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Asymptotic behavior of the number of solutions for non-Archimedean Diophantine approximations with restricted denominators ☆

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Abstract

We consider metric results for the asymptotic behavior of the number of solutions of Diophantine approximation inequalities with restricted denominators for Laurent formal power series with coefficients in a finite field. We especially consider approximations by rational functions whose denominators are powers of irreducible polynomials, and study the strong law of large numbers for the number of solutions of the inequalities under consideration.

Keywords: Laurent formal power series; Metric Diophantine approximation; Strong law of large numbers

1. Introduction

The metric theory of Diophantine approximation and, particularly, the asymptotic behavior of the number of solutions of Diophantine approximation inequalities has given rise to substantial literature in the real case, see e.g. [5,10]. Such results can also be naturally extended to the case

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of Laurent formal power series with coefficients in a finite field. Let us quote e.g. [6] and [8] who discuss the strong law of large numbers in metric theory of Diophantine approximation in positive characteristics. In the present paper, we consider specific inequalities with restricted denominators (powers of irreducible polynomials) with an approximation function which does not only depend on the degree of the denominator.

As usual, let $\mathbb{F}_q$ be a finite field of cardinality q, and we denote by

$$\mathbb{F}[X], \quad \mathbb{F}(X), \quad \mathbb{F}\big(\big(X^{-1}\big)\big), \quad \mathbb{L}$$

the set of polynomials (with $\mathbb{F}_q$-coefficients), the set of rational functions, the set of Laurent formal power series, and the set of Laurent formal power series of negative degree, respectively. Here we define the degree of $f \neq 0$

$$f = a_n X^n + a_{n-1} X^{n-1} + \cdots$$

with $a_n \neq 0$ by $\deg f = n$. We define as usually $\deg 0 = -\infty$. We consider the topology on $\mathbb{L}$ induced by the (ultra-)metric $d(f,g) = |f-g|$ for $f, g \in \mathbb{L}$, where $|f| := q^{\deg f}$. We denote by m the Haar probability measure on $\mathbb{L}$. We recall that measure of cylinders is just the product measure. Indeed, for any $\hat{a}_1, \ldots, \hat{a}_k \in \mathbb{F}_q$,

$$m\big\{f \in \mathbb{L}\colon\ f = a_1 X^{-1} + \cdots + a_k X^{-k} + \cdots,\ a_1 = \hat{a}_1,\ \ldots,\ a_k = \hat{a}_k\big\} = \frac{1}{q^k}.$$

We consider for a given formal power series $f \in \mathbb{L}$ the solutions $\frac{P}{Q}$ with P, Q polynomials with coefficients in $\mathbb{F}_q$ of

$$\left| f - \frac{P}{Q} \right| < \frac{\Psi(Q)}{|Q|}, \quad P, Q\text{: coprime},\ Q\text{: monic.} \tag{1}$$

In the case where the approximation function Ψ depends only on the degree of Q, i.e., if Ψ has form $\Psi(Q) = \frac{1}{q^{n+l_n}}$ if $\deg Q = n$, with l_n being a nonnegative integer, then the strong law of large numbers holds whenever $\sum \frac{1}{q^{l_n}} = \infty$. Moreover, some limit theorems can be obtained under a mild condition on l_n, see [4,6,2,3]. Note that although it is not explicitly stated as such in [6], the proof of the Khintchine type theorem stated in [6] implies results on the asymptotic behavior of the number of solutions for non-Archimedean Diophantine approximations. It is furthermore proved in [8] that the strong law of large numbers also holds even if we do not assume the coprimeness of P and Q.

However, it does not seem to be very easy to get the strong law of large numbers for Ψ not depending only on the degree of Q. Indeed, the only known result in the general case (Ψ not depending only on $\deg Q$) is a Duffin–Schaeffer type theorem, i.e., a generalized Khintchine type theorem (see [6]). In this paper, we thus consider some special cases of the approximation function Ψ:

(i) Ψ is positive only when Q is irreducible, otherwise it takes value 0,
(ii) Ψ is positive only when Q is the tth power, with t being fixed, of a single monic irreducible polynomial, otherwise it takes value 0,

(iii) Ψ is positive when Q is some power of a single monic irreducible polynomial, otherwise it takes value 0.

