$ for each $g \in G$ for some group $G = G_i$ (with suitable commutation relations with the maps of I'), we obtain a small category I (no longer directed) whose diagrams describe directed systems of group actions. Clearly, any orderable cover $\mathcal{J}'$ of I' induces an orderable cover $\mathcal{J}$ of I.

Example 1.8. Let I' consist of two parallel arrows $\phi_1, \phi_{-1} : i \to j$, $G_i = \mathbb{Z}/2$, and $G_j = 0$. Then the indexing category I has a single new non-identity map $f : i \to i$ and $\phi_k \circ f = \phi_{-k}$ $(k = \pm 1)$. Compare [14].

1.9. Model categories

Now let $\mathcal{C}$ be a simplicial model category (cf. [35, II, Section 1]), and let $\mathcal{C}^I$ denote the functor category of I-diagrams in $\mathcal{C}$. There are (at least) two relevant simplicial model category structures on $\mathcal{C}^I$:

(a) For general I and cofibrantly generated $\mathcal{C}$, we have the *diagram* model category structure, in which the weak equivalences and fibrations are defined objectwise, and the cofibrations are generated (under retracts, pushouts, and transfinite compositions) by the free maps (free on a generating cofibration at some $i \in I$) — cf. [25, Theorem 11.6.1].

(b) If I is a directed indexing category as above, it is in particular a (one-sided) Reedy category (cf. [25, Section 15.1.1]). Thus $\mathcal{C}^I$ has a *Reedy* model category structure, in which the weak equivalences are defined objectwise, the cofibrations are defined by attaching a suitable latching object, and the fibrations are defined by requiring that the structure maps to the matching objects are all fibrations (cf. [25, Section 15.3]).

Remark 1.10. In the cases where I is a Reedy category and $\mathcal{C}$ is cofibrantly generated, the identity $\mathrm{Id} : \mathcal{C} \to \mathcal{C}$ is a strong Quillen functor (actually a Quillen equivalence) between the two model category structures (see [25, Theorem 15.6.4]), considered as a right adjoint from the Reedy model structure to the diagram model structure. As a consequence, every Reedy fibration is an objectwise fibration (cf. [25, Proposition 15.3.11]), and conversely, every cofibration in the diagram model category is a Reedy cofibration. In both cases we use the same simplicial mapping space $\mathrm{map}_{\mathcal{C}^I}(X, Y)$, (sometimes denoted simply by $\mathrm{map}\,(X, Y)$), with

$$\mathrm{map}_{\mathcal{C}^I}(X, Y)_n := \mathrm{Hom}_{\mathcal{C}^I}(X \times \Delta[n], Y). \tag{1.11}$$

1.12. Diagrams over Z

For a fixed ground diagram $Z : I \to \mathcal{C}$, the comma category $\mathcal{C}^I/Z$ consists of diagrams $X : I \to \mathcal{C}$ over Z — that is, for each $i \in I$ we have maps $p_i : X_i \to Z_i$, natural in I. Once again $\mathcal{C}^I/Z$ has the two model category structures described above. The simplicial mapping space $\mathrm{map}_{\mathcal{C}^I/Z}(X, Y)$, defined as in (1.11), will usually be denoted simply by $\mathrm{map}_Z(X, Y)$. We may assume that Z is Reedy fibrant, so in particular (objectwise) fibrant.

1.13. Sketchable categories

Most of our results are valid for quite general simplicial model categories $\mathcal{C}$. However, as noted in the introduction, we shall be mainly interested in the case where $\mathcal{C} = s\mathcal{A}$ is the category of simplicial objects over some FP-sketchable category $\mathcal{A}$ (essentially: a category of (possibly graded) universal algebras — cf. [1, Section 1]). Note that any such $\mathcal{C}$ is cofibrantly generated — in fact, a resolution model category (see [9, Section 3]). Such an $\mathcal{A}$ will be called $\mathfrak{G}$*-sketchable* if it is equipped with a faithful forgetful functor to a category of graded groups (compare [10, Section 4.1]). The important property for our purposes is that a map $f : X \to Y$ in $\mathcal{C}$ is a fibration if and only if it is an epimorphism onto the basepoint component of Y (cf. [35, II, Section 3, Prop. 1]).

If we let $\mathcal{A} = \mathcal{G}p$, we obtain the homotopy category of pointed connected topological spaces (see [24, V, Section 6]), so our assumptions cover all the topological applications mentioned in the introduction.

In this context we may need to consider diagrams over a fixed ground diagram Z: following [36, Section 2] and [4, Section 3], for (diagrams of simplicial objects in) a $\mathfrak{G}$-sketchable category $\mathcal{A}$, one may identify Z-modules with abelian group objects over Z. Thus we may be forced to work in $\mathcal{C}^I/Z$ if we want to study cohomology with twisted coefficients.

1.14. Diagram completion

Any inclusion of categories $J \hookrightarrow I$ induces a forgetful *truncation* functor $\tau = \tau^I_J : \mathcal{C}^I \to \mathcal{C}^J$, and this has a right adjoint $\xi = \xi^I_J : \mathcal{C}^J \to \mathcal{C}^I$, which assigns to a diagram $Y : J \to \mathcal{C}$ the diagram $\xi Y : I \to \mathcal{C}$ with $\xi Y(i) := \lim_{i/J} Y$ for each $i \in I$ (where i/J is the obvious subcategory of the under category i/I). Note that $\xi Y(j) = Y_j$ for $j \in J$. Also, if $J \subseteq J' \subseteq I$ then $\xi^{J'}_J = \tau^I_{J'} \circ \xi^I_J$, $\xi^I_J = \xi^I_{J'} \circ \xi^{J'}_J$, and $\tau^I_J = \tau^{J'}_J \circ \tau^I_{J'}$, so we shall often omit the superscripts from these functors, with the second category understood from the context.

The resulting monad $\sigma_J := \xi_J \circ \tau_J : \mathcal{C}^I \to \mathcal{C}^I$ is called the *completion* at J, and we denote the augmentation of the adjunction by $\omega_J : Y \to \sigma_J Y$.

Moreover, given a fixed $Z \in \mathcal{C}^I$, the truncation functor $\hat{\tau}_J : \mathcal{C}^I/Z \to \mathcal{C}^J/\tau Z$ also has a right adjoint $\hat{\xi}_J : \mathcal{C}^J/\tau Z \to \mathcal{C}^I/Z$, with the limit $\hat{\xi}_J Y(i) := \lim_{i/J} Y$ taken over $\tau_J Z$ (that is, the diagram whose limit we take consists of $Y|_{i/J}$ mapping to $\tau_J Z$, where the latter includes also Z_i). Thus the completion at J in $\mathcal{C}^I/Z$ is:

$$\hat{\sigma}_J Y(j) = \sigma_J Y(j) \times_{\sigma_J Z(j)} Z_j, \tag{1.15}$$

where the structure map $\sigma_J q : \sigma_J Y \to \sigma_J Z$ is induced by the functoriality of limits. Once again, there will be an augmentation $\hat{\omega}_J : Y \to \hat{\sigma}_J Y$.

Example 1.16. If $I = [\mathbf{n}]$ is linear (Example 1.4) and $J = [\mathbf{k}]$ is an initial (right) segment, then for any tower $Y : [\mathbf{n}] \to \mathcal{C}$ we have:

$$\sigma_J Y(i) = \begin{cases} Y_i & \text{if } i \leq k \\ Y_k & \text{if } i \geq k. \end{cases}$$

Example 1.17. If I is the commutative square of Example 1.5, then $\sigma_{I_3} Y$ is the pullback diagram

$$\begin{tikzcd} Y_2 \times_{Y_1} Y_3 \arrow[r] \arrow[d] & Y_3 \arrow[d, "Y(b)"] \\ Y_2 \arrow[r, "Y(a)"'] & Y_1, \end{tikzcd} \tag{1.18}$$

while $\hat{\sigma}_{I_3} Y(3)$ is the further pullback

$$\begin{tikzcd} \hat{\sigma}_{I_3} Y(3) \arrow[r] \arrow[d] & Y_2 \times_{Y_1} Y_3 \arrow[d] \\ Z_4 \arrow[r] & Z_2 \times_{Z_1} Z_3. \end{tikzcd} \tag{1.19}$$

Example 1.20. If $I = \Delta' \subseteq \Delta^{\mathrm{op}}$ is the indexing category for restricted simplicial objects Y (without degeneracies), and J is its truncation to dimensions $< n$, then $\sigma_J Y(n) = M_n Y$ is the classical matching object of [12, X, Section 4.5].

1.21. Maps of diagrams

Given a fixed Reedy fibrant ground diagram $Z : I \to \mathcal{C}$, consider the simplicial mapping space $\mathrm{map}_Z(X, Y)$ as in Section 1.12 for $X, Y \in \mathcal{C}^I/Z$, where X is cofibrant and Y is fibrant.

In the cases of interest to us, Y will be an abelian group object in $\mathcal{C}^I/Z$, so the homotopy groups of $\mathrm{map}_Z(X, Y)$ are the cohomology groups of X with coefficients in Y (see [9, Section 5] for further details). In order to build our restriction tower, we need an appropriate orderable cover $\mathcal{J}$ of I (Definition 1.1), yielding a filtration

$$I \supseteq \cdots \supseteq I_n \supseteq I_{n-1} \supseteq \cdots .$$

Let $M_n := \mathrm{map}_{\mathcal{C}^{I_n}/\tau_n Z}(\tau_n X, \tau_n Z)$ for each $n \in N$, where $\tau_n X$ is the restriction of a diagram $X \in \mathcal{C}^I$ to I_n. The inclusions $I_{n-1} \hookrightarrow I_n$ and $I_n \hookrightarrow I$ induce maps $\rho_n : M_n \to M_{n-1}$ and $\hat{\rho}_n : M \to M_n$ which fit into a tower:

$$\begin{tikzcd} \mathrm{map}_Z(X, Y) \arrow[dr, "\hat{\rho}_{n+1}"] \arrow[drr, bend left, "\hat{\rho}_n"] \arrow[drrr, bend left, "\hat{\rho}_{n-1}"] & & & & \\ \cdots \arrow[r] & M_{n+1} \arrow[r, "\rho_{n+1}"] & M_n \arrow[r, "\rho_n"] & M_{n-1} \arrow[r, "\rho_{n-1}"] & \ldots M_0 \end{tikzcd} \tag{1.22}$$

with

$$\mathrm{map}_Z(X, Y) \cong \lim_n M_n. \tag{1.23}$$

2. A tower of fibrations

To determine when (1.22) is a tower of fibrations (so that (1.23) is a homotopy limit), we need the following:

Definition 2.1. Let I be an indexing category, $\mathcal{C}$ a model category, and $Z \in \mathcal{C}^I$. Given an orderable cover $\mathcal{J} = \{J_\nu\}_{\nu \in N}$ of I with associated filtration $(I_n) = (J[n])_{n \in \mathbb{Z}}$, let $\tau_k : \mathcal{C}^I \to \mathcal{C}^{I_k}$ and $\tau_k^m : \mathcal{C}^{I_m} \to \mathcal{C}^{I_k}$ denote the truncation functors, with adjoints indexed accordingly. A diagram $Y \in \mathcal{C}^I/Z$ is called *$\mathcal{J}$-fibrant* if for each $n \in \mathbb{Z}$, the augmentation $\hat{\omega}_{n+1} : \tau_{n+1} Y \to \hat{\sigma}_n^{n+1} Y = \hat{\sigma}_{I_n}^{I_{n+1}} Y$ is a fibration in $\mathcal{C}^{I_{n+1}}/\sigma_n^{n+1} Z = \mathcal{C}^{I_{n+1}}/\sigma_{I_n}^{I_{n+1}} Z$.

