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A Construction of Algebras with Large Global Dimensions

KUNIO YAMAGATA

Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan

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In [G], Green gave an example of a family of finite dimensional algebras A_n $(n \geqslant 0)$, over an algebraically closed field, with exactly two isomorphism classes of simple modules and having global dimension n. In [H], making use of properties of Fibonacci sequences, Happel studied homological properties of those algebras. The algebra A_0 in the family is a semi-simple algebra with two simple modules and A_n is obtained by adding one arrow and zero-relations to the quiver of A_{n-1}. The aim of this paper is to generaize the construction of arbitrary algebras. More precisely, for an algebra A we construct a family of algebras $A_n (n \geqslant 0, A_0 = A)$ having the same number of simple modules as the number of simple A-modules, so that the global dimension of A_n is greater than that of A_{n-1} provided that the global dimension of A is finite, and Cartan determinants of all A_n are the same as that of A. Let A be a non-simple basic algebra and decompose it, say $A = \bigoplus_{i=1}^{m} P_i$, $m \geqslant 2$, as left A-modules, where the set of factors $\{P_i\}$ is considered having a linear order according to the natural numbers in the set of indicies. Then the algebras A_n are constructed from the ordered decomposition of A and our main theorem asserts that

$$n + \max\left\{\left[\frac{n-1}{m-1}\right] + 1, \text{gl dim } A\right\} \leqslant \text{gl dim } A_n \leqslant \left[\frac{n-1}{m-1}\right] + n + 1 + \text{gl dim } A.$$

Although our construction depends on the decomposition of A and the order of decomposition factors, the global dimension of every A_n is determined by the numbers m and n provided that the ring A is semi-simple or hereditary such that the last factor P_m is simple. In particular, by our construction the family of algebras given by Green is divided into two subfamilies constructed similarly from the semi-simple algebra A_0 and from the hereditary algebra A_1.

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In the first section defining conditions for a family of rings are given. In Section 2, the projective dimensions of simple modules over the rings satisfying the conditions are computed, then the main theorem is stated in Section 3. The invariance of Cartan determinants of the constructed algebras are proved in Section 4. In Section 5, as an application, Green's example is considered, and some examples are given to verify the possiblity of the global dimension of A_n.

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1. A Family of Algebras

Throughout this paper, all modules are left modules and the composite of morphisms are written from the right to the left. For a module M and a natrual number n, M^n stands for a direct sum of n copies of M, and M^0 denotes the zero module. By top M we understand the factor module of M by the Jacobson radical rad M. For a rational number r, $[r]$ is the largest integer not greater than r.

Let A be a ring with identity and $A_0 = A$. By a family $\{A_n, p_n, m\}$ we understand a set of rings A_n $(n \geqslant 0)$, ring-homomorphisms $p_n\colon A_n \to A_{n-1}$ and a natural number $m \geqslant 2$ such that $p_n(A_n) = A_{n-1}$, and A_n is a direct sum of left modules $P_{n,i}$, $A_n = \bigoplus_{i=1}^{m} P_{n,i}$, and the restrictions of p_n to $P_{n,i}$ induce the morphisms $p_{n,i}\colon P_{n,i} \to P_{n-1,i}$ for $n \geqslant 1$. For any integers n and j with $0 \leqslant j \leqslant n$, every $P_{n-j,i}$ is then considered as a left A_n-module via the composite $p_{n-j+1} \cdots p_n\colon A_n \to A_{n-j}$, so that the morphism $p_{n,i}$ is an A_n-homomorphism.

We consider the following two conditions for a family $\{A_n, p_n, m\}$;

(I) For every $n > 0$, there are the following eact sequences for A_n-modules:

$$0 \longrightarrow P_{n-1,1}^{\alpha_{n,m}} \xrightarrow{q_{n,m}} P_{n,m} \xrightarrow{p_{n,m}} P_{n-1,m} \longrightarrow 0$$

$$0 \longrightarrow P_{n,m}^{\alpha_{n,m-1}} \xrightarrow{q_{n,m-1}} P_{n,m-1} \xrightarrow{p_{n,m-1}} P_{n-1,m-1} \longrightarrow 0$$

$$\cdots$$

$$0 \longrightarrow P_{n,2}^{\alpha_{n,1}} \xrightarrow{q_{n,1}} P_{n,1} \xrightarrow{p_{n,1}} P_{n-1,1} \longrightarrow 0,$$

where, for $1 \leqslant i \leqslant m$, $\alpha_{n,i}$ are non-negative integers, and $p_{n,i}$, $q_{n,i}$ are A_n-homomorphisms for $1 \leqslant i \leqslant m$.

(II) For $1 \leqslant i \leqslant m$, the image of $q_{n,i}$ is in rad $P_{n,i}$ and $q'_{n,i}$ is a split-monomorphism, where $q'_{n,i}$ is the canonical morphism induced from $q_{n,i}$,

$$q_{n,i}: P_{n,i+1}^{\alpha_{n,i}} \xrightarrow{q'_{n,i}} \operatorname{rad} P_{n,i} \xrightarrow{\varepsilon_{n,i}} P_{n,i} \qquad \text{for} \quad 1 \leqslant i < m,$$

$$q_{n,m}: P_{n-1,1}^{\alpha_{n,m}} \xrightarrow{q'_{n,m}} \operatorname{rad} P_{n,m} \xrightarrow{\varepsilon_{n,m}} P_{n,m},$$

where $\varepsilon_{n,i}$ are inclusion maps.