More precisely, we consider the three following inequalities for coprime polynomials P and Q:

(i) $$\left|f-\frac{P}{Q}\right|<\frac{1}{q^{2n+l_Q}}, \quad \deg Q=n, \quad Q\colon \text{monic and irreducible},$$

where the sequence of nonnegative integers (l_Q) is given (see Section 2);

(ii) $$\left|f-\frac{P}{Q}\right|<\frac{1}{q^{(t+1)n+l_{Q_1}}}, \quad \deg Q_1=n, \quad Q_1\colon \text{monic and irreducible}, \quad Q=Q_1^t,$$

where t is a fixed positive integer and the sequence of nonnegative integers (l_{Q_1}) is given (see Section 3);

(iii) $$\left|f-\frac{P}{Q}\right|<\frac{1}{q^{(t+1)n+l_{Q_1}+l_t}}, \quad \deg Q_1=n, \quad Q_1\colon \text{monic and irreducible}, \quad Q=Q_1^t,$$

for some positive integer t, where both sequences of nonnegative integers (l_{Q_1}) and $(l_t\colon t\geqslant 1)$ are given and $\sum_{t\geqslant 1}\frac{1}{q^{l_t}}$ is assumed to be a convergent series (see Section 4).

Obviously, (ii) is a special case of (iii) and (i) is a special case of (ii). However, we estimate the asymptotic behavior of the number of solutions of these inequalities step by step, first for clarity, and secondly because (i) is interesting as a Diophantine approximation problem: this corresponds to the approximation of irrational numbers by rational numbers with prime denominators. Note that (ii) is somehow a natural generalization of (i). On the other hand, (iii) illustrates the difficulty of finding a sufficient condition for Ψ such that the strong law of large numbers holds: indeed, we have to add as an extra hypothesis that $\sum_{t\geqslant 1}\frac{1}{q^{l_t}}$ is assumed to be a convergent series.

The main tool of our proofs will be the following lemma, which is also used in [8]. We recall here that the notation $X\ll Y$ is equivalent to the notation $X=O(Y)$.

Lemma 1.1. *(See Sprindžuk [10, p. 45].) Let $(\xi_n(\omega)\colon n\geqslant 1)$ be a sequence of random variables defined on a probability space $(\Omega,\mathcal{B},P)$. Moreover let $(\eta_n\colon n\geqslant 1)$ and $(\hat{\eta}_n\colon n\geqslant 1)$ be sequences of real numbers such that*

(i) $0\leqslant\eta_n\leqslant\hat{\eta}_n\leqslant 1$ *for all* $n\geqslant 1$,
(ii) *for any positive integers* $N_1<N_2$

$$\int_\Omega\left(\sum_{n=N_1}^{N_2}\xi_n(\omega)-\eta_n\right)^2 dP\ll\sum_{n=N_1}^{N_2}\hat{\eta}_n.$$

Then, one has

$$\sum_{n=1}^{N}\xi_n(\omega)=\sum_{n=1}^{N}\eta_n+O\bigl(\Xi(N)^{1/2}\log^{\frac{3+\varepsilon}{2}}\Xi(N)\bigr)\quad\text{for } P\text{-a.e.},$$

where $\varepsilon>0$ is arbitrary and $\Xi(N)=\sum_{n=1}^{N}\hat{\eta}_n$.

This lemma can be considered as a refinement of Khintchine's theorem. The idea of this lemma (called Schmidt's method in [5]) was used in metric theory of classical Diophantine approximation in the 1950s (see e.g. [9]). As applications of Schmidt's method in the real case, one obtains asymptotic formulas for the number of solutions of Diophantine inequalities for restricted sets of denominators such as the set of prime numbers (see Theorem 18 in [10]) or sets of positive lower density (see e.g. Chapter 4 in [5]).

Note that we may apply this lemma if the approximation function Ξ is large. However, in such a case, the error term might be larger than the main term. This is one of the reasons we have chosen an approximation function Ξ of type (ii) and (iii) on the right-hand side of inequality (1).