Remark 2.2. Because we assumed the degree is strictly decreasing, I_{n+1} and I are the same so far as the augmentation map $\hat{\omega}_{n+1}$ is concerned. Thus if we assume for simplicity that $I = I_{n+1}$, then $\hat{\omega}_{n+1}$ may be identified with its adjoint map $Y \to \hat{\sigma}_n Y$ in $\mathcal{C}^{I_{n+1}}/\sigma_n^{n+1} Z = \mathcal{C}^I/\sigma_n Z$.

Proposition 2.3. *Assume $\mathcal{J} = \{J_\nu\}_{\nu\in N}$ is an orderable cover of I, $X \in \mathcal{C}^I/Z$ is cofibrant, and $Y \in \mathcal{C}^I/Z$ is a $\mathcal{J}$-fibrant abelian group object. Then*

$$F_{n+1} \to M_{n+1} \xrightarrow{\rho_{n+1}} M_n$$

is a fibration sequence of simplicial abelian groups for each $n \in \mathbb{Z}$, and the fiber F_{n+1} is $\mathrm{map}_{\mathcal{C}^{J_{n+1}}/Z|_{J_{n+1}}}(X|_{J_{n+1}}, \mathrm{Fib}(\omega_{n+1}))$*. Here* $\mathrm{Fib}(\omega_{n+1})$ *denotes the fiber (in $\mathcal{C}^{I_{n+1}}/\sigma_n^{n+1}Z$) of the augmentation $\omega_{n+1} : \tau_{n+1}Y \to \sigma_n^{n+1}Y = \sigma_{I_n}^{I_{n+1}}Y$.*

Proof. Assume for simplicity that $I = I_{n+1}(= J[n+1])$, with $\tau_n = \tau_{I_n} : \mathcal{C}^I \to \mathcal{C}^{I_n}$ and $\sigma_n(= \sigma_{J[n]})$ the completion at $I_n(= J[n])$ (as in Remark 2.2). Then there is a natural adjunction isomorphism:

$$\mathrm{map}_{\mathcal{C}^{I_n}/\tau_n Z}(\tau_n X, \tau_n Y) = \mathrm{map}_{\mathcal{C}^I/\sigma_n Z}(X, \hat{\sigma}_n Y),$$

under which ρ_n is identified with the map induced in $\mathrm{map}_{\sigma_n Z}(X, -)$ by $\hat{\omega}_{n+1} : Y \to \hat{\sigma}_n Y$. This $\hat{\omega}_{n+1}$ is a fibration in $\mathcal{C}^I/\sigma_n Z$ by Definition 2.1, and thus induces a fibration of mapping spaces, with fiber $\mathrm{map}_{\sigma_n Z}(X, \mathrm{Fib}(\hat{\omega}_{n+1}))$.

Thus, it suffices to identify the fiber instead as $\mathrm{map}_{\mathcal{C}^{J_{n+1}}/Z|_{J_{n+1}}}(X|_{J_{n+1}}, \mathrm{Fib}(\omega_{n+1}))$. However, since $\hat{\omega}_{n+1}(i) : Y_i \to \hat{\sigma}_n Y(i)$ is the identity for $i \in I_n$, the diagram $\mathrm{Fib}(\hat{\omega}_{n+1}) : I \to \mathcal{C}$ is trivial (over Z) when restricted to I_n, and since $\mathcal{J}$ was orderable, any map $f : X = \tau_{n+1}X \to \mathrm{Fib}(\hat{\omega}_{n+1})$ is determined uniquely by its restriction to J_{n+1} — in fact, to the discrete subcategory $\widetilde{J}_{n+1} := J_{n+1} \setminus I_n$.

The fact that Y is an abelian group object in $\mathcal{C}^I/Z$ implies, by definition, that for each $i \in I$ there is a commuting triangle:

$$\begin{tikzcd} Z_i \arrow[r, "s_i"] \arrow[d, "="'] & Y_i \arrow[dl, "q_i"] \\ Z_i & \end{tikzcd} , \tag{2.4}$$

natural in I. Thus $\mathrm{Fib}(\hat{\omega}_{n+1})(j)$ for $j \in J_{n+1}$ is by definition the pullback of:

$$\begin{tikzcd} & Z_j \arrow[d] \arrow[drr, "\sigma_n s_j \circ \omega_Z"'] \arrow[drrrr, "\mathrm{Id}"] & & & & \\ Y_j \arrow[r, "\hat{\omega}", "(\omega, q_j)"'] & \hat{\sigma}_n Y_j & = & \sigma_n Y(j) & \times_{\sigma_n Z(j)} & Z_j, \end{tikzcd} \tag{2.5}$$

and we readily check that this is the same as $\mathrm{Fib}(\omega_{n+1})(j)$, which is the pullback of:

$$\begin{tikzcd} & \sigma_n Z(j) \arrow[d, "\sigma_n s(j)"] \\ Y_j \arrow[r, "\omega_Y"] & \sigma_n Y(j). \end{tikzcd} \quad \square \tag{2.6}$$

2.7. Directed indexing diagrams

We shall now see how Proposition 2.3 applies when $\mathcal{J}$ is an orderable cover of a directed indexing category I (see Section 1.3).

Recall that in the Reedy model category structure (cf. Section 1.9) on $\mathcal{C}^I$, a map $f : X \to Y$ is a fibration if and only if

$$X_j \xrightarrow{(f,p)} Y_j \times_{\sigma_n Y(j)} \sigma_n X(j) \tag{2.8}$$

is a fibration in $\mathcal{C}$ for every $j \in \mathrm{Obj}\, I$ with $\deg(j) = n+1$, where $\sigma_n = \sigma_{I_n}$ is the completion at I_n. In $\mathcal{C}^I/Z$ we must replace σ_n by $\hat{\sigma}_n$ (Section 1.14), of course.

Lemma 2.9. *If I is a directed indexing category, any Reedy fibrant $Y \in \mathcal{C}^I/Z$ is $\mathcal{J}$-fibrant for the fine cover of I (Example 1.4).*

Proof. Once again we assume $I = I_{n+1}$ (Remark 2.2), so we must show that $\hat{\omega}_{n+1} : Y \to \hat{\sigma}_n Y$ is a fibration in $\mathcal{C}^I/\sigma_n Z$. Since $\hat{\omega}_{n+1}$ is the identity for $j \in I_n$, consider $j \in \widetilde{J}_{n+1} := I_{n+1} \setminus I_n$. Since Y is Reedy fibrant in $\mathcal{C}^I/Z$, $q : Y \to Z$ is a Reedy fibration in $\mathcal{C}^I$, and since $\mathcal{J}$ is fine, this means that

$$Y_j \xrightarrow{(\omega_{n+1}, q_j)} \sigma_n Y(j) \times_{\sigma_n Z(j)} Z_j = \hat{\sigma}_n Y(j) = \hat{\sigma}_n Y(j) \times_{\hat{\sigma}_n Y(j)} \hat{\sigma}_n Y(j)$$

is a fibration in $\mathcal{C}$ — which shows that (2.8) indeed holds for each $j \in I$. $\square$

Proposition 2.10. *Let $\mathcal{C} = s\mathcal{A}$ for some $\mathfrak{G}$-sketchable category $\mathcal{A}$ (Section 1.13), and let $\mathcal{J} = \{J_\nu\}_{\nu \in N}$ be an orderable cover of a directed indexing category I, with $Z \in \mathcal{C}^I$ Reedy fibrant. Then any abelian group object $Y \in \mathcal{C}^I/Z$ is weakly equivalent to a fibrant (objectwise) abelian group object which is $\mathcal{J}$-fibrant.*

Proof. Because I is directed, we may construct the desired $\mathcal{J}$-fibrant replacement $\bar{Y}$ – an abelian group object in $\mathcal{C}^I/Z$ – by induction on the degree of $j \in I$. Moreover, we assumed that Z is Reedy fibrant, so in particular objectwise fibrant (see Remark 1.10). Note that any abelian group object $p : V \to Z$ in $\mathcal{C}^I/Z$ is (objectwise) fibrant, since p has a section by (2.4) and Section 1.13; hence p has the right lifting property with respect to any acyclic cofibration.

We assume by induction on $\deg(j) = n + 1$ that both $\bar{\omega}_{n+1}(j) : \bar{Y}_j \to \hat{\sigma}_n\bar{Y}(j)$ and $\bar{q}_j : \bar{Y}_j \to Z_j$ are fibrations in $\mathcal{C}$. Since for each j, $\sigma_n Y(j)$ is defined as a limit, and an abelian group object structure on any V is a map $V \times_Z V \to V$ (over Z), by functoriality (and commutativity) of limits we see that $\sigma_n q : \sigma_n\bar{Y} \to \sigma_n Z$ is an abelian group object, too — so $\sigma_n q$ is an objectwise fibration in $\mathcal{C}^I$. But

$$\begin{tikzcd}
\hat{\sigma}_n\bar{Y}_j \arrow[r, "\pi_Z"] \arrow[d] & Z_j \arrow[d] \\
\sigma_n\bar{Y}(j) \arrow[r, "\sigma_n q"] & \sigma_n Z(j)
\end{tikzcd}$$

is a pullback square, by definition, so π_Z is a fibration in $\mathcal{C}$ by base change.

In the induction step, for each j of degree $n + 1$, we factor:

$$\bar{\hat{\omega}}_j : \bar{Y}_j \to \hat{\sigma}_n\bar{Y}(j) = \sigma_n\bar{Y}(j) \times_{\sigma_n Z(j)} Z_j$$

as

$$\bar{Y}_j \hookrightarrow \bar{Y}'_j \xrightarrow{\bar{\omega}'_j} \hat{\sigma}\bar{Y}(j)$$

(an acyclic cofibration followed by a fibration), and replace $\bar{Y}_j$ by $\bar{Y}'_j$. Both $\bar{\omega}'_j$ and $\bar{q}_j := \pi_Z \circ \bar{\omega}'_j : \bar{Y}_j \to Z_j$ are then fibrations in $\mathcal{C}$, as required. □

Remark 2.11. This actually works for some orderable covers of indexing categories which are not directed. For example, if we use the fine cover $\mathcal{J}$ for an indexing category I constructed as in Remark 1.7, we can still change any Y into a $\mathcal{J}$-fibrant one by induction on the degree in I', since we have not introduced any new objects

Example 2.12. In Example 1.8, for any $Y \in \mathcal{C}^I$, σY is given by:

$$\sigma Y(j) = Y_i \times Y_i \quad \rightrightarrows \quad Y_i = \sigma Y(i),$$

with horizontal maps $Y(\phi_{\pm 1})$ the two projections, and $f : \sigma Y(j) \to \sigma Y(j)$ the switch map. To make this $\mathcal{J}$-fibrant for the obvious (fine) cover, we just have to choose $\bar{Y}$ so that $\hat{\omega} : \bar{Y}_j \to \sigma\bar{Y}(j)$ is a $\mathbb{Z}/2$-equivariant fibration.

2.13. The dual construction

The approach described above is clearly best suited to directed indexing categories I where the degree function is non-negative. In the inverse case, the dual approach may be preferable:

Given a small indexing category I and a subcategory J, the truncation functor $\tau = \tau^I_J : \mathcal{C}^I \to \mathcal{C}^J$ also has a left adjoint $\zeta = \zeta^I_J : \mathcal{C}^J \to \mathcal{C}^I$, which assigns to a diagram $X : J \to \mathcal{C}$ the diagram $\zeta X : I \to \mathcal{C}$ with $\zeta X(i) := \mathrm{colim}_{J/i} X$ for each $i \in I$. We denote the resulting comonad on $\mathcal{C}^I$ by $\theta_J = \zeta_J \circ \tau_J$. Note that if $X \in \mathcal{C}^I/Z$, then $\theta_J X$ comes equipped with a map to $\theta_J Z \in \mathcal{C}^I/Z$, so we do not need the analogue of (1.15).