It should be noted that every $p_{n,i}: P_{n,i} \to P_{n-1,i}$ is a projective cover in the category of A_n-modules if the second condition is satisfied. We denote by $r'_{n,i}: \operatorname{rad} P_{n-1,i} \to \operatorname{rad} P_{n,i}$ a section of the epimorphism $\operatorname{rad} P_{n,i} \to \operatorname{rad} P_{n-1,i}$ induced from $p_{n,i}$. Then, under the two conditions, there are isomorphism

$$P_{n,i+1}^{\alpha_{n,i}} \oplus \operatorname{rad} P_{n-1,i} \xrightarrow{(q'_{n,i},\, r'_{n,i})} \operatorname{rad} P_{n,i} \qquad \text{for} \quad 1 \leqslant i < m,$$

$$P_{n-1,1}^{\alpha_{n,m}} \oplus \operatorname{rad} P_{n-1,m} \xrightarrow{(q'_{n,m},\, r'_{n,m})} \operatorname{rad} P_{n,m}.$$

By combining these isomorphisms we have the following lemma:

LEMMA 1.1. *For a family $\{A_n, p_n, m\}$ satisfying the conditions* (I), (II), *the following sequences are exact*:

$$0 \longrightarrow \left(\bigoplus_{j=1}^{n} P_{n-j+1,i+1}^{\alpha_{n-j+1,i}}\right) \oplus \operatorname{rad} P_{0,i} \xrightarrow{(\psi^n_{n,i},\, \rho_{n,i})} P_{n,i} \longrightarrow \operatorname{top} P_{n,i} \longrightarrow 0$$

$$0 \longrightarrow \left(\bigoplus_{j=1}^{n} P_{n-j,1}^{\alpha_{n-j+1,m}}\right) \oplus \operatorname{rad} P_{0,m} \xrightarrow{(\psi^n_{n,m},\, \rho_{n,m})} P_{n,m} \longrightarrow \operatorname{top} P_{n,m} \longrightarrow 0.$$

Here $1 \leqslant i < m$, and $P_{n,i} \to \operatorname{top} P_{n,i}$, $P_{n,m} \to \operatorname{top} P_{n,m}$ are canonical factor morphisms and, for $1 \leqslant i \leqslant m$ and $2 \leqslant j \leqslant n$,

$$\psi^1_{n,i} = \varepsilon_{n,i} q'_{n,i} (= q_{n,i}),$$

$$\psi^j_{n,i} = (\varepsilon_{n,i} q'_{n,i}, \varepsilon_{n,i} r'_{n,i} q'_{n-1,i}, \ldots, \varepsilon_{n,i} r'_{n,i} \cdots r'_{n-j+2,i} q'_{n-j+1,i}),$$

$$\rho_{n,i} = \varepsilon_{n,i} r'_{n,i} \cdots r'_{1,i}.$$

LEMMA 1.2. *Let $\{A_n, p_n, m\}$ be as in the above. Then the following sequences are exact*:

(i) $$0 \longrightarrow \left(\bigoplus_{k=1}^{j} P_{n-k+1,i+1}^{\alpha_{n-k+1,i}}\right) \xrightarrow{\psi^j_{n,i}} P_{n,i} \xrightarrow{\varphi^j_{n,i}} P_{n-j,i} \longrightarrow 0 \quad \textit{for} \quad i < m,$$

$$\text{(ii)} \qquad 0 \longrightarrow \left(\bigoplus_{k=1}^{j} P_{n-k,1}^{\alpha_{n-k+1,m}}\right) \xrightarrow{\psi^j_{n,m}} P_{n,m} \xrightarrow{\varphi^j_{n,m}} P_{n-j,m} \longrightarrow 0,$$

where $\varphi^j_{n,k} = p_{n-j+1,k}\, p_{n-j,k} \cdots p_{n,k}$ *for* $1 \leqslant k \leqslant m$.

Proof. We only prove the exactness of the sequences in (i), because (ii) is proved by a similar argument. Then the sequence for $j=1$ is nothing but the exact sequence given in (I).

For $j \geqslant 2$, the following square is commutative for $1 \leqslant i < m$ because of the definition of r':

$$\begin{array}{ccc} \operatorname{rad} P_{n-1,i} & \xrightarrow{r'_{n,i}} & \operatorname{rad} P_{n,i} \\ \Big\downarrow{\scriptstyle \varepsilon_{n-1,i}} & & \Big\downarrow{\scriptstyle \varepsilon_{n,i}} \\ P_{n-1,i} & \xrightarrow[p_{n,i}]{} & P_{n,i}. \end{array}$$

Then it holds that

$$\begin{aligned} &\varphi^{j-1}_{n,i} \cdot (j\text{th factor of } \psi^j_{n,i}) \\ &\quad = (p_{n-j+2} \cdots p_{n,i})(\varepsilon_{n,i}\, r'_{n,i} \cdots r'_{n-j+2,i}\, q'_{n-j+1,i}) \\ &\quad = (p_{n-j+2} \cdots p_{n-1,i})(\varepsilon_{n-1,i}\, r'_{n-1,i} \cdots r'_{n-j+2,i}\, q'_{n-j+1,i}) \\ &\quad = \cdots \\ &\quad = (p_{n-j+2}\, \varepsilon_{n-j+2,i}\, r'_{n-j+2,i}\, q'_{n-j+1,i}) \\ &\quad = \varepsilon_{n-j+1,i}\, q'_{n-j+1,i} \\ &\quad = q_{n-j+1,i}. \end{aligned}$$

We therefore have the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & X_j & \xrightarrow{=} & X_j & & & & \\ & & \downarrow & & \Big\downarrow{\scriptstyle \psi^{j-1}_{n,i}} & & & & \\ 0 & \longrightarrow & X_j \oplus P_{n-j+1,i+1}^{\alpha_{n-j+1,i}} & \xrightarrow{\psi^j_{n,i}} & P_{n,i} & \xrightarrow{\varphi^j_{n,i}} & P_{n-j,i} & \longrightarrow & 0 \\ & & \downarrow & & \Big\downarrow{\scriptstyle \varphi^{j-1}_{n,i}} & & \| & & \\ 0 & \longrightarrow & P_{n-j+1,i+1}^{\alpha_{n-j+1,i}} & \xrightarrow{q_{n-j+1,i}} & P_{n-j+1,i} & \xrightarrow{p_{n-j+1,i}} & P_{n-j,i} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Here $X_j = \bigoplus_{k=1}^{j-1} P_{n-k+1,i+1}^{\alpha_{n-k+1,i}}$, and the left vertical sequence is a canonically splittable exact sequence and the middle vertical sequence is exact by induction on j.