In all that follows, the denominators that we consider are assumed to be monic.

2. Metric Diophantine approximation by irreducible polynomial denominators

In this section, we consider an inequality of type (1) with restricted denominators that are supposed to be monic irreducible polynomials and a function $\Psi : \mathbb{F}_q[X] \to \mathbb{R}$ of the form $Q \mapsto (|Q| \cdot q^{l_Q})^{-1}$, where l_Q takes nonnegative integer values for Q monic irreducible, and infinite value otherwise. We thus consider the following inequality over $\mathbb{L}$:

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n+l_Q}} \tag{2}$$

where P and Q are coprime, $\deg Q = n$, and l_Q takes infinite value whenever Q is not monic irreducible.

Theorem 2.1. *For almost all $f \in \mathbb{L}$, the number of solutions of* (2) *with* $\deg Q \leqslant N$ *satisfies*

$$\Xi(N) + O\left(\Xi^{1/2}(N) \log^{\frac{3+\varepsilon}{2}} \Xi(N)\right)$$

with

$$\Xi(N) = \sum_{n=1}^{N} \sum_{Q:\ \deg Q = n} \frac{1}{q^{n+l_Q}}$$

for any $\varepsilon > 0$.

Remark 2.2. If $\Xi(N)$ does not diverge at ∞ as $N \to \infty$, then Theorem 2.1 means that there exist at most finitely many solutions for a.e. $f \in \mathbb{L}$.

We first need the following lemma:

Lemma 2.3. *We fix coprime monic polynomials Q and Q' such that $n = \deg Q$ and $m = \deg Q'$. Let l be a nonnegative integer. The number of pairs of non-zero polynomials (P, P') with* $\deg P < \deg Q$ *and* $\deg P' < \deg Q'$ *that satisfy*

$$\left| \frac{P}{Q} - \frac{P'}{Q'} \right| < \frac{1}{q^{m+l}}$$

is less than q^{n-l}.

Proof. Let (P, P') be a such a pair of polynomials. Since

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| = \left|\frac{PQ' - P'Q}{QQ'}\right|$$

we have

$$n + m - \deg(PQ' - P'Q) > m + l$$

and so

$$\deg(PQ' - P'Q) < n - l.$$

If $l \geqslant n$ then there exists no such pair of polynomials (P, P'). Hence, we may assume $n > l$.

Note that the number of non-zero polynomials of degree less than $n - l$ is equal to $q^{n-l} - 1$. Now suppose that there exist two pairs of polynomials (P_1, P_1') and (P_2, P_2') that satisfy

$$P_1 Q' - P_1' Q = P_2 Q' - P_2' Q$$

with

$$\deg P_1, \deg P_2 < \deg Q \quad \text{and} \quad \deg P_1', \deg P_2' < \deg Q'.$$

Since $(Q, Q') = 1$, we deduce from

$$(P_1 - P_2) Q' = \left(P_1' - P_2'\right) Q$$

that we have $P_1 = P_2$ and $P_1' = P_2'$. Thus the number of pairs of non-zero polynomials (P, P') such that

$$\left|\frac{P}{Q} - \frac{P'}{Q'}\right| < \frac{1}{q^{m+l}}$$

is less than q^{n-l}. $\square$

Proof of Theorem 2.1. For Q monic polynomial of $\deg Q = n$ and for P polynomial, we put

$$F_{\frac{P}{Q}} := \left\{ f \in \mathbb{L}\colon \left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n + l_Q}} \right\},$$

$$F_Q := \bigcup_{P\colon 0 \leqslant \deg P < \deg Q,\, \gcd(P,Q)=1} F_{\frac{P}{Q}},$$

and

$$F_n := \bigcup_{Q\colon \text{irr.}, \deg Q = n} F_Q.$$

Let us note that the sets F_Q's are disjoint for a given n. Furthermore, one checks that

$$m(F_{\frac{P}{Q}}) = \frac{1}{q^{2n+l_Q}}, \qquad m(F_Q) = \frac{q^n-1}{q^{2n+l_Q}}, \qquad m(F_n) = \sum_{Q:\ \text{irr., deg } Q=n} \frac{q^n-1}{q^{2n+l_Q}}.$$

Hence

$$m(F_n) \sim \sum_{Q:\ \text{irr., deg } Q=n} \frac{1}{q^{n+l_Q}}.$$

Note that we can give a more precise estimate for $m(F_n)$ but that we do not need more in the present proof.