We then say that a diagram $X \in \mathcal{C}^I/Z$ is *$\mathcal{J}$-cofibrant* for an orderable cover $\mathcal{J}$ if for each $n \in \mathbb{Z}$, the coaugmentation $\eta_{n+1} : \theta^{n+1}_n X = \theta^{I_{n+1}}_{I_n} X \to \tau_{n+1} X$ is a cofibration in $\mathcal{C}^{I_{n+1}}/\tau_{n+1}Z$. We then have:

Proposition 2.14. *Assume $\mathcal{J} = \{J_\nu\}_{\nu \in N}$ is an orderable cover of I, $X \in \mathcal{C}^I/Z$ is $\mathcal{J}$-cofibrant, and $Y \in \mathcal{C}^I/Z$ is a fibrant abelian group object. Then*

$$F_{n+1} \to \mathrm{map}_{\mathcal{C}^{I_{n+1}}/\tau_{n+1}Z}(\tau_{n+1}X, \tau_{n+1}Y) \xrightarrow{\rho_{n+1}} \mathrm{map}_{\mathcal{C}^{I_n}/\tau_n Z}(\tau_n X, \tau_n Y)$$

is a fibration sequence of simplicial abelian groups for each $n \in \mathbb{Z}$, and the fiber F_{n+1} is $\mathrm{map}_{\mathcal{C}^{J_{n+1}}/Z|_{J_{n+1}}}(\mathrm{Cof}(\eta_{n+1}), Y|_{J_{n+1}})$.

Here $\mathrm{Cof}(\eta_{n+1})$ denotes the cofiber (over $\tau_{n+1}Z$) of the coaugmentation $\eta_{n+1} : \theta^{n+1}_n X \to \tau_{n+1}X$.

Proof. Dual to that of Proposition 2.3. □

Note that if I is a directed indexing category, we need no special assumptions on $X, Y \in \mathcal{C}^I/Z$ (or $\mathcal{C}$) in order for the dual of Proposition 2.10 to hold, since all colimits are over Z to begin with. Thus, we can again build $\mathcal{J}$-cofibrant replacements by induction on degree to yield the following:

Proposition 2.15. *Let $\mathcal{C} = s\mathcal{A}$ for some $\mathfrak{G}$-sketchable category $\mathcal{A}$, and let $\mathcal{J} = \{J_\nu\}_{\nu\in N}$ be an orderable cover of a directed indexing category I. Then any $X \in \mathcal{C}^I/Z$ is weakly equivalent to a cofibrant object (with respect to the model structure of Section 1.9*(a)*), which is $\mathcal{J}$-cofibrant.*

3. The two truncation spectral sequences

As noted above, for a suitable model category $\mathcal{C}$ and any indexing category I, given $Z \in \mathcal{C}^I$ and $X, Y \in \mathcal{C}^I/Z$ with X cofibrant and Y a fibrant abelian group object, the homotopy groups of $\mathrm{map}_Z(X, Y)$ are the cohomology groups $H^*(X/Z, Y)$ (suitably indexed). Thus if $\mathcal{J}$ is some orderable cover of I such that Y is $\mathcal{J}$-fibrant, the homotopy spectral sequence for the tower of fibrations (cf. [24, VII, Section 6]) of (fibrant) simplicial sets (1.22) yields a spectral sequence with $E^2_{k,n} = \pi_{k+n}\,\mathrm{Fib}(\rho_n) \Longrightarrow \pi_{k+n}\,\mathrm{map}_Z(X, Y)$. To identify the E^2-term, we need the following:

Definition 3.1. Consider an orderable cover $\mathcal{J} = \{I', J\}$ of a diagram I (where we have in mind $I = I_{n+1}$, $I' = I_n$, and $J = J_{n+1}$). If Y is an abelian group object in $\mathcal{C}^I/Z$ which is $\mathcal{J}$-fibrant, then we have a fibration sequence

$$\mathrm{Fib}(\hat{\omega}) \to Y \xrightarrow{\hat{\omega}} \hat{\sigma} Y,$$

of abelian group objects over Z, where $\hat{\sigma}$ is the completion at I'.

We define the *relative cohomology* of the pair (I, J) to be the total left derived functor of $\mathrm{Hom}_{\mathcal{C}^J/Z|_J}(-, \mathrm{Fib}(\hat{\omega}))$, (into simplicial abelian groups), denoted by $H(X/Z; \hat{\omega})$. In particular, the ith *relative cohomology group* for (I, J) is $H^i(X/Z; \hat{\omega}) := \pi_i H(X/Z; \hat{\omega})$.

Remark 3.2. Note that in most applications the abelian group object $Y \in \mathcal{C}^I/Z$ will be an nth dimensional Eilenberg–Mac Lane object (over Z), in which case it is customary to re-index the relative cohomology groups so that $H^n(X/Z; \hat{\omega}) := \pi_0 H(X/Z; \hat{\omega})$.

Observe, however, that our setup allows Y to consist of Eilenberg–Mac Lane objects of varying dimensions, with the maps $Y(f)$ representing cohomology operations. In this general setting, no canonical re-indexing exists.

Fact 3.3. *Given I, J, I' and Y, Z as above, for any (cofibrant) $X \in \mathcal{C}^I/Z$ there is a long exact sequence in cohomology*

$$\to H^i((X/Z)|_J; \hat{\omega}) \to H^i(X/Z; Y) \to H^i((X/Z)|_{I'}; Y|_{I'}) \to H^{i+1}((X/Z)|_J; \hat{\omega}) \to \tag{3.4}$$

Theorem 3.5. *For any simplicial model category $\mathcal{C}$, directed indexing category I, and diagrams $Z : I \to \mathcal{C}$, $X \in \mathcal{C}^I/Z$, abelian group object $Y \in \mathcal{C}^I/Z$, and left-orderable cover $\mathcal{J}$ of I there is a first quadrant spectral sequence with:*

$$E^2_{s,t} = H^{t+s}((X/Z)|_{J_t}; \hat{\omega}) \Longrightarrow H^{s+t}(X/Z; Y)$$

and $d^2 : E^2_{s,t} \to E^2_{s-2,t+1}$.

Proof. Replace Z by a weakly equivalent Reedy fibrant diagram in $\mathcal{C}^I$, then X by a weakly equivalent cofibrant object in $\mathcal{C}^I/Z$, and then use Proposition 2.10 to replace Y by a weakly equivalent $\mathcal{J}$-fibrant abelian group object in $\mathcal{C}^I/Z$. Proposition 2.3 then implies that (1.22) is a tower of fibrations, and the associated homotopy spectral sequence has the specified relative cohomology groups as the homotopy groups of the fibers (which are the E^2-term of the spectral sequence, in our indexing). □

The spectral sequence need not converge, in general, without some cohomological connectivity assumptions on the subdiagrams (unless the cover $\mathcal{J}$ is finite, of course).

Remark 3.6. If $\mathcal{J}$ is the fine cover, the E^2-term simplifies to:

$$E^2_{s,t} = \prod_{j\in\widetilde{J}_t} H^{t+s}(X_j/Z_j, \hat{\phi}_j),$$

where $\hat{\phi}_j : Y_j \to \lim_{\phi:j\to i} Y_i$ is the structure map.

Using the approach of Section 2.13, we also obtain a dual spectral sequence:

Theorem 3.7. *For $\mathcal{C}$, I, Z, X, and Y as in Theorem 3.5, and $\mathcal{J}$ right-orderable, there is a first quadrant spectral sequence with:*

$$E^2_{s,t} = H^{s+t}(\eta_t; Y) \Longrightarrow H^{s+t}(X/Z; Y).$$

Remark 3.8. Note that $H^*(\eta_t; Y) := H^*(\mathrm{Cof}(\eta_t)/Z|_{J_t}; Y)$ is just the usual cohomology of the map of diagrams $\eta_t : \theta^t_{t-1}X \to \tau_s X$ (see Section 2.13). This fits into the usual long exact sequence of a pair, dual to that of (3.4).

When X is cofibrant, Z and Y are constant, and $\mathrm{colim}_I X = \mathrm{hocolim}_I X$ – for example, when I is a partially ordered set, so $\mathrm{colim}_I X = \bigcup_{i \in I} X_i$ – then $H^*(X/Z; Y) = H^*(\mathrm{colim}_I X/Z; Y)$, and the dual spectral sequence is simply the usual Mayer–Vietoris spectral sequence for the cover X of $\mathrm{colim}_I X$ (cf. [38, Section 5], and compare [12, XII, 4.5], [40, Section 10], and [39]).

Example 3.9. Let I be the commuting square as in Example 1.5:

Given a diagram of abelian group objects $Y : I \to \mathcal{C}$, the successive fibers $\mathrm{Fib}(\omega_{n+1})$ (see Proposition 2.3) are:

$$\begin{tikzcd}
\mathrm{Ker}(Y(c)) \cap \mathrm{Ker}(Y(d)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd}$$

for $\omega_4 : Y = \tau_4 Y \to \sigma_3 Y$;

$$\begin{tikzcd}
 & \mathrm{Ker}(Y(b)) \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd}$$

for $\omega_3 : \tau_3 Y \to \sigma_2 Y$;

$$\begin{tikzcd}
\mathrm{Ker}(Y(a)) \arrow[r] & 0
\end{tikzcd}$$

for $\omega_2 : \tau_2 Y \to \sigma_1 Y$; and the single object Y_1 for $\omega_1 : \tau_1 Y \to \sigma_0 Y$.

Thus the E^2-term for the spectral sequence consists of only four non-trivial lines:

$$E^2_{s,t} \cong \begin{cases} H^{s+4}(X_4; \mathrm{Ker}(Y(c)) \cap \mathrm{Ker}(Y(d))) & \text{if } t = 4; \\ H^{s+3}(X_3; \mathrm{Ker}(Y(b))) & \text{if } t = 3; \\ H^{s+2}(X_2; \mathrm{Ker}(Y(a))) & \text{if } t = 2; \\ H^{s+1}(X_1; Y_1) & \text{if } t = 1; \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

If we had used the fine cover, by Remark 3.6 we would instead have:

$$E^2_{s,t} \cong \begin{cases} H^{s+3}(X_4; \mathrm{Ker}(Y(c)) \cap \mathrm{Ker}(Y(d))) & \text{if } t = 3; \\ H^{s+2}(X_3; \mathrm{Ker}(Y(a))) \oplus H^{s+2}(X_2; \mathrm{Ker}(Y(b))) & \text{if } t = 2; \\ H^{s+1}(X_1; Y_1) & \text{if } t = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.11. The square can be thought of as a single morphism in the category of arrows, so that we could analyze it as in [9, Section 4], where $H^*(X; Y)$ is shown to fit into a long exact sequence with ordinary cohomology groups in the remaining two slots. See Section 7.11.

4. An approach through local cohomology

The towers of Section 2 were constructed by covering a given indexing category I by truncated subcategories, obtained by omitting successive initial (or terminal) objects. We now present an alternative approach, using subcategories obtained by omitting *internal* objects of I. As we shall see, the resulting towers differ in nature from those considered above.

Definition 4.1. An indexing category I will be called *strongly directed* if:

i. It is *directed* in the sense of having no maps $f : i \to i$ but the identity.
ii. It has a non-empty *weakly initial* subcategory (necessarily discrete) consisting of all objects with no incoming maps, as well as a non-empty *weakly final* subcategory consisting of all objects with no outgoing maps.

iii. It is *locally finite* (that is, all Hom-sets are finite).
iv. I (that is, its underlying undirected graph) is *connected*.

Definition 4.2. We refer to $(\mathcal{C}, I, Z, X, Y)$ as *admissible* if:

(a) $\mathcal{C}$ is a simplicial model category;
(b) I is strongly directed;
(c) $Z \in \mathcal{C}^I$ is Reedy fibrant (hence objectwise fibrant);
(d) $X, Y \in \mathcal{C}^I/Z$ with X cofibrant and Y a fibrant abelian group object.

Definition 4.3. For any categories $\mathcal{C}$ and I and diagrams $Z \in \mathcal{C}^I$ and $X, Y \in \mathcal{C}^I/Z$, the product of simplicial sets

$$\mathcal{D}_{\mathcal{C}^I/Z}(X, Y) := \prod_{i\in I} \operatorname{map}_{\mathcal{C}/Z_i}(X_i, Y_i)$$

will be called the *space of discrete transformations* from X to Y over Z.