To show the exactness of the middle horizontal sequence, it suffices to show that $\operatorname{Ker} \varphi_{n,i}^j = \operatorname{Im} \psi_{n,i}^j$, because $\varphi_{n,i}^j$ is epimorphic and $\psi_{n,i}^j$ is monomorphic obviously. Now we have that

$$\operatorname{Ker} \varphi_{n,i}^j = (\varphi_{n,i}^{j-1})^{-1} (\operatorname{Ker} p_{n-j+1,i}) = (\varphi_{n,i}^{j-1})^{-1} (\operatorname{Im} q_{n-j+1,i}),$$

and, from the above diagram, $\operatorname{Im} q_{n-j+1,i} = \varphi_{n,i}^{j-1}(\operatorname{Im} \psi_{n,i}^j)$ so that $(\varphi_{n,i}^{j-1})^{-1} (\operatorname{Im} q_{n-j+1,i}) = \operatorname{Im} \psi_{n,i}^j + \operatorname{Ker} \varphi_{n,i}^{j-1}$. Therefore, since $\operatorname{Ker} \varphi_{n,i}^{j-1} = \psi_{n,i}^{j-1}(X_j) \subseteq \operatorname{Im} \psi_{n,i}^j$ clearly, we have that $\operatorname{Ker} \varphi_{n,i}^j = \operatorname{Im} \psi_{n,i}^j$.

2. Projective Dimensions of Simple A_n-Modules

From now on we assume that every family $\{A_n, p_n, m\}$ satifies the conditions (I), (II) and assume that any $\alpha_{n,i}$ is not zero. Unless otherwise stated, modules are A_n-modules for an arbitrarily fixed n. The projective dimension of a module M over a ring A is denoted by $\operatorname{pd}_A M$ of $\operatorname{pd} M$ simply.

Lemma 2.1.

$$\operatorname{pd}_{A_n} P_{n-j,i} = \left[\frac{i+j-2}{m-1}\right] + j \qquad \text{for} \quad 0 < j \leqslant n.$$

In particular, in the case when $m = 2$,

$$\operatorname{pd}_{A_n} P_{n-j,1} = 2j - 1, \qquad \operatorname{pd}_{A_n} P_{n-j,2} = 2j.$$

Proof. (1) First, for $i = 1$ we prove the following statements:

(i) $\operatorname{pd}_{A_n} P_{n-k,1} < \operatorname{pd}_{A_n} P_{n-l,1}$ for $k < l$, and

(ii) $\operatorname{pd}_{A_n} P_{n-k,1} = \left[\dfrac{k-1}{m-1}\right] + k$ for $k > 0$.

Let $k = t(m-1) + s$, where $t \geqslant 0$ and $0 \leqslant s < m-1$. In the case when $0 < k \leqslant m-1$, it follows from Lemma 1.2(i) that $\operatorname{pd} P_{n-k,1} = k + \operatorname{pd} P_{n,k+1} = k (= [(k-1)/(m-1)] + k)$ and so, in particular,

$$\operatorname{pd} P_{n-k,1} > \operatorname{pd} P_{n-k+k',1} \qquad \text{for} \quad 0 < k' \leqslant k.$$

For $k > m-1$, it follows from Lemma 1.2(i) again that

$$\operatorname{pd} P_{n-k,1} = (m-1) + \max\{\operatorname{pd} P_{n-k+(m-1),m}, \ldots, \operatorname{pd} P_{n,m}\}$$

and hence, from Lemma 1.2(ii),

$$\operatorname{pd} P_{n-k,1} = m + \max\{\operatorname{pd} P_{n-k+(m-1),1}, ..., \operatorname{pd} P_{n-1,1}\}.$$

In particular, we have that $\operatorname{pd} P_{n-k,1} > \operatorname{pd} P_{n-k+k',1}$ for $m-1 \leqslant k' \leqslant k$. Thus, in case $s=0$, taking account of the fact that $m-1 = k-(t-1)(m-1)$ and

$$\left[\frac{k}{m-1}\right] = \left[\frac{k-1}{m-1}\right] + 1,$$

it holds that

$$\operatorname{pd} P_{n-k,1} = (t-1)\,m + \operatorname{pd} P_{n-(m-1),1} = (t-1)\,m + (m-1) = t+k-1$$

so that $\operatorname{pd} P_{n-(m-1),1} = [(k-1)/(m-1)] + k$.

In case $s>0$, since $m>2$ and

$$\left[\frac{k}{m-1}\right] = \left[\frac{k-1}{m-1}\right],$$

we have that

$$\operatorname{pd} P_{n-k,1} = tm + \operatorname{pd} P_{n-s,1} = tm + s = t + k = \left[\frac{k-1}{m-1}\right] + k.$$

(2) For $i \neq m$, it follows from Lemma 1.2 that

$$\begin{aligned}\operatorname{pd} P_{n-j,i} &= m - i + \max\{\operatorname{pd} P_{n-(i+j-m),m}, ..., \operatorname{pd} P_{n-1,1}\}\\ &= m - i + 1 + \max\{\operatorname{pd} P_{n-(i+j-m),1}, ..., \operatorname{pd} P_{n-1,1}\},\end{aligned}$$

and hence, from (i) and (1) above, $\operatorname{pd} P_{n-j,i} = m-i+1+\operatorname{pd} P_{n-(i+j-m),1}$. Thus, to show the lemma in this case, it suffices to substitute $i+j-m$ for k in (ii) above, because

$$\left[\frac{i+j-2}{m-1}\right] - 1 = \left[\frac{i+j-m-1}{m-1}\right].$$

For $i=m$, it follows from Lemma 1.2 and the above (i) that

$$\operatorname{pd} P_{n-j,m} = 1 + \max\{\operatorname{pd} P_{n-j,1}, ..., \operatorname{pd} P_{n-1,1}\} = 1 + \operatorname{pd} P_{n-j,1}.$$

Hence the assertion follows from the above (ii) by substituting j for k, because

$$\left[\frac{j-1}{m-1}\right] = \left[\frac{m+j-2}{m-1}\right] - 1.$$

In the above proof we showed that $\operatorname{pd} P_{n-k,1} < \operatorname{pd} P_{n-l,1}$ for $0 \leqslant k \leqslant l$. But we can show a more precise relation between $\operatorname{pd} P_{n-j,i}$ for all i and j. To show this we denote by $m(j)$ the natural number satisfying the property that $1 \leqslant m(j) < m$ and $(m(j)+j-m)/(m-1)$ is an integer for every j with $0 \leqslant j \leqslant n$. Note that, for every j, the number $m(j)$ is uniquely determined and

$$\left[\frac{l+j-2}{m-1}\right] = \left[\frac{k+j-2}{m-1}\right] + 1$$

for $m > 2$ and $0 < k \leqslant m(j) < l \leqslant m$. Then the following lemma follows easily from Lemma 2.1 and the trivial inequalities

$$\left[\frac{i+(j+1)-2}{m-1}\right] - \left[\frac{i+j-2}{m-1}\right] \leqslant 1.$$

LEMMA 2.2. (1) $0 \leqslant \operatorname{pd}_{A_n} P_{n-(j+1),i} - \operatorname{pd}_{A_n} P_{n-j,i} \leqslant 1$ *for* $0 \leqslant j < n$, $1 \leqslant i \leqslant m$.