Let us apply Lemma 1.1 by setting, for all n, $\xi_n := \chi_{F_n}$, the indicator function of the set F_n, and $\eta_n := \hat{\eta}_n := \int \xi_n \, dm$. Condition (i) of Lemma 1.1 is satisfied. Let us consider now condition (ii). It is enough to show

$$\sum_{n=N_1}^{N_2} \sum_{m=N_1}^{N_2} m(F_n \cap F_m) - m(F_n)m(F_m) \ll \sum_{n=N_1}^{N_2} m(F_n)$$

for any positive integers $N_1 < N_2$. For this, it is sufficient to show that

$$\sum_{m=N_1}^{n-1} m(F_n \cap F_m) - m(F_n)m(F_m) \ll m(F_n).$$

Now we see that

$$\begin{aligned}
&\sum_{m=N_1}^{n-1} m(F_n \cap F_m) - m(F_n)m(F_m) \\
&\quad = \sum_{Q:\ \deg Q=n} \sum_{m=N_1}^{n-1} \sum_{Q':\ \deg Q'=m} m(F_Q \cap F_{Q'}) - m(F_Q)m(F_{Q'}) \\
&\quad = \sum_{Q} \sum_{m=N_1}^{n-1} \sum_{Q'} \sum_{P} \sum_{P'} m(F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}}) - m(F_{\frac{P}{Q}})m(F_{\frac{P'}{Q'}}).
\end{aligned}$$

Let us distinguish two cases according to the value of $2n + l_Q$ with respect to that of $2m + l_{Q'}$.

- We first assume that $2n + l_Q \geqslant 2m + l_{Q'}$. Then, $m(F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}}) = \frac{1}{q^{2n+l_Q}}$ whenever $F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}} \neq \emptyset$. In this case, it follows that $|\frac{P}{Q} - \frac{P'}{Q'}| < \frac{1}{q^{2m+l_{Q'}}}$. So by Lemma 2.3, we see that the number of pairs (P, P') with $F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}} \neq \emptyset$ is less than $q^{n-m-l_{Q'}}$. We thus deduce that

$$m(F_Q \cap F_{Q'}) < \frac{q^{n-m-l_{Q'}}}{q^{2n+l_Q}} = \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}}.$$

- If $2n + l_Q < 2m + l_{Q'}$, then we have $m(F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}}) = \frac{1}{q^{2m+l_{Q'}}}$ whenever $F_{\frac{P}{Q}} \cap F_{\frac{P'}{Q'}} \neq \emptyset$ and $|\frac{P}{Q} - \frac{P'}{Q'}| < \frac{1}{q^{2n+l_Q}}$ by the same way.

In either case, we get

$$\begin{aligned}
&\sum_{m=N_1}^{n-1} m(F_n \cap F_m) - m(F_n)m(F_m) \\
&\quad < \sum_{Q} \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}} - \frac{(1-\frac{1}{q^n})}{q^{n+l_Q}} \frac{(1-\frac{1}{q^m})}{q^{m+l_{Q'}}} \\
&\quad < \sum_{Q} \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}} \left(\frac{1}{q^n} + \frac{1}{q^m}\right) \\
&\quad < \sum_{Q} \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \cdot \frac{1}{q^{m+l_{Q'}}} \cdot \frac{2}{q^m}.
\end{aligned}$$

Finally, we estimate

$$\sum_{Q} \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}} \frac{1}{q^m} < \sum_{Q} \frac{1}{q^{n+l_Q}} \sum_{m=N_1}^{n-1} \frac{1}{q^{m+l_{Q'}}}$$

based on the fact that there exist at most q^m polynomials Q', which yields

$$\sum_{Q} \sum_{m=N_1}^{n-1} \sum_{Q'} \frac{1}{q^{n+l_Q}} \frac{1}{q^{m+l_{Q'}}} \frac{1}{q^m} \ll \sum_{Q} \frac{1}{q^{n+l_Q}} \sim m(F_n).$$