We shall generally abbreviate this to $\mathcal{D}_Z(X, Y)$. Note that these are maps of functors only for the discrete indexing category I^δ, with no non-identity maps.

4.4. *The primary tower*

In the spirit of Section 1, for any finite indexing category I we construct a finite sequence of full subcategories

$$I_1 \subset I_2 \subset \cdots I_n = I \tag{4.5}$$

of I, starting with I_1, whose objects are the weakly initial and final sets.

As before, this can be done in several ways (ultimately yielding variant spectral sequences). In any case, we can refine (4.5) so that for each k, I_{k-1} is obtained from I_k by omitting a single internal object i_k (where *internal* means that it is neither weakly initial nor weakly final).

If $(\mathcal{C}, I, Z, X, Y)$ is admissible, the inclusions of categories $\iota_{k-1} : I_{k-1} \hookrightarrow I_k$ induce a finite tower of simplicial abelian groups:

$$\operatorname{map}_{\mathcal{C}^{I_n}/Z}(X, Y) \to \cdots \to \operatorname{map}_{\mathcal{C}^{I_k}/Z}(X, Y) \xrightarrow{\iota_{k-1}^*} \operatorname{map}_{\mathcal{C}^{I_{k-1}}/Z}(X, Y) \to \cdots, \tag{4.6}$$

analogous to (1.22).

4.7. *The auxiliary fibration*

Unfortunately, (4.6) is not, in general, a tower of fibrations, so we cannot use it directly to obtain a useable spectral sequence for the cohomology of a diagram. To do so, we must replace it (up to homotopy) by a tower of fibrations, with $\operatorname{map}_Z(X, Y)$ as its homotopy inverse limit. The resulting spectral sequence (abutting to the homotopy groups of $\operatorname{map}_Z(X, Y)$), will have the homotopy groups of the homotopy fibers of the maps ι_k^* as its E^2-term. In fact, instead of constructing the replacement directly, we make use of the following observation:

For any indexing category I and diagrams $X, Y : I \to \mathcal{C}$, the set $\operatorname{Nat}_{\mathcal{C}^I}(X, Y)$ of diagram maps (natural transformations) from X to Y fits into an equalizer diagram:

$$\operatorname{Nat}_{\mathcal{C}^I}(X, Y) \hookrightarrow \prod_{i\in I} \operatorname{Hom}_{\mathcal{C}}(X_i, Y_i) \rightrightarrows \prod_{i,j\in I} \prod_{\eta\in \operatorname{Hom}_I(i,j)} \operatorname{Hom}_{\mathcal{C}}(X_i, Y_j). \tag{4.8}$$

Here the two parallel arrows map to each factor indexed by $\eta : i \to j$ in I by the appropriate projection, followed by either $Y(\eta)_* : \operatorname{Hom}_{\mathcal{C}}(X_i, Y_i) \to \operatorname{Hom}_{\mathcal{C}}(X_i, Y_j)$, or $X(\eta)^* : \operatorname{Hom}_{\mathcal{C}}(X_j, Y_j) \to \operatorname{Hom}_{\mathcal{C}}(X_i, Y_j)$, respectively.

In the case where Y is an abelian group object in $\mathcal{C}^I$ (or $\mathcal{C}^I/Z$), this describes $\operatorname{Nat}_{\mathcal{C}^I}(X, Y)$ as the kernel of the difference ξ of the two parallel arrows. By considering mapping spaces rather than Hom-sets, we obtain a left-exact sequence of simplicial abelian groups:

$$0 \to \operatorname{map}(X, Y) \to \mathcal{D}(X, Y) \xrightarrow{\xi} \prod_{i,j\in I} \prod_{\eta:i\to j} \operatorname{map}(X_i, Y_j), \tag{4.9}$$

and similarly for $\operatorname{map}_Z(X, Y)$.

However, (4.9) is not generally a fibration sequence, except when the underlying graph of I is a tree (the proof of [9, Prop. 4.23], where I consists of a single map, generalizes to this case). Nevertheless, for strongly directed indexing categories I (Definition 4.1), we can define a subspace $L_I(X, Y)$ (see Definition 5.5) inside the right-hand space of (4.9), such that ξ factors through a fibration Ψ (see Lemma 5.9), and:

$$0 \to \operatorname{map}_Z(X, Y) \to \mathcal{D}_Z(X, Y) \xrightarrow{\Psi} L_I(X, Y) \tag{4.10}$$

is thus a fibration sequence.

For such an I we obtain an auxiliary tower:

$$L_{I_n}(X,Y) \xrightarrow{p_{n-1}} L_{I_{n-1}}(X,Y) \to \cdots \to L_{I_2}(X,Y) \xrightarrow{p_1} L_{I_1}(X,Y) \tag{4.11}$$

(see Notation 5.10). We shall show that the maps p_k are fibrations (see Proposition 6.2), with a fiber which we identify as $F_k := \mathcal{H}_c^{I_k}(X/Z,Y)$ (cf. Definition 7.4).

4.12. *The auxiliary fibers*

Since all of these constructions will be natural, for each k the inclusion of categories $i_{k-1} : I_{k-1} \hookrightarrow I_k$ will induce a commuting square of fibrations:

$$\begin{tikzcd}
\mathcal{D}_{\mathcal{C}^{I_k}/Z}(X,Y) \arrow[r, "\Psi_k"] \arrow[d, "\pi_{k-1}"'] & L_{I_k}(X,Y) \arrow[d, "p_{k-1}"] \\
\mathcal{D}_{\mathcal{C}^{I_{k-1}}/Z}(X,Y) \arrow[r, "\Psi_{k-1}"] & L_{I_{k-1}}(X,Y),
\end{tikzcd}$$

where the left vertical map π_{k-1} is the projection onto the appropriate factors. Thus we will have a homotopy-commutative diagram:

$$\begin{tikzcd}
\mathrm{Fib}(i_{k-1}^*) \arrow[r] \arrow[d] & \prod_{i \in I_k \setminus I_{k-1}} \mathrm{map}_{\mathcal{C}/Z_i}(X_i, Y_i) \arrow[r] \arrow[d] & \mathcal{H}_c^{I_k}(X/Z,Y) \arrow[d] \\
\mathrm{map}_{\mathcal{C}^{I_k}/Z}(X,Y) \arrow[r] \arrow[d, "i_{k-1}^*"'] & \mathcal{D}_{\mathcal{C}^{I_k}/Z}(X,Y) \arrow[r, "\Psi_k"] \arrow[d, "\pi_{k-1}"'] & L_{I_k}(X,Y) \arrow[d, "p_{k-1}"] \\
\mathrm{map}_{\mathcal{C}^{I_{k-1}}/Z}(X,Y) \arrow[r] & \mathcal{D}_{\mathcal{C}^{I_{k-1}}/Z}(X,Y) \arrow[r, "\Psi_{k-1}"] & L_{I_{k-1}}(X,Y)
\end{tikzcd} \tag{4.13}$$

in which all rows and columns are fibration sequences up to homotopy.

Since the homotopy groups of $\Pi_i \, \mathrm{map}_{\mathcal{C}/Z_i}(X_i, Y_i)$ are a direct product of cohomology groups of the individual spaces in the diagram X, the top row of (4.13) allows us to identify the successive homotopy fibers of maps of the primary tower (4.6) in terms of those of the auxiliary tower (4.11). Taking $k = n$, we see also that $\mathrm{map}_Z(X,Y)$ is in fact the homotopy limit of the primary tower.

4.14. *A modified primary tower*

Using standard methods, we can change (4.6) into a tower with the same homotopy limit, but simpler successive fibers: For $1 \le k \le n$ we define $q_k : \mathcal{D}_Z(X,Y) \to L_{I_k}(X,Y)$ to be the composite fibration:

$$\mathcal{D}_Z(X,Y) \xrightarrow{\Psi_I} L_I(X,Y) \xrightarrow{p_k \circ \cdots \circ p_{n-1}} L_{I_k}(X,Y),$$

and denote the fiber of q_k by $\mathcal{E}^I_{\mathcal{C}^{I_k}/Z}(X,Y)$.

The induced maps $r_k : \mathcal{E}_Z^{I_k}(X,Y) \to \mathcal{E}_Z^{I_{k-1}}(X,Y)$ then fit into a tower:

$$\mathcal{E}_Z^{I_n}(X,Y) \xrightarrow{r_{n-1}} \cdots \xrightarrow{r_2} \mathcal{E}_Z^{I_2}(X,Y) \xrightarrow{r_1} \mathcal{E}_Z^{I_1}(X,Y). \tag{4.15}$$

As in Section 4.12, we see that the homotopy fiber of r_k is the loop space of the fiber $F_k := \mathcal{H}_c^{I_k}(X/Z,Y)$ of p_k, while the homotopy limit of (4.15) is $\mathcal{E}_Z^I(X,Y) = \mathrm{map}_Z(X,Y)$. Therefore, if we take the homotopy spectral sequence for the tower (4.15), rather than that for (4.6), we get the same abutment, and a closely related E^2-term.

Definition 4.16. For $(\mathcal{C}, I, Z, X, Y)$ as above and J a subcategory of I, we denote by $\mathcal{E}^J_{\mathcal{C}^I/Z}(X,Y) = \mathcal{E}^J_Z(X,Y)$ the sub-simplicial set of $\mathcal{D}_Z(X,Y)$ consisting of transformations which are natural when restricted to J-diagrams. In other words, these are elements σ of $\mathcal{D}_Z(X,Y)$ which make

$$\begin{tikzcd}
X_i \arrow[r, "X(f)"] \arrow[d, "\sigma_i"'] & X_j \arrow[d, "\sigma_j"] \\
Y_i \arrow[r, "Y(f)"'] & Y_j
\end{tikzcd} \tag{4.17}$$

commute, for any morphism $f : i \to j$ in J.

For example, $\mathcal{E}_Z^{I_1}(X, Y)$, consists of those transformations which are natural only with respect to morphisms of maximal length. On the other hand, $\mathcal{E}_Z^{I}(X, Y)$ is simply $\mathrm{map}_Z(X, Y)$.

Note that any inclusion of subcategories $J' \to J$ of I induces an injection of simplicial sets $r_{J'}^{J} : \mathcal{E}_Z^{J}(X, Y) \to \mathcal{E}_Z^{J'}(X, Y)$, since any transformation natural over J must be natural over the subcategory J'.

Lemma 4.18. *For* $(I_k)_{k=1}^{n}$ *as in (4.5), we can identify* $\mathcal{E}_Z^{I_k}(X, Y)$ *of Section 4.14 with* $\mathcal{E}_{\mathcal{C}^I/Z}^{I_k}(X, Y)$*, and* $r_k : \mathcal{E}_Z^{I_k}(X, Y) \to \mathcal{E}_Z^{I_{k-1}}(X, Y)$ *with* $r_{I_{k-1}}^{I_k}$*.*

Proof. Follows from Definition 4.16. □

5. The auxiliary tower

Suppose $(\mathcal{C}, I, Z, X, Y)$ is admissible. In order to construct the auxiliary tower (4.11), we need a number of definitions:

Definition 5.1. Assuming $(\mathcal{C}, I, Z, X, Y)$ is admissible:

(a) For any composable sequence $f_\bullet$ of k non-identity morphisms in I (i.e., a k-simplex of the reduced nerve of I, $\mathcal{N}(I)$, where identities are excluded) its *diagonal* mapping space is

$$M(f_\bullet) := \mathrm{map}_{Z_{t(f_k)}}(X_{s(f_1)}, Y_{t(f_k)}).$$

In particular, for $f : a \to b$ in I we have $M(f) := \mathrm{map}_{Z_b}(X_a, Y_b)$.

