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Endomorphisms of Modules Over Semi-Prime Rings

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1. It has been shown by Feller and Swokowski [*1*] that if R is an integral domain which has a right and left quotient ring, then the endomorphism ring $E(M)$ of a finitely generated torsion-free R-module M is a prime ring which in turn has a right and left quotient ring. The purpose of this note is to generalize the above result to the situation in which R is a semi-prime ring. We also prove some simple results for the more general case of a semi-prime ring which has only a right quotient ring.

2. DEFINITION. *Let R be a ring. An element $c \in R$ is regular if c is neither a left nor a right zero-divisor. We shall say that an R-module M is torsion-free if no nonzero element of M is annihilated by a regular element of R.*

Now let R be a ring with regular elements. An extension ring Q of R is said to be a *right quotient ring* of R if (a) every regular element of R has an inverse in Q and (b) every element of Q has the form ac^{-1} with $a, c \in R$ and c regular. It is well known that R has a right quotient ring if and only if given $a, c \in R$ with c regular, there exist $a_1, c_1 \in R$ with c_1 regular such that $ac_1 = ca_1$. Left quotient rings are defined similarly.

Suppose R is a ring which has a right quotient ring Q, and let M be a torsion-free right R-module. Then M has a quotient Q-module MQ. The construction of MQ from M is precisely analogous to the construction of Q from R, and the elements of MQ have the form mc^{-1} with $m \in M$ and c a regular element of R. The following theorem is due to Feller and Swokowski [*1*] if R is an integral domain. However, only trivial modifications to their proof are needed to establish the more general result.

THEOREM 1. *Let R be a ring with a right and left quotient ring Q. Let M be a finitely generated torsion-free right R-module such that MQ is Q-free and every regular element of $E(MQ)$ is invertible. Then $E(M)$ has a right and left quotient ring isomorphic to $E(MQ)$.*

We refer to Goldie [*3*] for the definition of the dimension of a module and a uniform module. A ring R is *semi-prime* if R has no nonzero nilpotent right or left ideals, and R is *prime* if the zero ideal is a prime ideal of R. It has been shown by Goldie [3] that a ring R has a right quotient ring Q which is semi-simple artinian if and only if R is a finite-dimensional semi-prime ring with maximum condition on annihilator right ideals. We then say that R is a semi-prime ring with right quotient conditions. Further, Q is simple if and only if R is prime; we then say that R is a prime ring with right quotient conditions. An integral domain which has a right quotient division ring is called a *right Öre domain*. We need the following lemma

LEMMA 1. *Let R be a semi-prime ring with identity and right quotient conditions, and let $e \neq 0$ be an idempotent in R. Then $e R e$ is a semi-prime ring with right quotient conditions.*

Proof. Let N be a right ideal of $e R e$, and denote by NR the right ideal generated by N in R. Then

$$NR = NRe + NR(1 - e) = N + NR(1 - e).$$

Thus for any positive integer α,

$$(NR)^{\alpha} = N^{\alpha} + N^{\alpha}R(1 - e).$$

Hence, if N is nilpotent so is NR, and then $NR = 0$, $N = 0$, and eRe is semi-prime. Let S be a subset of eRe, and denote by $\rho(S)$, $r(S)$ the right annihilators of S in eRe, R, respectively. If T is a subset of eRe with $T \supseteq S$ and $r(T) = r(S)$, then

$$\rho(T) = eRe \cap r(T) = eRe \cap r(S) = \rho(S).$$

Since the maximum condition holds for annihilator right ideals of R, we see that the maximum condition holds for annihilator right ideals of eRe. Now suppose that $I_1 \oplus \cdots \oplus I_n$ is a direct sum of right ideals of eRe, and put

$$K = (I_1R + \cdots + I_{m-1}R) \cap I_mR \text{ for } m \leqslant n.$$

Then $Ke = 0$, so that $K^2 = (eK)^2 = 0$. Thus $K = 0$ and the sum $I_1R + \cdots + I_nR$ is direct. Since R is finite-dimensional, it follows that eRe is finite-dimensional, and we have shown that eRe is a semi-prime ring with right quotient conditions.

THEOREM 2. *Let R be a semi-prime ring with right and left quotient conditions, and suppose M is a finitely generated torsion-free R-module. Then the endomorphism ring $E(M)$ is also a semi-prime ring with right and left quotient conditions. If R is a prime ring, so is $E(M)$.*

Proof. Let Q be the quotient ring of R. Then Q is semi-simple artinian, and hence there is a finitely generated Q-module N^* such that $MQ \oplus N^*$

is a free Q-module. Choose a finitely generated torsion-free R-module N such that $N^* = NQ$. Then by Theorem 1, $E(M \oplus N)$ has a right and left quotient ring isomorphic to $E(MQ \oplus NQ)$. Now $E(MQ \oplus NQ)$ is semi-simple artinian, so $E(M \oplus N)$ is a semi-prime ring with right and left quotient conditions. Let e be the projection $M \oplus N \to M$ in $E(M \oplus N)$. Then

$$eE(M \oplus N)e \cong E(M),$$

and hence by Lemma 1 and its left-right dual, $E(M)$ is a semi-prime ring with right and left quotient conditions. If R is prime, then Q is simple, so $E(MQ \oplus NQ)$ is simple. Hence $E(M \oplus N)$ is a prime ring, and we can show as in Lemma 1 that

$$E(M) \cong eE(M \oplus N)e$$

is a prime ring.