Consequently, we get

$$\sum_{m=N_1}^{n-1} m(F_n \cap F_m) - m(F_n)m(F_m) \ll m(F_n),$$

which completes the proof. $\square$

As an application, we now consider the particular case where l_Q vanishes, i.e., l_Q takes zero value if Q is monic irreducible, and l_Q takes infinite value otherwise:

Corollary 2.4. *For almost all $f \in \mathbb{L}$, one has*

$$\begin{aligned}
&\operatorname{Card}\left\{1 \leqslant n \leqslant N \colon \exists Q \text{ irr. with } \deg Q = n \text{ and } \exists P \text{ s.t. } \left|f - \frac{P}{Q}\right| < \frac{1}{q^{2n}}\right\} \\
&\quad = \log N + O\left(\log^{1/2} N \cdot \log^{(3+\varepsilon)/2} \log N\right), \quad \text{for any } \varepsilon > 0.
\end{aligned}$$

Proof. We first note that the number $l(n)$ of monic irreducible polynomials of degree n is equivalent to $\frac{q^n}{n}$. Indeed, it is well known (see e.g. [7]) that

$$l(n) = 1/n \sum_{d|n} \mu(n/d) q^d$$

where μ is the Möbius function. Let $r(n) = \sum_{d|n,\, d<n} \mu(n/d) q^d$. One has $l(n) = \frac{1}{n} q^n + \frac{1}{n} r(n)$. Furthermore, $|r_n| \leqslant \sum_{d=1}^{\lfloor n/2 \rfloor} q^d = q \frac{q^{\lfloor n/2 \rfloor} - 1}{q-1}$. Consequently,

$$|r_n| \leqslant \frac{q^{\lfloor n/2 \rfloor + 1}}{q-1}, \tag{3}$$

and thus $l(n) \sim q^n/n$.

We then deduce Corollary 2.4 from Theorem 2.1 by noting that

$$\Xi(N) = \sum_{n=1}^{N} \sum_{Q\colon \text{ irr., deg } Q = n} \frac{1}{q^n} = \sum_{n=1}^{N} \frac{1}{n} + \sum_{n=1}^{N} \frac{1}{n} \frac{r_n}{q^n},$$

and then, by applying (3). $\square$

We denote by $\frac{P_n}{Q_n}$ the nth convergent of the continued fraction expansion of $f \in \mathbb{L}$. It is well known that if

$$\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2}$$

holds for $\frac{P}{Q}$, then

$$\frac{P}{Q} = \frac{P_n}{Q_n}$$

for some $n \geqslant 0$, see [1] for example. Example 2 in [6] claimed that

$$\text{Card}\left\{ 1 \leqslant n \leqslant N\colon \exists Q \text{ irr. with } \deg Q = n \text{ and } \exists P \text{ s.t. } \left| f - \frac{P}{Q} \right| < \frac{1}{q^{2n}} \right\} = \infty$$

for almost all $f \in \mathbb{L}$. Thus we see that there exist infinitely many convergents with irreducible Q_n's for almost all $f \in \mathbb{L}$. As a by-product of Corollary 2.4, we can obtain the more precise result:

Corollary 2.5. *For almost all $f \in \mathbb{L}$,*

$$\text{Card}\{1 \leqslant n \leqslant N\colon\ Q_n \textit{ irr.}\} \sim \log N$$

where $\frac{P_n}{Q_n}$ stands for the nth convergent of the continued fraction expansion of f.