(b) For each $k \geq 1$, let $\mathrm{Diag}_Z^k(X, Y) := \prod_{f_\bullet \in \mathcal{N}(I)_k} M(f_\bullet)$. In particular, we denote $\mathrm{Diag}_Z^1(X, Y) = \prod_{f \in I} M(f)$ by$\mathrm{Diag}_Z(X, Y)$.

(c) Any map into the product $\mathrm{Diag}_Z^k(X, Y)$ is defined by specifying its projection onto each factor $M(f_\bullet)$, indexed by $f_\bullet \in \mathcal{N}(I)_k$.

In particular, we have two maps of interest $\mathrm{Diag}_Z^{k-1}(X, Y) \to \mathrm{Diag}_Z^k(X, Y)$:

(i) X^*, for which the $f_\bullet$-component is the composite

$$\mathrm{Diag}_Z^{k-1}(X, Y) \xrightarrow{\text{proj}} M(f_2, \ldots, f_k) \xrightarrow{X(f_1)^*} M(f_\bullet).$$

(ii) Y_*, for which the $f_\bullet$-component is the composite

$$\mathrm{Diag}_Z^{k-1}(X, Y) \xrightarrow{\text{proj}} M(f_1, \ldots, f_{k-1}) \xrightarrow{Y(f_k)_*} M(f_\bullet).$$

(d) By iterating the maps $\Phi^1 := Y_* + X^* : \mathrm{Diag}_Z^{k-1}(X, Y) \to \mathrm{Diag}_Z^k(X, Y)$ for various $k > 1$ we obtain maps:

$$\Phi^j : \mathrm{Diag}_Z^k(X, Y) \to \mathrm{Diag}_Z^{k+j}(X, Y)$$

for each $j \geq 1$. Setting $\Phi^0 := \mathrm{Id} : \mathrm{Diag}_Z^1(X, Y) \to \mathrm{Diag}_Z^1(X, Y)$, we may combine these to define:

$$\Phi : \mathrm{Diag}_Z(X, Y) \to \prod_{k=1}^{n} \mathrm{Diag}_Z^k(X, Y)\,.$$

For any $f_\bullet \in \mathcal{N}(I)_k$, we write $\Phi_{f_\bullet}$ for Φ composed with the projection onto $M(f_\bullet)$.

(e) For any $f_\bullet = (f_1, \ldots, , f_k) \in \mathcal{N}(I)_k$, let $c(f_\bullet) := f_k \circ f_{k-1} \circ \cdots \circ f_1$ denote the composition in I. We then have a map $\kappa_{f_\bullet} : \prod_{k=1}^{n} \mathrm{Diag}_Z^k(X, Y) \to M(c(f_\bullet))$, which is just the projection onto $M(f_\bullet) \overset{=}{\to} M(c(f_\bullet))$.

Remark 5.2. If $(g, f) \in \mathcal{N}(I)_2$, is a composable pair in I, then by the definition of Φ we have

$$\Phi_{(g,f)} = Y(f) \circ \Phi_g + \Phi_f \circ X(g).$$

More generally, if $h_\bullet = (g_\bullet, f_\bullet) \in \mathcal{N}(I)_{k+j}$ is the concatenation of $g_\bullet \in \mathcal{N}(I)_k$ and $f_\bullet \in \mathcal{N}(I)_j$, then:

$$\Phi_{(g_\bullet, f_\bullet)} = Y(c(f_\bullet))_* \Phi_{g_\bullet} + X(c(g_\bullet))^* \Phi_{f_\bullet}\,. \tag{5.3}$$

Note also that

$$(Y_* + X^*) \circ (Y_* + X^*) = Y_*Y_* + Y_*X^* + X^*X^* : \mathrm{Diag}_Z^k(X, Y) \to \mathrm{Diag}_Z^{k+2}(X, Y)$$

and so inductively:

$$\Phi^j = (Y_* + X^*)^j = \sum_{i=0}^{j} (Y_*)^{j-i}(X^*)^i : \mathrm{Diag}_Z^k(X, Y) \to \mathrm{Diag}_Z^{k+j}(X, Y)\,. \tag{5.4}$$

Definition 5.5. Let K_I denote the indexing category with

- objects: $\mathbf{0}$, $\mathbf{1}$, and $\mathrm{Arr}(I) := \mathcal{N}(I)_1$,
- morphisms: one arrow $\phi : \mathbf{0} \to \mathbf{1}$, and an arrow $k_{f_\bullet} : \mathbf{1} \to c(f_\bullet) \in \mathrm{Arr}(I)$ for each $f_\bullet \in \mathcal{N}(I)$.

If $(\mathcal{C}, I, Z, X, Y)$ is admissible, define a diagram of simplicial abelian groups $V_I : K_I \to s\mathcal{A}$ by setting $V_I(\mathbf{0}) = \mathrm{Diag}_Z(X, Y)$, $V_I(\mathbf{1}) = \prod_{k=1}^{n} \mathrm{Diag}_Z^k(X, Y)$, and $V_I(f) = M(f)$, with $V_I(\phi) = \Phi$ and $V_I(k_{f_\bullet}) = \kappa_{f_\bullet}$. Then set $L_I(X, Y) := \lim_{K_I} V_I$.

This limit can be described more concretely as follows: write $\mathrm{Indec}(I)$ for the collection of indecomposable maps in I, and let $\mathcal{L}_I(X, Y)$ denote the subspace of $\prod_{f \in \mathrm{Indec}(I)} M(f)$ consisting of tuples $\varphi_\bullet$ satisfying

$$\sum_{i=0}^{k} Y(f_k \circ \cdots \circ f_{i+1})\varphi_{f_i} X(f_{i-1} \circ \cdots \circ f_1) = \sum_{i=0}^{l} Y(g_l \circ \cdots \circ g_{i+1})\varphi_{g_i} X(g_{i-1} \circ \cdots \circ g_1) \tag{5.6}$$

whenever $c(f_\bullet) = c(g_\bullet)$.

Lemma 5.7. *The simplicial abelian group $L_I(X, Y)$ is isomorphic to $\mathcal{L}_I(X, Y)$.*

Proof. The limit condition for $\varphi \in L_I(X, Y)$ implies that the value of φ_f for any decomposable f is uniquely determined by the values of φ_{f_i} for f_i indecomposable, by the recursive formula (5.3). □

Remark 5.8. As a consequence of the previous lemma, for (full) subcategories $J \subset I$ we have natural inclusion maps $i_J : L_J(X, Y) \to \prod_{f \in \mathrm{Indec}(J)} M(f)$.

We now investigate the properties of $L_I(X, Y)$ and its associated fibrations. First, note that there are two maps $X^*, Y_* : \mathcal{D}_Z(X, Y) \to \mathrm{Diag}_Z(X, Y)$, which project to precomposition and postcomposition respectively on appropriate factors and we show:

Lemma 5.9. *The difference map $\xi := Y_* - X^* : \mathcal{D}_Z(X, Y) \to \mathrm{Diag}_Z(X, Y)$ factors through a map $\Psi : \mathcal{D}_Z(X, Y) \to L_I(X, Y)$ with kernel* $\mathrm{map}_Z(X, Y)$.

Proof. Note that the sum (5.4), applied to an element in the image of the difference map

$$Y_* - X^* : \mathcal{D}_Z(X, Y) \to \mathrm{Diag}_Z(X, Y),$$

is telescopic, so we are left with: $(Y_*)^k - (X^*)^k$. Since X and Y are in $\mathcal{C}^I$, for any $f_\bullet \in \mathcal{N}(I)_k$ the composite:

$$\mathcal{D}_Z(X, Y) \to \mathrm{Diag}_Z(X, Y) \to \prod_{k=1}^{n} \mathrm{Diag}_Z^k(X, Y) \xrightarrow{\kappa_{f_\bullet}} M(c(f_\bullet))$$

sends any σ to $Y(f)\sigma_{s(f)} - \sigma_{t(f)}X(f)$. As a consequence, we get an identical value for any $g_\bullet \in \mathcal{N}(I)_j$ with $c(f_\bullet) = c(g_\bullet)$. Thus, the universal property of the limit implies the difference map factors through the limit $L_I(X, Y)$.

To identify the kernel of Ψ, we instead consider the difference map:

$$Y_* - X^* : \mathcal{D}_Z(X, Y) \to \mathrm{Diag}_Z(X, Y).$$

Clearly $\Psi(\sigma) = 0$ if and only if $Y(f)\sigma_{s(f)} - \sigma_{t(f)}X(f) = 0$, for every morphism f in I — that is, precisely when σ is a natural transformation of $\mathcal{C}^I$. Since both X and Y are diagrams over Z, and each σ_f is a map over Z_f, σ is in that case actually a natural transformation over Z. □

Notation 5.10. In order to describe the behavior of the L-construction with respect to the inclusion of a subcategory $\iota : J \to I$, note that we can define two different diagrams of simplicial abelian groups indexed by K_J (Definition 5.5):

One is V_J, whose limit is $L_J(X, Y)$. The second, which we denote by $V_{I,J}$, has $V_{I,J}(\mathbf{0}) = \mathrm{Diag}_Z(X, Y)$, $V_{I,J}(\mathbf{1}) = \prod_{k=1}^{n} \mathrm{Diag}_Z^k(X, Y)$, as for V_I, (and $V_{I,J}(f) = M(f)$ for $f \in \mathrm{Arr}(J)$). If we set $L_{I,J}(X, Y) := \lim_{K_J} V_{I,J}$, we see that there is a canonical map $\tau : L_I(X, Y) \to L_{I,J}(X, Y)$ (since fewer constraints are imposed in defining the second limit as a subset of $\prod_{f \in \mathrm{Indec}(I)} M(f)$).

On the other hand, we have a morphism of K_J-diagrams from $\xi : V_{I,J} \to V_J$, obtained by projecting the larger products $\mathrm{Diag}_Z^k(X, Y)$ onto $\mathrm{Diag}_{Z|_J}^k\left(X|_J, Y|_J\right)$ for each $k \geq 1$. This induces a map on the limits $\xi_* : L_{I,J}(X, Y) \to L_J(X, Y)$, and we define the *restriction map* $(p =)p^I_J : L_I(X, Y) \to L_J(X, Y)$ to be $p^I_J := \xi_* \circ \tau$.

Finally, note that there is an obvious restriction map $r : \mathcal{D}_{\mathcal{C}^I/Z}(X, Y) \to \mathcal{D}_{\mathcal{C}^J/Z}(X, Y)$, which is simply the projection onto the factors indexed by $\mathrm{Arr}(J)$.

From the definitions it is clear that the diagram:

$$\begin{tikzcd}
\mathcal{D}_{\mathcal{C}^I/Z}(X, Y) \arrow[r, "\Psi_I"] \arrow[d, "r"'] & L_I(X, Y) \arrow[d, "p^I_J"] \\
\mathcal{D}_{\mathcal{C}^J/Z}(X, Y) \arrow[r, "\Psi_J"'] & L_J(X, Y)
\end{tikzcd} \tag{5.11}$$

commutes.

The kernel of $p^I_J \circ \Psi_I$ will be the same as the kernel of $\Psi_J \circ r^I_J$, by the commutativity of (5.11). However, by Lemma 5.9, the kernel of Ψ_J is the space of J-natural transformations. Thus the kernel of the composite $p^I_J \circ \Psi_I$ will be the space $\mathcal{D}_{\mathcal{C}^J/Z}(X, Y)$.

Lemma 5.12. *Given $J \subseteq I$ and $f \in \mathrm{Indec}(J)$ with $f = c(f_\bullet)$ for $f_\bullet = (f_k, f_{k-1}, \ldots, f_1) \in \mathcal{N}(I)_k$ with $f_i \in \mathrm{Indec}(I)$ $(i = 1, \ldots k)$, the following diagram commutes:*

$$\begin{tikzcd}
L_I\,(X,Y) \arrow[r, "p^I_J"] \arrow[d, "i_I"] & L_J\,(X,Y) \arrow[d, "i_J"] \\
\prod_{f \in \mathrm{Indec}(I)} M(f) \arrow[d, "\mathrm{proj}"] & \prod_{f \in \mathrm{Indec}(J)} M(f) \arrow[d, "\mathrm{proj}"] \\
M(f_1) \times \cdots \times M(f_k) \arrow[r, "\Phi^k_{f_\bullet}"] & M(f)
\end{tikzcd} \tag{5.13}$$

where the maps i_I and i_J are the inclusions of Remark 5.8.