3. The hypothesis that R has a left quotient ring is now abandoned, and we make the following definition.

DEFINITION. *Let R be a ring and M, N be R-modules. M and N are said to be subisomorphic if there exist R-isomorphisms $M \to N$ and $N \to M$. Two rings, S, T are said to be subisomorphic if there exist ring isomorphisms $S \to T$ and $T \to S$.*

THEOREM 3. *Let R be a ring with a right quotient ring Q having minimum condition on right ideals. Let M, N be subisomorphic torsion-free right R-modules, suppose MQ, NQ are finitely generated Q-modules, and suppose $E(N)$ has a right (left) quotient ring S. Then $E(M)$ has a right (left) quotient ring T, and S, T are subisomorphic as rings.*

Proof. Let $f : M \to N$ and $g : N \to M$ be R-isomorphisms. The descending chain condition for submodules holds in MQ and NQ, so f has an inverse in $\mathrm{Hom}(NQ, MQ)$ and g an inverse in $\mathrm{Hom}(MQ, NQ)$. Now let $a, c \in E(M)$ with c regular. Then fag and fcg belong to $E(N)$. Moreover, fcg is regular, because $xfcg = 0$ with $x \in E(N)$ gives $xfc = xfcgg^{-1} = 0$, and then we have $xf = 0$, so that $x = xff^{-1} = 0$. Similarly $fcgx = 0$ gives $x = 0$. Now $E(N)$ has a right quotient ring, so there exist elements a_1, c_1 in $E(N)$ with c_1 regular, such that $fagc_1 = fcga_1$. We then have $agc_1 = cga_1$, and hence $agc_1f = cga_1f$. As before, we find that gc_1f is regular in $E(M)$ so $E(M)$ has a right quotient ring T. The function $T \to S$ given by $ac^{-1} \to fag(fcg)^{-1}$ is an isomorphism because $fag(fcg)^{-1} = fac^{-1}f^{-1}$. Similarly, we can write down an isomorphism $S \to T$, so S and T are subisomorphic.

LEMMA 2. *Let S, T be subisomorphic rings and suppose that S is semi-simple artinian and that every regular element of T is invertible. Then T is semi-simple artinian.*

Proof. We may suppose that $S \supseteq T \supseteq S'$, where $S \cong S'$. Suppose S' is a direct sum $e_1S' \oplus \cdots \oplus e_nS'$ of minimal right ideals, where $e_1, \cdots e_n$ are orthogonal idempotents whose sum is 1. Then also

$$T = e_1T \oplus \cdots \oplus e_nT$$

and

$$S = e_1S \oplus \cdots \oplus e_nS.$$

Since $S \cong S'$, it follows that the ideals $e_\alpha S$ are minimal also. Hence either $e_iS \cong e_jS$ or $e_iSe_j = e_jSe_i = 0$. Therefore either $e_iT \cong e_jT$ or $e_iTe_j = e_jTe_i = 0$. In this latter case we have

$$\text{Hom}(e_iT, e_jT) = \text{Hom}(e_jT, e_iT) = 0.$$

It is therefore sufficient to show that each e_iTe_i is a division ring, for then T is a direct sum of complete matrix rings over division rings. Let $t \in T$ with $e_1te_1 \neq 0$. Now e_1Se_1 has no zero divisors, so $e_1te_1 + e_2 \cdots + e_n$ is a regular element of T. Hence there exists an element $t' \in T$ such that $e_1te_1t' = e_1$, and then $(e_1te_1)(e_1t'e_1) = e_1$. Similarly e_iTe_i is a division ring for each i, and this completes the proof.

LEMMA 3. *Let R be a semi-prime ring with right quotient conditions, and suppose U, V are uniform right ideals of R. Then either U contains a copy of V and vice versa, or else* $\text{Hom}(U, V) = \text{Hom}(V, U) = 0$.

Proof. If $uv = 0$ with $u \in U$ and v a nonzero element of V, then $uV = 0$ (see Lemma 3.3 of [3]). Hence either $UV = 0$ or else there exists $u \in U$ with $uV \cong V$. However, if $UV = 0$, then $(VU)^2 = V(UV)U = 0$, so that $VU = 0$. Thus either U contains a copy of V and vice versa, or $UV = VU = 0$. Suppose $UV = 0$ and let $f: U \to V$ be an R-homomorphism. Then $f(U)V = f(UV) = 0$, so $[f(U)]^2 = 0$, and $f(U) = 0$. Hence $\text{Hom}(U, V) = 0$, and similarly $\text{Hom}(V, U) = 0$.