(2) *For* $0 < j \leqslant n$, $\operatorname{pd}_{A_n} P_{n-j,1} = \cdots = \operatorname{pd}_{A_n} P_{n-j,m(j)}$ *and*

$$1 + \operatorname{pd}_{A_n} P_{n-j,1} = \operatorname{pd}_{A_n} P_{n-j,m(j)+1} = \cdots = \operatorname{pd}_{A_n} P_{n-j,m}.$$

Proof. It is clear from Lemma 2.1 for the case when $m = 2$, and from the note above Lemma 2.2 for the other case.

PROPOSITION 2.1. *Let n be a positive integer.*

(1) *If M is an A_i-module for $0 \leqslant i \leqslant n$, then*

$$\operatorname{pd}_{A_n} M \leqslant \operatorname{gl\,dim} A_i + \operatorname{pd}_{A_n} P_{i,m}.$$

(2) *If M is an A-module, then*

$$\operatorname{pd}_{A_n} M \leqslant \operatorname{gl\,dim} A + \left[\frac{n-1}{m-1}\right] + n + 1.$$

Proof. (1) We can assume that $\operatorname{gl\,dim} A_i$ is finite. Then, for a projective resolution of an A_i-module M,

$$0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0,$$

it follows from Lemma 2.2 that

$$\operatorname{pd}_{A_n} P_j \leqslant \operatorname{pd}_{A_n} A_i \leqslant \operatorname{pd}_{A_n} P_{i,m} \qquad \text{for} \quad 0 \leqslant j \leqslant k.$$

Hence the desired inequality follows from the general fact that $\operatorname{pd}_{A_n} M \leqslant k + \max\{\operatorname{pd}_{A_n} P_j \mid 0 \leqslant j \leqslant k\}$.

(2) is clear from (1) and Lemma 2.1.

PROPOSITION 2.2. *For a positive integer n,*

(1) $\mathrm{pd}_{A_n}(\mathrm{top}\, P_{n,i}) = \max\{\mathrm{pd}_{A_n} P_{0,i}, 1 + \mathrm{pd}_{A_n}(\mathrm{rad}\, P_{0,i})\}$ *for* $1 \leqslant i \leqslant m$,

(2) $\max\{\mathrm{pd}_{A_n}(\mathrm{top}\, P_{n,i}) \mid 1 \leqslant i \leqslant m\}$

$$= \max\left\{\left[\frac{n-1}{m-1}\right] + n + 1, 1 + \mathrm{pd}_{A_n}(\mathrm{rad}\, A)\right\}.$$

Proof. (1) For $1 \leqslant i < m$, it follows from Lemmas 1.1 and 2.2 that

$$\begin{aligned}\mathrm{pd}(\mathrm{top}\, P_{n,i}) &= 1 + \max\{\mathrm{pd}\, P_{n-j,i+1}, \mathrm{pd}(\mathrm{rad}\, P_{0,i}) \mid 0 \leqslant j < n\} \\ &= 1 + \max\{\mathrm{pd}\, P_{1,i+1}, \mathrm{pd}(\mathrm{rad}\, P_{0,i})\}.\end{aligned}$$

Therefore, by Lemmas 1.2 and 2.2, we have that

$$\mathrm{pd}(\mathrm{top}\, P_{n,i}) = \max\{\mathrm{pd}\, P_{0,i}, 1 + \mathrm{pd}(\mathrm{rad}\, P_{0,i})\} \qquad 1 \leqslant i < m.$$

Similarly, by Lemmas 1.1 and 2.2 it holds that

$$\begin{aligned}\mathrm{pd}(\mathrm{top}\, P_{n,m}) &= 1 + \max\{\mathrm{pd}\, P_{n-j,1}, \mathrm{pd}(\mathrm{rad}\, P_{0,m}) \mid 0 < j \leqslant n\} \\ &= \max\{1 + \mathrm{pd}\, P_{0,1}, 1 + \mathrm{pd}(\mathrm{rad}\, P_{0,m})\} \\ &= \max\{\mathrm{pd}\, P_{0,m}, 1 + \mathrm{pd}(\mathrm{rad}\, P_{0,m})\}.\end{aligned}$$

(2) This is an immediate consequence of Lemma 2.1 because, from (1) and Lemma 2.2, $\max\{\mathrm{pd}(\mathrm{top}\, P_{n,i}) \mid 1 \leqslant i \leqslant m\} = \max\{\mathrm{pd}\, P_{0,m}, 1 + \mathrm{pd}(\mathrm{rad}\, A)\}$.

3. GLOBAL DIMENSION OF A_n

From condition (I) all rings A_n in the family $\{A_n, p_n, m\}$ are semi-primary (left artinian) if so is A. In this section we assume that A is *semi-primary* as well as the both conditions (I), (II) for a given family $\{A_n, p_n, m\}$.

Now let I_n be the kernel of $p_n\colon A_n \to A_{n-1} (n > 0)$, and $I_{n,i}$ the kernel of $p_{n,i}\colon I_n = \bigoplus_{i=1}^{m} I_{n,i}$. The first syzyzy of an A_n-module M is denoted by $\Omega_n(M)$ or $\Omega_n^1(M)$, and $\Omega_n^k(M)$ is the $k(>1)$th syzyzy.