Proof. Suppose $\varphi_\bullet$ is an element of $L_I\,(X, Y)$, while $f = c(f_\bullet)$ is a maximal decomposition (so each f_i is indecomposable). Then φ_f lies in $\mathrm{Diag}^1_Z\,(X, Y)$, so $\Phi_{\varphi_f} = \varphi_f$ lands in $M(f)$. However, $(\varphi_{f_k}, \ldots, \varphi_{f_1}) \in M(f_k) \times \cdots \times M(f_1)$ maps to $\sum_{i=0}^k Y(f_k \circ \cdots \circ f_{i+1})\varphi_{f_i} X(f_{i-1} \circ \cdots \circ f_1)$ also in $M(c(f_\bullet)) = M(f)$. Thus, $\varphi_\bullet \in L_I\,(X, Y) = \mathcal{L}_I\,(X, Y)$ (see Lemma 5.7) implies the value of φ_f for any decomposable f is uniquely determined by the values of φ_{f_i} for f_i indecomposable, using formula (5.6). □

Note that if f is also indecomposable in I, the bottom map of (5.13) is $\mathrm{Id} : M(f) \to M(f)$. The choice of decomposition of f in I is also irrelevant, by Definition 5.5.

6. Fibrations in the auxiliary tower

As noted in Section 4.7, the auxiliary tower (4.11) was constructed with two goals in mind: to replace (4.6) by a tower of fibrations (with the same homotopy limit), and to identify the homotopy fibers of the successive maps in (4.6). In this section we show that the map Ψ of Lemma 5.9 is indeed a fibration, and that the auxiliary tower is a tower of fibrations. First, we need the following:

Definition 6.1. Any strongly directed indexing category I has two filtrations, defined inductively:

(a) The filtration $\{\mathcal{F}_i\}_{i=0}^n$ on I is defined by decomposition length from the left, so $\mathcal{F}_0$ consists of weakly initial objects in I and $\mathcal{F}_{n+1}$ consists of indecomposable maps with sources in $\mathcal{F}_n$, (including their targets).
(b) The filtration $\{\mathcal{G}_i\}_{i=0}^n$ is similarly defined by decomposition length from the right, so $\mathcal{G}_0$ consists of the weakly terminal objects in I and $\mathcal{G}_{n+1}$ consists of indecomposable maps with targets in $\mathcal{G}_n$, (including their sources).

Proposition 6.2. *If $(\mathcal{C}, I, Z, X, Y)$ is admissible, the induced difference map:*

$$\Psi : \mathcal{D}_Z(X, Y) \to L_I\,(X, Y)$$

of Lemma 5.9 is a fibration of simplicial abelian groups.

Proof. By [35, II, Section 3, Prop. 1], it suffices to show that Ψ surjects onto the zero component of $L_I\,(X, Y)$. Thus, given $0 \sim \varphi_\bullet \in L_I\,(X, Y)$, we must produce an element $\sigma_\bullet \in \mathcal{D}_Z(X, Y)$ with $\Psi(\sigma_\bullet) = \varphi_\bullet$; i.e., for every $f : a \to b$ in I we want:

$$\sigma_b \circ X(f) = Y(f) \circ \sigma_a - \varphi_f. \tag{6.3}$$

Note that since Y is an abelian group object in $\mathcal{C}^I/Z$, the zero map $X \to Y$ is the unique map in $\mathcal{C}^I/Z$ that factors through the section $s : Z \to Y$ (which exists by (2.4) and Section 1.13).

We proceed by induction on the filtration $\{\mathcal{F}_i\}_{i=0}^n$ of I of Definition 6.1. To begin, for each $c \in \mathcal{F}_0$, we may choose $\sigma_c : X_c \to Y_c$ to be 0.

Assume by induction that we have constructed maps $\sigma_c : X_c \to Y_c$ for each $c \in \mathcal{F}_i$, satisfying (6.3) for every f in $\mathcal{F}_i$, and with each $\sigma_c \sim 0$. Note that for any $f : b \to c$, in $\mathcal{F}_{i+1}$ the map:

$$\nu\,(f) := Y(f) \circ \sigma_b - \varphi_f : X_b \to Y_c \tag{6.4}$$

is well-defined (since necessarily $b \in \mathcal{F}_i$). This is our candidate for $\sigma_c \circ X(f)$, and $\nu\,(f) \sim -Y(f) \circ \sigma_b \sim 0$ by the assumption on φ together with the induction hypothesis (considering naturality of the section $Z \to Y$).

Moreover, given any $g : a \to b$ (necessarily in $\mathcal{F}_i$), we have $\varphi_g = Y(g) \circ \sigma_a + \sigma_b \circ X(f)$ by (6.3), so from $\varphi_\bullet \in L_J(X, Y)$ it follows that:

$$\begin{aligned} \nu(f \circ g) &= Y(f \circ g) \circ \sigma_a - \varphi_{f \circ g} \\ &= Y(f \circ g) \circ \sigma_a - [Y(f) \circ \varphi_g + \varphi_f \circ X(g)] \\ &= Y(f \circ g) \circ \sigma_a - [Y(f) \circ (Y(g) \circ \sigma_a - \sigma_b \circ X(g)) + \varphi_f \circ X(g)] \\ &= \nu(f) \circ X(g). \end{aligned} \tag{6.5}$$

Now given $c \in \mathcal{F}_{i+1} \setminus \mathcal{F}_i$, set:

$$\hat{X}_c := \operatorname*{colim}_{b \in I/c} X_b.$$

Since $X \in \mathcal{C}^I$ is cofibrant, it is Reedy cofibrant (Remark 1.10), which implies that the canonical map $\varepsilon_c : \hat{X}_c \to X_c$ is a cofibration. Moreover, (6.5) implies that the maps $\nu(f)$ defined above induce a map $\hat{\nu}_c : \hat{X}_c \to Y_c$. Since all the maps in question are nullhomotopic by construction, the diagram:

$$\begin{tikzcd} \hat{X}_c \arrow[r, "\varepsilon_c"] \arrow[dr, "\hat{\nu}_c"'] & X_c \arrow[d, "0"] \\ & Y_c \end{tikzcd}$$

commutes up to homotopy. Hence by [9, Cor. 4.20] there is a map $\sigma : X_c \to Y_c$ in $\mathcal{C}/Z_c$ making the diagram

$$\begin{tikzcd} \hat{X}_c \arrow[r, "\varepsilon_c"] \arrow[dr, "\hat{\nu}_c"'] & X_c \arrow[d, "\sigma"] \\ & Y_c \end{tikzcd} \tag{6.6}$$

commute, and we choose this to be σ_c. By construction $\sigma_c \circ X(f) = \nu(f)$ for every $f : b \to c$, so (6.3) is satisfied. This completes the induction. □

Proposition 6.7. *If $(\mathcal{C}, I, Z, X, Y)$ is admissible, let J be a subcategory of I obtained by omitting a terminal object c. Then the restriction map $p^I_J : L_I(X, Y) \to L_J(X, Y)$ is a fibration.*

Proof. As in the previous proof, we must inductively define a lift $\sigma_\bullet \in L_I(X, Y)$ for a nullhomotopic $\varphi_\bullet \in L_J(X, Y)$. Under these conditions, p^I_J is simply a forgetful functor, so this means $\sigma_g = \varphi_g$ for g a morphism of J and we must define $\sigma_\ell : X_d \to Y_c$ whenever $\ell : d \to c$ is a morphism in I, in a manner compatible with the definition of $\varphi_\bullet$. Note that $\varphi_\bullet$ determines the composite $Y(f) \circ \Phi^I_{g_\bullet} =: \psi(g_\bullet, f)$.

Following the approach of the previous proof, we will define $\nu(g_\bullet, f)$ for any $e \xrightarrow{g_\bullet} d \xrightarrow{f} c$ in I, where f is indecomposable, so as to satisfy three properties:

First, we require that our choices be *coherent*:

$$\nu(g_\bullet \circ h_\bullet, f) = \nu(g_\bullet, f) \circ X(c(h_\bullet)), \tag{6.8}$$

which will allow us to build a homotopy-commutative triangle using a colimit construction.

Second, we need our choices to be *consistent*:

$$\nu(g_\bullet, f) = \nu\left(g'_\bullet, f'\right) + \psi(g_\bullet, f) - \psi\left(g'_\bullet, f'\right) \quad \text{whenever } f \circ g_\bullet = f' \circ g'_\bullet \text{ in } I, \tag{6.9}$$

which is needed so that we eventually obtain an element $\sigma_\bullet \in L_I(X, Y)$. In fact, our construction will also work when $g_\bullet = \emptyset$, which will yield $\sigma(f) = \nu(\emptyset, f)$.

Finally, we require that each $\nu(g_\bullet, f) \sim 0$.

We now proceed to choose $\nu(g_\bullet, f)$ for $e \xrightarrow{g_\bullet} d \xrightarrow{f} c$ with $e \in \mathcal{F}_i$ (Definition 6.1) by induction on $i \geq 0$:

For each $\ell : e \to c$ in I with $e \in \mathcal{F}_0$, choose some decomposition $e \xrightarrow{g_\bullet} d \xrightarrow{f} c$ (with $\ell = c(g_\bullet, f)$ and f indecomposable), and an arbitrary nullhomotopic $0 = \nu(g_\bullet, f) : X_e \to Y_c$. For any other decomposition $\ell = c(g'_\bullet, f')$, the map $\nu\left(g'_\bullet, f'\right)$ is then determined by (6.9).

Assume that ν has been defined for every $e \in \mathcal{F}_i$ so that (6.8) and (6.9) hold (wherever applicable). For each $e \in \mathcal{F}_{i+1} \setminus \mathcal{F}_i$ and map $\ell : e \to c$, consider the over-category $\mathcal{F}_i/e$ (which is non-empty by definition of $\mathcal{F}_{i+1}$) and set $\hat{X}_e := \operatorname{colim}_{a \in \mathcal{F}_i/e} X_a$. Because the diagram X is cofibrant, hence Reedy cofibrant (Remark 1.10) in $\mathcal{C}^I$, the canonical map $\varepsilon_e : \hat{X}_e \hookrightarrow X_e$ is a cofibration.

Again choose some decomposition $e \xrightarrow{g_\bullet} d \xrightarrow{f} c$ of ℓ. The maps $\nu\left(g_\bullet \circ h_\bullet, f\right) : X_a \to Y_c$, for each composable sequence $h_\bullet : a \to e$ in $\mathcal{F}_i/e$ induce a (necessarily nullhomotopic) map $\hat{\mu}_e : \hat{X}_e \to Y_c$ by (6.8). Since:

$$\begin{tikzcd}
\hat{X}_e \arrow[r, "\varepsilon_e"] \arrow[dr, "\hat{\nu}_{(g_\bullet, f)}"'] & X_e \arrow[d, "0"] \\
& Y_c
\end{tikzcd}$$

then commutes up to homotopy, we apply [9, Cor. 4.20] to find

$$\begin{tikzcd}
\hat{X}_e \arrow[r, "\varepsilon_e"] \arrow[dr, "\hat{\nu}_{(g_\bullet, f)}"'] & X_e \arrow[d, "\nu(g_\bullet, f)"] \\
& Y_c
\end{tikzcd}$$

making the diagram commute.

For any other decomposition $e \xrightarrow{g'_\bullet} d' \xrightarrow{f'} c$ of ℓ, use (6.9) to define $\nu\left(g'_\bullet, f'\right)$. This completes the induction step.