We denote by $r(x)$ the right annihilator of the element x. Then we have

LEMMA 4. *Let R be a semi-prime ring with right quotient conditions, and let I be a nonzero right ideal of R. Then there is a right ideal J in I and an element x in J such that E(J) is a direct sum of complete matrix rings over various right Öre domains, and* $I \cap r(x) = 0$.

Proof. Let U_1 be a uniform right ideal in I. By Lemma 3, there is an element u_1 in U_1 such that $U_1 \cap r(u_1) = 0$. If $I \cap r(u_1) = 0$, take $J = U_1$, $x = u_1$. If $I \cap r(u_1) \neq 0$, let U_2 be a uniform right ideal in $I \cap r(u_1)$, and if possible, choose U_2 to be isomorphic to U_1. If this is not possible, then however U_2 is chosen, $\text{Hom}(U_1, U_2) = \text{Hom}(U_2, U_1) = 0$, by Lemma 3. We have $U_1 \cap U_2 = 0$; let u_2 be an element of U_2 with $U_2 \cap r(u_2) = 0$.

Now $u_1U_2 = 0$, and we therefore obtain

$$(U_1 \oplus U_2) \cap r(u_1 + u_2) = 0.$$

If $I \cap r(u_1 + u_2) = 0$, take $J = U_1 \oplus U_2$, $x = u_1 + u_2$. Otherwise, let U_3 be a uniform right ideal in $I \cap r(u_1 + u_2)$ with $U_3 \simeq U_2$ if possible, and so on. Now R is a finite-dimensional ring, so the length of the direct sum $U_1 \oplus U_2 \oplus \cdots$ is finite. Hence for some integer k, we have $I \cap r(u_1 + \cdots + u_k) = 0$, $I \supset U_1 \oplus \cdots \oplus U_k$. Take $J = U_1 \oplus \cdots \oplus U_k$ and $x = u_1 + \cdots + u_k$. Now by construction $J = L_1 \oplus \cdots \oplus L_n$, where each L_i is a direct sum of isomorphic uniform right ideals and $\mathrm{Hom}(L_i, L_j) = 0$ for all $j \neq i$. Hence $E(J)$ is a direct sum $E(L_1) \oplus \cdots \oplus E(L_n)$ of two-sided ideals. Also $E(L_i)$ is isomorphic to a complete matrix ring over $E(U_\alpha)$ for some α. However, $E(U_\alpha)$ is a right Öre domain (see [*2*]), so $E(J)$ is a direct sum of complete matrix rings over right Öre domains.

Theorem 4. *Let R be a semi-prime ring with right quotient conditions, and let I be a nonzero right ideal in R. Then E(I) is a semi-prime ring with right quotient conditions.*

Proof. Choose the right ideal J and the element x as in Lemma 4. Since $I \cap r(x) = 0$, it follows that $xI \simeq I$. Now x belongs to J, so xI is contained in J and hence I and J are subisomorphic. $E(J)$ is a direct sum of complete matrix rings over right Öre domains, so in particular $E(J)$ is a semi-prime ring with right quotient conditions. Thus by Theorem 3 and Lemma 2, $E(I)$ has a right quotient ring which is semi-simple artinian. Therefore $E(I)$ is a semi-prime sing with right quotient conditions.

4. Theorem 5. *Let U be a uniform right ideal in the semi-prime ring R, where R has right and left quotient conditions. Then E(U) is a right and left Öre domain.*

Proof. If U is finitely generated, the result follows by Theorem 2. If U is not finitely generated, U contains a finitely generated ideal V (indeed a principal ideal), and by Lemma 3, U and V are subisomorphic. Then by Theorem 3, $E(U)$ has a right and left quotient ring.

This result is known if R is a prime ring. See Goldie [*3*].

Theorem 6. *Let R be a semi-prime ring with right and left quotient conditions, and suppose I is a nonzero right ideal in R. Then E(I) is a semi-prime ring with right and left quotient conditions.*

Proof. As in Theorem 4, but making use of Theorem 5.

5. We now show by means of an example that the hypothesis of left

quotient conditions in Theorem 2 cannot be removed, even if we abandon the requirement that $E(M)$ have left quotient conditions.

Let R be a right Öre domain which is not a left Öre domain, and let Q be the right quotient ring of R. Choose nonzero elements a, b of R such that $Ra \cap Rb = 0$, and put $U = a^{-1}R + b^{-1}R$. Then $U \subset Q$, so U is a uniform torsion-free R-module, and of course U is finitely generated. Clearly $\mathrm{Hom}(R, U) \neq 0$; let $f \in \mathrm{Hom}(U, R)$. Then $f(a^{-1})a = f(b^{-1})b$. Since $Ra \cap Rb = 0$, this means that $f = 0$ and hence $\mathrm{Hom}(U, R) = 0$. Therefore if we put $M = U \oplus R$, it follows that $g(U) \subseteq U$ for any endomorphism g of M. Let I be the set of all endomorphisms of M which map the second summand R into U and U into zero. Then if $h \in I$ and $g \in E(M)$, $gh(U) = 0$. Also the image of the second summand R under gh is in $g(U)$ and hence in U. Clearly $I^2 = 0$, so I is a nonzero nilpotent ideal in $E(M)$.

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