LEMMA 3.1. *Let M be an A_n-module, and assume that M is a direct sum of A_n-modules M_0 and $\rho(M)\colon M = M_0 \oplus \rho(M)$, such that $I_n M_0 = 0$ and $\rho(M)$ is a direct sum of projective A_n-modules and projective A_{n-1}-modules. Then, for every $k > 0$, $\Omega_n^k(M)$ is a direct sum of A_n-modules $\Omega_{n-1}^k(M_0)$ and*

$\rho(\Omega_n^k(M))$: $\Omega_n^k(M) = \Omega_{n-1}^k(M_0) \oplus \rho(\Omega_n^k(M))$, *such that* $\rho(\Omega_n^k(M))$ *is a direct sum of projective* A_n*-modules and projective* A_{n-1}*-modules.*

Proof. It suffices to show the lemma for $k=1$; we may assume that $\rho(M)=0$, because $\Omega_n(M) = \Omega_n(M_0) \oplus \Omega_n(\rho(M))$ and, by (I), $\Omega_n(\rho(M))$ is a direct sum of projective A_n-modules and projective A_{n-1}-modules.

Now let $0 \to \Omega_n(M) \to P \xrightarrow{\phi} M \to 0$ be a projective cover of an A_n-module M. Then $I_n P \subset \Omega_n(M)$ because $I_n M = 0$ by assumption, and $I_n P$ is a direct summand of rad P by (II). Hence $\Omega_n(M)$ is a direct sum of $I_n P$ and an A_n-module N,

$$\Omega_n(M) = N \oplus I_n P,$$

so that there is a canonical exact sequence of A_n-modules $0 \to N \to P/I_n P \xrightarrow{\bar{\phi}} M \to 0$. It follows that $I_n N = 0$ and $\Omega_{n-1}(M) = N$, because $\bar{\phi}$ is a projective cover of M in the category of A_{n-1}-modules. Since, by (I) $I_n P$ is a direct sum of projective A_n-modules and projective A_{n-1}-modules, we conclude that the above decomposition of $\Omega_n(M)$ has the desired property.

LEMMA 3.2. *Let* M *be a non-zero* A_{n-1}*-module. Then*

$$\mathrm{pd}_{A_{n-1}} M + 1 \leqslant \mathrm{pd}_{A_n} M.$$

In particular, $\mathrm{pd}_A M + n \leqslant \mathrm{pd}_{A_n} M$ *for any* A*-module* M *and any* $n \geqslant 0$.

Proof. We may assume that $\mathrm{pd}_{A_{n-1}} M = k$ is finite, because, by Lemma 3.1, $\mathrm{pd}_{A_n} M$ is infinite if so is $\mathrm{pd}_{A_{n-1}} M$. Then $\Omega_{n-1}^k(M)$ is a non-zero projective A_{n-1}- module and hence $\mathrm{pd}_{A_n}(\Omega_{n-1}^k(M)) \geqslant 1$ by (I). Therefore $\mathrm{pd}_{A_n}(\Omega_n^k(M)) \geqslant 1$, because $\Omega_{n-1}^k(M)$ is a direct summand of $\Omega_n^k(M)$ by Lemma 3.1. This implies that $\mathrm{pd}_{A_{n-1}} M + 1 \leqslant \mathrm{pd}_{A_n} M$.

The other assertion of the lemma is clear.

PROPOSITION 3.1. *For* $0 \leqslant j \leqslant n$,

$$\mathrm{gl\,dim}\, A_{n-j} + j \leqslant \mathrm{gl\,dim}\, A_n \leqslant \mathrm{gl\,dim}\, A_{n-j} + \mathrm{pd}_{A_n} P_{n-j,m}.$$

Proof. Since A_{n-j} is semi-primary, it is well-known that the global dimension of A_{n-j} is the maximal projective dimension of simple A_{n-j}-modules. So the proposition is clear from Lemma 3.2 and Proposition 2.1.

Now the following main theorem is an immediate consequence of Propositions 2.2(2), 2.1(2), and Lemma 3.2.

THEOREM 3.1. *Let* A *be a semi-primary ring. Then for any* $n \geqslant 0$,

$$\mathrm{gl\,dim}\, A_n = \max\left\{\left[\frac{n-1}{m-1}\right] + n + 1,\ 1 + \mathrm{pd}_{A_n}(\mathrm{rad}\, A)\right\}$$

and

$$n+\max\left\{\left[\frac{n-1}{m-1}\right]+1, \operatorname{gl\,dim} A\right\} \leqslant \operatorname{gl\,dim} A_n \leqslant \left[\frac{n-1}{m-1}\right]+n+1+\operatorname{gl\,dim} A.$$

COROLLARY 3.1. *For a semi-primary ring A, the following assertions hold*:

(1) *If* $\operatorname{gl\,dim} A=0$, *then* $\operatorname{gl\,dim} A_n=[(n-1)/(m-1)]+n+1$.

(2) *If* $m=2$, *then*

$$\max\{2n, n+\operatorname{gl\,dim} A\} \leqslant \operatorname{gl\,dim} A_n \leqslant 2n+\operatorname{gl\,dim} A.$$

Proof. (1) If $\operatorname{gl\,dim} A=0$, then $\operatorname{rad} A=0$ and hence the assertion is clear from the theorem. (2) is trivial from the second assertion of the theorem.

If A is semi-simple, the assertion (1) in Corollary 3.1 shows that $\operatorname{gl\,dim} A_n$ do not depend on the decomposition of A_0: $A_0=\bigoplus_{i=1}^{m} P_{0,i}$. This is not true for non-semisimple rings A. But, in the case when A is hereditary, we can show the global dimension of A_n more precisely.

PROPOSITION 3.2. *Assume that A is a connected, non-semisimple hereditary semi-primary ring. Then for $n>0$*,

$$\left[\frac{n-1}{m-1}\right]+n+1 \leqslant \operatorname{gl\,dim} A_n \leqslant \left[\frac{n-1}{m-1}\right]+n+2,$$

in particular, if $P_{0,m}$ is a simple A-module,

$$\operatorname{gl\,dim} A_n=\left[\frac{n-1}{m-1}\right]+n+2.$$

Proof. The first part is trivial from the second part of the theorem. In case $P_{0,m}$ is a simple A-module, there is an i such that $P_{0,m}$ is a direct summand of $\operatorname{rad} P_{0,i}$, because A is connected and non-semisimple. Hence $\operatorname{pd}(\operatorname{rad}(P_{0,i}))=\operatorname{pd} P_{0,m}$ by Lemma 2.2 and so, by Proposition 2.2, $\operatorname{pd}(\operatorname{top} P_{0,i})=\max\{\operatorname{pd} P_{0,i}, 1+\operatorname{pd}(\operatorname{rad} P_{0,i})\}=1+\operatorname{pd} P_{0,m}$. Thus we know from the first part of this proposition that $\operatorname{gl\,dim} A_n=1+\operatorname{pd} P_{0,m}$.