We have thus defined $\nu\left(g_\bullet, f\right) : X_e \to Y_c$ satisfying (6.8) and (6.9) for every $e \xrightarrow{g_\bullet} d \xrightarrow{f} c$ in I/c. In particular, we can choose $\sigma\left(f\right) = \nu\left(\emptyset, f\right) : X_d \to Y_c$ for each indecomposable $f : d \to c$ in I and see that $\sigma_\bullet \in L_I\left(X, Y\right)$ (by Lemma 5.7) is the desired lift. □

Corollary 6.10. *If $\left(\mathcal{C}, I, Z, X, Y\right)$ is admissible, let J be a full subcategory of I obtained by omitting an object c such that all maps out of c are indecomposable. Then $p^I_J : L_I\left(X, Y\right) \to L_J\left(X, Y\right)$ is a fibration.*

Proof. As in the proof of Proposition 6.7 we can construct $\sigma\left(f\right)$ for each $f : d \to c$ in I, such that we have $\hat{\nu} : \operatorname{colim}_{d \in I/c} X_d \to Y_c$, as well as $\hat{\epsilon}_c : \operatorname{colim}_{d \in I/c} X_d \to X_c$. For any $g : c \to b$, in I (indecomposable by assumption), we also have a map $\hat{\varphi} : \operatorname{colim}_{d \in I/c} X_d \to X_b$ induced by $\varphi_\bullet$. Note that by (5.3) we must have:

$$\sigma\left(g\right) \circ X(\hat{\epsilon}_c) = \Phi^I_{(g,f)} - Y(g) \circ \sigma\left(f\right) = \hat{\varphi} - Y(g) \circ \hat{\nu},$$

and since $X(f)$ is a cofibration, we may choose the extension $\sigma\left(g\right)$ as in (6.6). □

Definition 6.11. If I is a strongly directed indexing category, let $\mathcal{J} = \{J_k\}_{k \in N}$ be a fine orderable cover (Example 1.4) of I subordinate to the filtration $\mathcal{G}$ (Definition 6.1), such that $J_k \setminus J_{k-1}$ consists of a single object of I for each $k \in N$. Let $\mathcal{C} = s\mathcal{A}$ for some $\mathfrak{G}$-sketchable category $\mathcal{A}$ (Section 1.13), with $Z \in \mathcal{C}^I$ fibrant. A fibrant abelian group object $Y \in \mathcal{C}^I/Z$ is called *strongly fibrant* if it is $\mathcal{J}$-fibrant with respect to the model category structure of Section 1.9(a).

Remark 6.12. Note that this definition is independent of the choice of the refinement $\mathcal{J}$ of $\mathcal{G}$. Furthermore, by Proposition 2.10, any abelian group object $Y \in \mathcal{C}^I/Z$ is weakly equivalent to one which is strongly fibrant.

Proposition 6.13. *Suppose $\left(\mathcal{C}, I, Z, X, Y\right)$ is admissible, and that Y is strongly fibrant. Assume that J is obtained from I by omitting an object c such that all maps into c are indecomposable. Then the restriction map $p^I_J : L_I\left(X, Y\right) \to L_J\left(X, Y\right)$ is a fibration.*

Proof. Dual to the proofs of Proposition 6.7 and Corollary 6.10. The strong fibrancy is needed since in the model category we use for diagrams ordinary fibrancy is merely objectwise, while strong fibrancy is dual to Reedy cofibrancy for our purposes. □

Proposition 6.14. *If $\left(\mathcal{C}, I, Z, X, Y\right)$ is admissible, Y is strongly fibrant, and J is obtained from I by omitting any object c, then the restriction map $p^I_J : L_I\left(X, Y\right) \to L_J\left(X, Y\right)$ is a fibration.*

Proof. Consider any composable sequence:

$$d \xrightarrow{h_\bullet} c \xrightarrow{g} b \xrightarrow{f_\bullet} a \tag{6.15}$$

in I. As above, $0 \sim \varphi_\bullet \in L_J\left(X, Y\right)$ will determine the map

$$\psi\left(h_\bullet, g, f_\bullet\right) := Y(c((g, f_\bullet))) \circ \Phi^I_{h_\bullet} + \Phi^I_{f_\bullet} \circ X(c((h_\bullet, g))) \tag{6.16}$$

and we use $\nu\left(h_\bullet, g, f_\bullet\right) : X_d \to Y_a$, to denote the candidate for $Y(c(f_\bullet)) \circ \sigma\left(g\right) \circ X(c(h_\bullet))$ which we will construct.

As before we require *coherence*:

$$\nu\left(h_\bullet \circ \ell_\bullet, g, k_\bullet \circ f_\bullet\right) = Y(c(k_\bullet)) \circ \nu\left(h_\bullet, g, f_\bullet\right) \circ X(c(\ell_\bullet)) \tag{6.17}$$

for any

$$e \xrightarrow{\ell_\bullet} d \xrightarrow{h_\bullet} c \xrightarrow{g} b \xrightarrow{f_\bullet} a \xrightarrow{k_\bullet} z$$

in I; and *consistency*:

$$\nu\left(h'_\bullet, g', f'_\bullet\right) = \psi\,(h_\bullet, g, f_\bullet) + \nu\,(h_\bullet, g, f_\bullet) - \psi\left(h'_\bullet, g', f'_\bullet\right) \tag{6.18}$$

whenever $c(h'_\bullet, g', f'_\bullet) = c(h_\bullet, g, f_\bullet)$.

We choose the maps ν satisfying (6.17) and (6.18) by two successive inductions:

- The first is by induction on i, the filtration degree of d in $\{\mathcal{F}_i\}_{i=0}^m$ (by composition length from the left): this is done as in the proof of Proposition 6.7, until finally we have $\nu\,(h, g, f_\bullet)$ for every $d \xrightarrow{h} c \xrightarrow{g} b \xrightarrow{f_\bullet} a$, where h is indecomposable and a is terminal in I (by coherence this extends back to any $d \xrightarrow{h_\bullet} c$).
- The second is by induction on j, the filtration degree of a in $\{\mathcal{G}_j\}_{j=0}^n$ (by composition length from the right), as in the proof of Proposition 6.13 (which is why we need Y to be strongly fibrant).

At the end of the two induction processes we have chosen $\nu\,(h, g) : X_d \to Y_b$ for h and g indecomposable. We can then choose $\sigma\,(h) = \nu\,(h) : X_d \to Y_c$ as in the last step of the proof of Proposition 6.7, and finally choose $\sigma\,(g) = \nu\,(g) : X_c \to Y_b$ as in the proof of Corollary 6.10. This completes the construction of a lift $\sigma_\bullet \in L_I\,(X, Y)$ for $\varphi_\bullet$ as required. □

Corollary 6.19. *Suppose $(\mathcal{C}, I, Z, X, Y)$ is admissible, Y is strongly fibrant, and J is any full subcategory of I with the same weakly initial and final objects. Then the restriction map $p : L_I\,(X, Y) \to L_J\,(X, Y)$ is a fibration.*

Proof. By induction on the number of objects in $I \setminus J$, using Proposition 6.14. □

7. Identifying the fibers

As we have just seen, if I is a good indexing category, under our standard assumptions on Z, X, and Y the auxiliary tower (4.11) is a tower of fibrations of simplicial abelian groups. It remains to identify the fibers of the restriction maps $p : L_I\,(X, Y) \to L_J\,(X, Y)$, for a subcategory J of I; this will allow us to determine those of the primary tower (4.6) (or, more directly, those of the modified tower (4.15)). We consider only the case when $I \setminus J$ consists of a single internal object c.

Lemma 7.1. *If $(\mathcal{C}, I, Z, X, Y)$ is admissible and Y is strongly fibrant, then $\varphi_\bullet \in \mathrm{Ker}(p) \subseteq L_I\,(X, Y)$ if and only if*

(a) *$\phi_f = 0$ for each morphism f of I which does not begin or end in c.*

(b) *for any $d \xrightarrow{g} c \xrightarrow{f} b$ in I with f and g indecomposable:*

$$Y(f) \circ \varphi_g + \varphi_f \circ X(g) = 0. \tag{7.2}$$

Proof. This follows from Lemma 5.12. □

Remark 7.3. The lemma implies that $(\varphi_f, -\varphi_g)$ defines a map from $X(g)$ to $Y(f)$. Note also that if φ_f is an arrow over $Z_{t(f)}$, the same is true of its negative; the remainder of the diagram for a map over $Z(f)$ already commutes because X and Y are diagrams over Z. Thus $(\varphi_f, -\varphi_g)$ is a map of arrows over $Z(f)$.

Definition 7.4. If $(\mathcal{C}, I, Z, X, Y)$ is admissible, we define the *local cohomology* of $X \in \mathcal{C}^I/Z$ at an object $c \in I$, denoted by $\mathcal{H}_c(X/Z, Y)$, to be the total derived functors into simplicial abelian groups of $\mathrm{map}_{\phi_c}(\psi_c, \rho_c)$ applied to X, where $\psi_c : \mathrm{hocolim}_{d \in I/c} X_d \to X_c$, $\rho_c : Y_c \to \mathrm{holim}_{b \in c/I} Y_b$, and $\phi_c : Z_c \to \mathrm{holim}_{b \in c/I} Z_b$, are the structure maps. The ith *local cohomology group* of $X \in \mathcal{C}^I/Z$ at c is defined to be $\mathcal{H}_c^i(X/Z, Y) := \pi_i \mathcal{H}_c(X/Z, Y)$.

Remark 7.5. In many cases, the local cohomology at c can be identified explicitly as the André–Quillen cohomology of an appropriate (small) diagram.

Proposition 7.6. *If $(\mathcal{C}, I, Z, X, Y)$ is admissible, Y is strongly fibrant, and $J = I \setminus \{c\}$, then Ker(p) is weakly equivalent (as a simplicial abelian group) to $\mathcal{H}_c(X/Z, Y)$.*

Proof. To obtain the total derived functors, in this case, we must replace X by a weakly equivalent cofibrant, hence Reedy cofibrant object, which implies that $\mathrm{hocolim}_{d \in I/c} X_d$ is simply the colimit, and ψ_c is a cofibration. By Remark 6.12, we can replace Y by a weakly equivalent strongly fibrant abelian group object in $\mathcal{C}^I/Z$, which implies that $\mathrm{holim}_{b \in c/I} Y_b$ is the limit, and ρ_c is a fibration. With these choices, $\mathcal{H}_c^I(X/Z, Y)$ is simply the mapping space $\mathrm{map}_{\phi_c}(\psi_c, \rho_c)$, which is isomorphic to *Ker*(p) in Lemma 7.1 (using the sign of Remark 7.3). □

Theorem 7.7. *If $(\mathcal{C}, I, Z, X, Y)$ is admissible, for any ordering $(c_i)_{i=1}^{\infty}$ of the objects of I, there is a natural first quadrant spectral sequence with:*

$$E^2_{s,t} = \mathcal{H}^{s+1}_{c_t}(X/Z; Y) \Longrightarrow H^{s+t+1}(X/Z; Y),$$

with $d_2 : E^2_{s,t} \to E^2_{s-2,t+1}$.

Proof. We may replace Y by a weakly equivalent strongly fibrant abelian group object, by Remark 6.12. By Corollary 6.19, (4.15) is then a tower of fibrations, so it has an associated homotopy spectral sequence. To identify the E^2-term, note that the homotopy groups of the homotopy fibers of the tower are the local cohomology groups in Proposition 7.6, suitably indexed (see Remark 3.2). □

Remark 7.8. Note that $p^I_J : L_I(X, Y) \to L_J(X, Y)$ is a fibration for any full subcategory $J \subseteq I$ with the same weakly initial and final objects (Corollary 6.19), and we can similarly describe the fiber of p^I_J as a sort of local cohomology $\mathcal{H}^I_J(X/Z, Y)$, and thus identify the E^2-term of the spectral sequence obtained from a fairly arbitrary cover of I.