Remark. By $\mathrm{L}(M)$ we denote the Loewy length of a module M, and assume that $\mathrm{L}(P_{n-1,i})=\mathrm{L}(A_{n-1})$ as A_{n-1}-modules for some $1\leqslant i\leqslant m$. Then it is easily seen from the conditions (I) and (II) that $\mathrm{L}(P_{n-1,1})<\mathrm{L}(P_{n,m})$ and $\mathrm{L}(P_{n-1,i})\leqslant \mathrm{L}(P_{n,i})<\mathrm{L}(P_{n,i-1})$ for $1<i\leqslant m$. Therefore we know that $\mathrm{L}(A_n)<\mathrm{L}(A_{n+1})$ for any $n\geqslant 0$, and remember that $\operatorname{gl\,dim} A_n<$

gl dim A_{n+1}. On the other hand, in [KK] Kirkman and Kuzmanovich gave an example of a family of algebras with two isomorphism classes of simple modules and with Loewy lengths smaller than 5, but having arbitrary large global dimensions. This example shows that the global dimension cannot be bounded by the Loewy length and the number of simple modules. And it is well known that the number of simple modules are not bounded by the global dimension and the Loewy length. The other question of this type related to these three numbers is whether the Loewy length is not bounded by the other two numbers, either: the global dimension and the number of isomorphism classes of simple modules. But we conjecture that the Loewy length is bounded by the others. To state it more precisely, for integers $s(>0)$, $g(\geqslant 0)$, let $\mathscr{A}(s, g)$ be the class of algebras with exactly s isomorphism classes of simple modules and having global dimensions g. Then

Conjecture. Loewy lengths of algebras in $\mathscr{A}(s, g)$ are bounded.

It should be noted that the conjecture is true for the class of quasi-hereditary semi-primary rings [DR, Statement 9]. But it is still open even for the class of algebras with global dimension 3.

4. Cartan Determinant

Let A be a *left artinian* and basic ring, and assume that a family $\{A_n, p_n, m\}$ satisfies the condition (I) with $\alpha_{n,i} \geqslant 0$ and the following condition (II′) which is weaker than (II):

(II′) Ker $p_n \subset \operatorname{rad} A_n$ for any $n > 0$.

Let e_i be the idempotents of A corresponding to the projections $A_0 \to P_{0,i}$ for the decomposition $A_0 = \bigoplus_{i=1}^{m} P_{0,i}$, and $e_i = \sum_{k=1}^{l(i)} e_{i,k}$ a sum of orthogonal primitive idempotents. Then, by the condition (II′), the complete set of orthogonal primitive idempotents $\{e_{i,k} \mid i, k\}$ of A is lifted to every A_n. So we denote again by $\{e_{i,k} \mid i, k\}$ a lifted complete set of orthogonal primitive idempotents of A_n. Thus $A_n = \bigoplus_{n=1}^{m} P_{n,i}$ and $P_{n,i} = A_n e_i = \bigoplus_{k=1}^{l(i)} A_n e_{i,k}$. Let $p_{n,i,k}: A_n e_{i,k} \to A_{n-1} e_{i,k}$ be the restriction of p_n, so $p_n = \bigoplus_{i,k} p_{n,i,k}$. Since A_n is left artinian, we can consider the dimension type of $A_n e_{i,k}$, $\underline{\dim}\, A_n e_{i,k}$, which is an element of $\mathbb{Z}^l$ $(l = \sum_{i=1}^{m} l(i))$. The Cartan matrix $\mathrm{C}(A_n)$ of A_n is the matrix with $\mathrm{C}_i(A_n)$ in the ith row block, where for $1 \leqslant i \leqslant m$

$$\mathrm{C}_i(A_n) = \begin{pmatrix} \underline{\dim}\, A_n e_{i,1} \\ \vdots \\ \underline{\dim}\, A_n e_{i,l(i)} \end{pmatrix}.$$

PROPOSITION 4.1. $\det C(A_n) = \det C(A)$.

Proof. We shall prove that $\det C(A_n) = \det C(A_{n-1})$. Consider the following exact sequences of A_n-modules:

$$0 \to \operatorname{Ker} p_{n,i,k} \to A_n e_{i,k} \to A_{n-1} e_{i,k} \to 0.$$

Then, $\underline{\dim}\, A_n e_{i,k} = \underline{\dim}\, A_{n-1} e_{i,k} + \underline{\dim} \operatorname{Ker} p_{n,i,k}$. Since there are A_n-isomorphism

$$p_{n-1,1}^{\alpha_{n,m}} \simeq \bigoplus_{k=1}^{l(m)} \operatorname{Ker} p_{n,m,k}, \qquad P_{n,i+1}^{\alpha_{n,i}} \simeq \bigoplus_{k=1}^{l(i)} \operatorname{Ker} p_{n,i,k},$$

for $1 \leqslant i < m$, it holds that

(i) $$\underline{\dim}\, A_n e_{m,k} \in \underline{\dim}\, A_{n-1} e_{m,k} + \sum_{j=1}^{l(1)} \mathbb{Z}\, \underline{\dim}\, A_{n-1} e_{1,j},$$

(ii) $$\underline{\dim}\, A_n e_{i,k} \in \underline{\dim}\, A_{n-1} e_{i,k} + \sum_{j=1}^{l(i+1)} \mathbb{Z}\, \underline{\dim}\, A_n e_{i+1,j}, \qquad 1 \leqslant i < m.$$