We shall not attempt to define $\mathcal{H}^I_J(X/Z, Y)$ in general. Observe, however, that if J is discrete (i.e., there are no non-identity maps between its objects $c_1, \ldots, c_n$), then

$$\mathcal{H}^I_J(X/Z, Y) \cong \prod_{i=1}^{n} \mathcal{H}_{c_i}(X/Z, Y). \tag{7.9}$$

Example 7.10. For the commuting square of Example 3.9, we now get a cover for I consisting of $I_3 = I$, $I_2 = I \setminus \{3\}$ — i.e., a commuting triangle:

$$\begin{tikzcd}
4 \arrow[d, "c"'] \arrow[dr, "b\circ d"] & \\
2 \arrow[r, "a"'] & d
\end{tikzcd}$$

$I_1 = \{4 \xrightarrow{a\circ c} 1\}$, and $I_0 = \{4\}$.

Given a diagram of abelian group objects $Y : I \to \mathcal{C}$, the local cohomology groups which form the E^2-term of the spectral sequence of Theorem 7.7 are:

$$E^2_{s,t} \cong \begin{cases} H^{s+3}(X(d); Y(b)) & \text{if } t = 2; \\ H^{s+2}(X(c); Y(a)) & \text{if } t = 1; \\ H^{s+1}(X_4; Y_1) & \text{if } t = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Once more we could unite the first and second rows by omitting I_2 from our cover, as in Example 3.9, by (7.9).

7.11. *A comparison*

In the simplest case, when $I = [\mathbf{1}]$ (a single map):

$$\begin{tikzcd}
& X_2 \arrow[r, "X\phi"] \arrow[d, "f_2"] \arrow[ddl, "p_2"'] & X_1 \arrow[d, "f_1"'] \arrow[ddr, "p_1"] & \\
& Y_2 \arrow[r, "Y\phi"'] \arrow[dl, "q_2"] & Y_1 \arrow[dr, "q_1"'] & \\
Z_2 \arrow[rrr, "Z\phi"] & & & Z_1,
\end{tikzcd}$$

we have the "defining fibration sequence":

$$\mathrm{map}\,(X, Y) \to \mathrm{map}\,(X_2, Y_2) \times \mathrm{map}\,(X_1, Y_1) \xrightarrow{\xi} \mathrm{map}\,(X_2, Y_1) \tag{7.12}$$

of [9, Prop. 4.20] (where all mapping spaces are taken in the appropriate comma categories).

Projecting the total space of (7.12) onto the second factor yields the following interlocking diagram of horizontal and vertical fibration sequences:

$$\begin{tikzcd}
\operatorname{map}(X_2, \mathrm{Fib}(Y\phi)) \arrow[r] \arrow[d, "i_*"'] & \operatorname{map}(X, Y) \arrow[r] \arrow[d] & \operatorname{map}(X_1, Y_1) \arrow[d, "\mathrm{Id}"] \\
\operatorname{map}(X_2, Y_2) \arrow[r] \arrow[d, "\phi_*"'] & \operatorname{map}(X_2, Y_2) \times \operatorname{map}(X_1, Y_1) \arrow[r, "\pi"] \arrow[d, "\xi"] & \operatorname{map}(X_1, Y_1) \arrow[d] \\
\operatorname{map}(X_2, Y_1) \arrow[r, "\mathrm{Id}"] & \operatorname{map}(X_2, Y_1) \arrow[r] & *
\end{tikzcd} \tag{7.13}$$

We see that the spectral sequence of Theorem 3.5 reduces to the long exact sequence in homotopy for the top horizontal fibration sequence in (7.13) , while the long exact sequence of Fact 3.3 is obtained from the left vertical fibration sequence in (7.13).

Remark 7.14. This actually works for any linear order $I = [\mathbf{n}]$ (Example 1.4):

Given $X, Y \in \mathcal{C}^I/Z$, if we set $I' := [\boldsymbol{n-1}]$ (so $J := \{n \xrightarrow{\phi_n} n-1\}$) and let $\tau = \tau^I_{I'} : \mathcal{C}^I/Z \to \mathcal{C}^{I'}/Z|_{I'}$, then (7.12) yields a fibration sequence:

$$\operatorname{map}(X, Y) \to \operatorname{map}(X_n, Y_n) \times \operatorname{map}(\tau X, \tau Y) \xrightarrow{\xi} \operatorname{map}(X_n, Y_{n-1})$$

which again induces a interlocking diagram of fibrations:

$$\begin{tikzcd}
\operatorname{map}(X_n, \mathrm{Fib}(Y\phi_n)) \arrow[r] \arrow[d, "i_*"'] & \operatorname{map}(X, Y) \arrow[r] \arrow[d] & \operatorname{map}(\tau X, \tau Y) \arrow[d, "\mathrm{Id}"] \\
\operatorname{map}(X_n, Y_n) \arrow[r] \arrow[d, "(\phi_n)_*"'] & \operatorname{map}(X_n, Y_n) \times \operatorname{map}(\tau X, \tau Y) \arrow[r, "\pi"] \arrow[d, "\xi"] & \operatorname{map}(\tau X, \tau Y) \arrow[d] \\
\operatorname{map}(X_n, Y_{n-1}) \arrow[r, "\mathrm{Id}"] & \operatorname{map}(X_n, Y_{n-1}) \arrow[r] & *
\end{tikzcd}$$

as in (7.13). Note that the long exact sequences in homotopy (i.e., cohomology) of the central vertical fibrations (for various values of n) provide an alternative inductive approach for calculating the cohomology of X, which can again be formalized in a spectral sequence (though in this case the fibers are the unknown quantity).

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References

[1] J.V Adámek, J Rosický, Locally Presentable and Accessible Categories, Cambridge U. Press, Cambridge, UK, 1994.
[2] D.W. Anderson, A generalization of the Eilenberg–Moore spectral sequence, Bull. AMS 78 (5) (1972) 784–786.
[3] H.J. Baues, G. Wirsching, The cohomology of small categories, J. Pure Appl. Algebra 38 (1985) 187–211.
[4] J.M. Beck, Triples, algebras and cohomology, Repr. Theory Appl. Cats. 2 (2003) 1–59.
[5] B. Bendiffalah, D. Guin, Cohomologie de diagrammes d'algèbres triangulaires, in: Colloquium on Homology and Representation Theory (Vaquerías, 1998), Bol. Acad. Nac. Cienc. (Córdoba) 65 (2000) 61–71.
[6] D.J. Benson, J.F. Carlson, Diagrammatic methods for modular representations and cohomology, Comm. Algebra 15 (1987) 53–121.
[7] D. Blanc, Higher homotopy operations and the realizability of homotopy groups, Proc. London Math. Soc. 70 (1995) 214–240.
[8] D. Blanc, W.G. Dwyer, P.G. Goerss, The realization space of a Π-algebra: A moduli problem in algebraic topology, Topology 43 (2004) 857–892.
[9] D. Blanc, M.W. Johnson, J.M. Turner, On realizing diagrams of Π-algebras, Algebraic Geom. Topol. 6 (2006) 763–807.
[10] D. Blanc, G. Peschke, The fiber of functors between categories of algebras, J. Pure Appl. Algebra 207 (2006) 687–715.
[11] A.K. Bousfield, On the homology spectral sequence of a cosimplicial space, Amer. J. Math. 109 (2) (1987) 361–394.
[12] A.K. Bousfield, D.M. Kan, Homotopy Limits, Completions, and Localizations, in: Lec. Notes Math., vol. 304, Springer, Berlin, New York, 1972.
[13] A.M. Cegarra, Cohomology of diagrams of groups. The classification of (co)fibred categorical groups, Int. Math. J. 3 (2003) 643–680.
[14] T. Datuashvili, Cohomologically trivial internal categories in categories of groups with operations, Appl. Categ. Structures 3 (1995) 221–237.
[15] J.F. Davis, W. Lück", The p-chain spectral sequence, K-Theory 30 (2003) 71–104.
[16] E. Dror-Farjoun, Homotopy and homology of diagrams of spaces, in: H.R. Miller, D.C. Ravenel (Eds.), Algebraic Topology (Seattle, Wash., 1985), in: Lec. Notes Math., vol. 1286, Springer, Berlin, New York, 1987, pp. 93–134.
[17] G. Dula, R. Schultz, Diagram Cohomology and Isovariant Homotopy Theory, in: Mem. Amer. Math. Soc., vol. 110, Providence, RI, 1994.
[18] W.G. Dwyer, D.M. Kan, Hochschild–Mitchell cohomology of simplicial categories and the cohomology of simplicial diagrams of simplicial sets, Nederl. Akad. Wetensch. Indag. Math. 50 (1988) 111–120.
[19] W.G. Dwyer, D.M. Kan, J.H. Smith, Homotopy commutative diagrams and their realizations, J. Pure Appl. Algebra 57 (1989) 5–24.

[20] H.R. Fischer, F.L. Williams, Borel–LePotier diagrams–calculus of their cohomology bundles, Tohoku Math. J. (2) 36 (1984) 233–251.
[21] M. Gerstenhaber, A. Giaquinto, S.D. Schack, Diagrams of Lie algebras, J. Pure Appl. Algebra 196 (2005) 169–184.
[22] M. Gerstenhaber, S.D. Schack, On the deformation of algebra morphisms and diagrams, Trans. Amer. Math. Soc. 279 (1983) 1–50.
[23] M. Gerstenhaber, S.D. Schack, Algebraic cohomology and deformation theory, in: M. Gerstenhaber, M. Hazewinkel (Eds.), Deformation Theory of Algebras and Structures and Applications (Il Ciocco, 1986), in: NATO ASI, Series C, vol. 247, Kluwer, Dordrecht, 1997, pp. 11–264.
[24] P.G. Goerss, J.F. Jardine, Simplicial Homotopy Theory, in: Prog. in Math., vol. 174, Birkhäuser, Boston, Stuttgart, 1999.
[25] P.S. Hirschhorn, Model Categories and their Localizations, AMS, Providence, RI, 2002.
[26] S. Illman, Equivariant Singular Homology and Cohomology, I, in: Mem. AMS, vol. 156, Am. Math. Soc, Providence, RI, 1975.
[27] M.A. Jibladze, T.I. Pirashvili, Cohomology of algebraic theories, J. Algebra 137 (2) (1991) 253–296.
[28] J.P. May, Equivariant Homotopy and Cohomology Theory, in: Reg. Conf. Ser. Math., vol. 91, Am. Math. Soc., Providence, RI, 1996.
[29] B. Mitchell, Rings with several objects, Adv. Math. 8 (1972) 1–161.
[30] I. Moerdijk, J.-A. Svensson, The equivariant Serre spectral sequence, Proc. AMS 118 (1993) 263–267.
[31] P. Olum, Homology of squares and factoring of diagrams, in: Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, WA, 1968), Springer, Berlin, New York, 1969, pp. 480–489.
[32] P. Pavešić, Diagram cohomologies using categorical fibrations, J. Pure Appl. Algebra 112 (1996) 73–90.
[33] R.J. Piacenza, Cohomology of diagrams and equivariant singular theory, Pac. J. Math. 91 (1980) 435–443.
[34] R.J. Piacenza, Diagrams of simplicial sets, complexes and bundles, Tamkang J. Math. 15 (1984) 83–94.
[35] D.G. Quillen, Homotopical Algebra, in: Lec. Notes Math., vol. 20, Springer, Berlin, New York, 1963.
[36] D.G. Quillen, On the (co-)homology of commutative rings, in: Applications of Categorical Algebra, in: Proc. Symp. Pure Math., vol. 17, AMS, Providence, RI, 1970, pp. 65–87.
[37] M. Robinson, Cohomology of diagrams of algebras, preprint, 2008. arXiv:0802.3651.
[38] G.B. Segal, Categories and cohomology theories, Topology 13 (1974) 293–312.
[39] J. Słomińska, Some spectral sequences in Bredon cohomology, Cahiers Top. Géom. Diff. Cat. 33 (1992) 99–133.
[40] R.M. Vogt, Homotopy limits and colimits, Math. Z. 134 (1973) 11–52.