From (i) above, we know that the kth row of $C_m(A_n)$ is obtained by adding multiples of rows of $C_1(A_{n-1})$ by scalars to the kth row of $C_m(A_{n-1})$. Hence, by elementary row operations, we have the following matrix $C^1(A_n)$:

$$C^1(A_n) := \begin{pmatrix} C_1(A_{n-1}) \\ \vdots \\ C_{m-1}(A_{n-1}) \\ C_m(A_n) \end{pmatrix}.$$

Since, from (ii), the kth row of $C_{m-1}(A_n)$ is also obtained by adding some scalar multiples of rows of $C_m(A_n)$ to the kth row of $C_{m-1}(A_{n-1})$, we have the following matrix by elementary row operations:

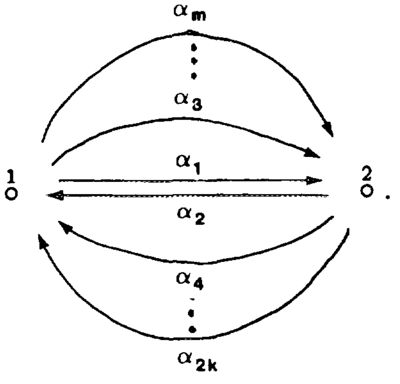
$$C^2(A_n) := \begin{pmatrix} C_1(A_{n-1}) \\ \vdots \\ C_{m-2}(A_{n-1}) \\ C_{m-1}(A_n) \\ C_m(A_n) \end{pmatrix}.$$

By repeating this method starting from $C_m(A_{n-1})$, we finally have the matrix $C^m(A_n)$ which is equal to $C(A_n)$. Thus we conclude that $\det C(A_n) = \det C(A_{n-1})$.

5. Examples

Let k be an algebraically closed field. All algebras in this section are k-algebras.

5.1. Let Λ_n be the algebra defined by the following quiver with n arrows and relations: $\alpha_j\alpha_i = 0$ for $j \leqslant i$, where $n = 2k$ if $m = 2k-1$ for $k \geqslant 1$ or $n = 2k+1$ if $m = 2k+1$ for $k \geqslant 0$:



Let Λ_0 be the semi-simple algebra whose quiver has two vertices $\overset{1}{\circ}$ and $\overset{2}{\circ}$ but without arrows. Those algebras appear in [G].

Now let $f_n: \Lambda_n \to \Lambda_{n-2}$ $(n \geqslant 2)$ be the algebra-morphism such that $f_n(\alpha_1) = 0$, $f_n(\alpha_2) = 0$ and $f_n(\alpha_{i+2}) = \alpha_i$ for $i > 0$, by which Λ_{n-2} is considered as a Λ_n-module. Let e_1, e_2 be the idempotents corresponding to the vertices $\overset{1}{\circ}$, $\overset{2}{\circ}$, so that $\alpha_{2i} = e_1\alpha_{2i}e_2$, $\alpha_{2i-1} = e_2\alpha_{2i-1}e_1$, and

$$\operatorname{rad} \Lambda_n e_1 = \bigoplus_{i \geqslant 1} R_{n,\,2i-1}, \qquad \operatorname{rad} \Lambda_n e_2 = \bigoplus_{i \geqslant 1} R_{n,\,2i},$$

where $R_{n,j}$ are k-spaces spanned by the paths $\{\cdots\alpha_{l_3}\alpha_{l_2}\alpha_{l_1} \mid j = l_1 < l_2 < l_3 < \cdots\}$, which are left ideals of Λ_n. Let $h_n: \Lambda_{n-2}e_1 \to \Lambda_n e_1 : \alpha_i \mapsto \alpha_{i+2}$. Then we have the following exact sequences of Λ_n-modules:

$$0 \longrightarrow \Lambda_{n-2}e_1 \xrightarrow{g_{n,2}} \Lambda_n e_2 \xrightarrow{f_{n,2}} \Lambda_{n-2}e_2 \longrightarrow 0$$

$$0 \longrightarrow \Lambda_n e_2 \xrightarrow{g_{n,1}} \Lambda_n e_1 \xrightarrow{f_{n,1}} \Lambda_{n-2}e_1 \longrightarrow 0.$$

Here $f_{n,1}$, $f_{n,2}$ are restrictions of f_n, and

$$g_{n,2}(\lambda e_1) = h_n(\lambda e_1)\,\alpha_2 \quad \text{for } \lambda \in \Lambda_{n-2}, \qquad g_{n,1}(\lambda e_2) = \lambda e_2 \alpha_1 \quad \text{for } \lambda \in \Lambda_n.$$

Moreover, the canonical monomorphisms

$$g'_{n,2}: A_{n-2}e_1 \to \text{rad } A_n e_2,$$
$$g'_{n,1}: A_n e_2 \to \text{rad } A_n e_1,$$

are splittable, because $g_{n,2}(A_{n-2}e_1) = R_{n,2}$ and $g_{n,1}(A_n e_2) = R_{n,1}$. Under those preparations, we can now show global dimensions of the algebras A_n.

$$\text{(Green)} \qquad \text{gl dim } A_n = n.$$

Proof. (1) Let $A_n = A_{2n}$ $(n \geqslant 0)$. Then, by the change of notations that $P_{n,i} = A_n e_i$ $(i = 1, 2)$, $p_n = f_{2n}: A_n \to A_{n-1}$, $q_{n,1} = g_{2n,1}$, and $q_{n,2} = g_{2n,2}$, we observed above that the family $\{A_n, p_n, m = 2\}$ satisfies conditions (I), (II). It therefore follows from Corollary 3.1 that gl dim $A_n = 2n$.

(2) Let $A_n = A_{2n+1}$ $(n \geqslant 0)$. Then, by the data

$$P_{n,i} = A_n e_i \quad (i = 1, 2), \qquad p_n = f_{2n+1}: A_n \to A_{n-1},$$
$$q_{n,1} = g_{2n+1,1} \qquad \text{and} \qquad q_{n,2} = g_{2n+1,2},$$

the family $\{A_n, p_n, m = 2\}$ satisfies (I) and (II). Hence, by Proposition 3.2, we know that gl dim $A_n = 2n + 1$.

Remark. If we restrict conditions (I) and (II) to the case where $m = 2$, then the arguments in the earlier sections will be very simplified so that it will imply a simple proof for Green's example (see [SS]).

5.2. As for the inequalities in Theorem 3.1 for the case where $m = 2$, i.e., in Corollary 3.1, we can consider the following four cases:

(a) $2n < n + \text{gl dim } A = \text{gl dim } A_n < 2n + \text{gl dim } A$,

(b) $2n = n + \text{gl dim } A < \text{gl dim } A_n < 2n + \text{gl dim } A$,

(c) $n + \text{gl dim } A < 2n = \text{gl dim } A_n < 2n + \text{gl dim } A$,

(d) $\max\{2n, n + \text{gl dim } A\} \leqslant \text{gl dim } A_n = 2n + \text{gl dim } A$.

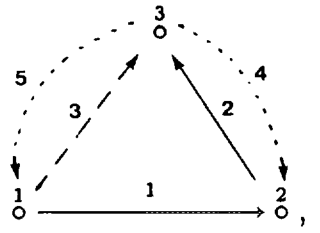
We shall show by examples that all the cases happen. We use the notations given in Section 1. Arrows of quivers are denoted by $\rightarrow$, $\dashrightarrow$ or $\cdot\cdot\rightarrow$, and the numbers above arrows denote subscripts of arrows (e.g., $u \xrightarrow{k} v$ stands for an arrow $\alpha_k: u \to v$). *Relations in quivers are always given as* $\alpha_j \alpha_i = 0$ *for* $j > i$.

Let A be the algebra defined by the following quiver with relations

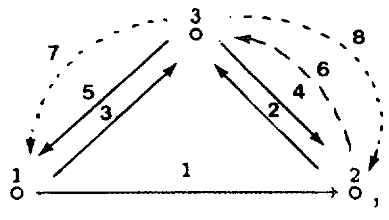
$$A: \overset{1}{\circ} \xrightarrow{1} \overset{2}{\circ} \xrightarrow{2} \overset{3}{\circ}, \qquad \alpha_2 \alpha_1 = 0,$$

and let A_i $(i = 1, 2, 3)$ be the following algebras with relations

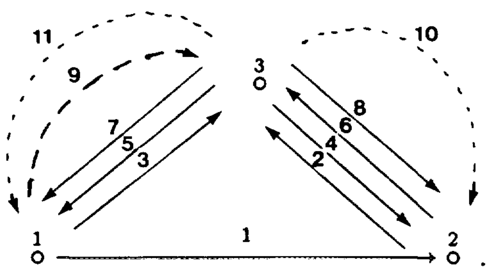
A_1 :



A_2 :



A_3 :



Now consider the following decompositions for $i = 1, 2, 3$

$$P_{i,1} = A_i e_3, \qquad P_{i,2} = A_i e_1 \oplus A_i e_2, \qquad \text{and} \qquad \alpha_{i,1} = \alpha_{i,2} = 1.$$

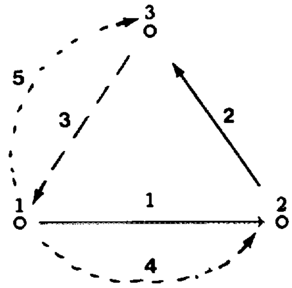
Then, in condition (I), the arrows denoted by ——→ correspond to the embedding $q_{i,2}$, and the arrows · ·→ to $q_{i,1}$. By defining $p_{i,1}$, $p_{i,2}$ naturally, it is easily seen that the family of algebras satisfies conditions (I), (II). Moreover, gl dim $A_0 = 2$, gl dim $A_1 = 3$, gl dim $A_2 = 5$, and gl dim $A_3 = 6$, so that A_1, A_2, and A_3 satisfy the above inequalities (a), (b), and (c), respectively.

For (d), consider another decomposition that

$$P_{0,1} = Ae_1, \qquad P_{0,2} = Ae_2 \oplus Ae_3,$$

and let A_1 be the following algebra with relations and take the decomposition that $P_{1,1} = A_1e_1$, $P_{1,2} = A_1e_2 \oplus A_1e_3$;

A_1 :



Here α_3 corresponds to the embedding $q_{1,2}$, and α_4, α_5 correspond to $q_{1,1}$. Then the family $\{A_0, A_1, p_1, m=2\}$ satisfies (I) and (II), and gl dim $A_1 = 4 = 2n + \text{gl dim } A$. Thus the last inequality (d) holds.

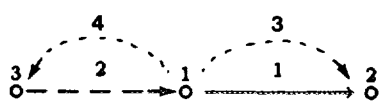
5.3. Two algebras, denoted by A_1, in 5.2 have the different global dimensions. This shows that gl dim A_n depends on the decomposition of A_0: $A_0 = \bigoplus_{i=1}^m P_{0,i}$. The next example shows that gl dim A_n depends on the embedding $q_{n,m}$, too.

Let A be the algebra defined by the quiver

$$A: \overset{1}{\circ} \xrightarrow{\ 1\ } \overset{2}{\circ} \quad \overset{3}{\circ},$$

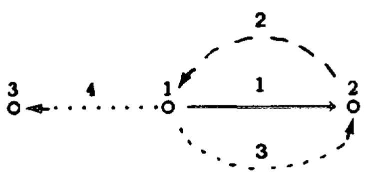
and $P_{0,1} = Ae_1$, $P_{0,2} = Ae_2 \oplus Ae_3$.

First, consider the algebra A_1 such that

A_1  $\alpha_j\alpha_i = 0$ *for* $j > i$.

Let $q_{1,2}$ and $q_{1,1}$ be the canonical embeddings induced from α_2 (i.e., the multiplication of α_2) and from α_3, α_4, respectively. Then gl dim $A_1 = 2$ and the Loewy length $\mathrm{L}(A_1) = 3$.

Next, let A_1 be the algebra such that

A_1  $\alpha_j\alpha_i = 0$ *for* $j > i$, and let $q_{1,2}$ and $q_{1,1}$ correspond to α_2 and α_3, α_4, respectively. Then $\operatorname{gl\,dim} A_1 = 3$ and $L(A_1) = 4$.

5.4. The second algebra A_1 in 5.2, namely, the algebra with $\operatorname{gl\,dim} A_1 = 4$, shows that the condition in Proposition 3.2 that $P_{0,m}$ is simple cannot be omitted.

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