Proof. For almost all $f \in \mathbb{L}$, we have

$$\deg Q_N \sim \frac{q}{q-1} N,$$

see [1] for example. From Corollary 2.4 and the above criterion on $\frac{P}{Q}$ and $\frac{P_n}{Q_n}$, we have

$$\mathrm{Card}\{1 \leqslant n \leqslant N\colon\ Q_n \text{ irr.}\} \sim \log \deg Q_N \sim \log N$$

for almost all $f \in \mathbb{L}$. □

3. Fixed powers of irreducible denominators

The aim of this section is to extend the set of admissible denominators in (1). We thus fix a positive integer $t \geqslant 2$ and consider denominators Q of the form $Q = Q_1^t$ with monic irreducible Q_1. We first discuss what could be a reasonable inequality (1) with respect to the strong law of large numbers. We thus also fix $k \geqslant 1$ and consider

$$\left| f - \frac{P}{Q} \right| < \frac{1}{q^{(t+k)n}}, \quad Q = Q_1^t, \quad \deg Q_1 = n, \quad Q_1\colon \text{irreducible}, \quad \gcd(P, Q) = 1. \tag{4}$$

Let

$$F_n^{(t,k)} := \left\{ f \in \mathbb{L}\colon\ f \text{ satisfies (4) for some } Q = Q_1^t,\ \deg Q_1 = n \right\}.$$

One has

$$m\left(F_n^{(t,k)}\right) \leqslant \sum_{Q\colon\ Q = Q_1^t,\, \deg Q_1 = n,\, Q_1 \text{ irr.}} m\left(F_Q^{(t,k)}\right) = \sum_Q \sum_P m\left(F_{\frac{P}{Q}}^{(t,k)}\right)$$

where

$$F_{\frac{P}{Q}}^{(t,k)} := \left\{ f \in \mathbb{L}\colon\ \left| f - \frac{P}{Q} \right| < \frac{1}{q^{(t+k)n}} \right\} \quad \text{with } \gcd(P, Q) = 1,$$

$$F_Q^{(t,k)} := \bigcup_{P\colon\ 0 \leqslant \deg P < \deg Q,\, \gcd(P,Q)=1} F_{\frac{P}{Q}}^{(t,k)}.$$

Since Q_1 is monic and $\deg Q_1 = n$, then the number of polynomials Q of the form Q_1^t is less than q^n, and we deduce

$$m\left(F_n^{(t,k)}\right) < \frac{q^n \cdot q^{tn}}{q^{(t+k)n}} = \frac{1}{q^{(k-1)n}}. \tag{5}$$

If $k \geqslant 2$, we see that $\sum m(F_n^{(t,k)}) < \infty$. Thus, by the Borel–Cantelli lemma, there exist at most finitely many solutions of (4) for a.e. f. Note that the bound for $k = 0$ is too large for a Diophantine approximation inequality. On the other hand, if $k = 1$, we see that $\sum m(F_Q^{(t,k)}) = \infty$.

Hence, the only case which is of interest with respect to the strong law of large numbers is the $k = 1$ case that we study below.

We thus consider the following inequality

$$\left|f - \frac{P}{Q}\right| < \frac{1}{q^{(t+1)n+l_{Q_1}}}, \quad Q = Q_1^t, \quad Q_1 \text{ irreducible}, \quad \deg Q_1 = n \tag{6}$$

where P and Q are coprime, t is a fixed positive integer, and l_{Q_1} takes nonnegative integer values if Q_1 is a monic irreducible polynomial, and infinite value otherwise.

Theorem 3.1. *For almost all $f \in \mathbb{L}$, the number of solutions of* (6) *with* $\deg Q_1 \leqslant N$ *is equal to*

$$\Xi(N) + O\big(\Xi^{1/2}(N) \log^{\frac{3+\varepsilon}{2}} \Xi(N)\big), \quad \textit{for any } \varepsilon > 0$$

with

$$\Xi(N) = \sum_{n=1}^{N} \sum_{Q_1:\ Q_1 \text{ irr.}, \deg Q_1 = n} \frac{1}{q^{n+l_{Q_1}}}.$$

Proof. As in the proof of Theorem 2.1, we define for a given t and for $Q = Q_1^t$ with Q_1 monic irreducible of degree n

$$\begin{aligned} E_{\frac{P}{Q}} &= \left\{ f \in \mathbb{L}\colon\ \exists P \text{ s.t. } \left|f - \frac{P}{Q}\right| < \frac{1}{q^{(t+1)n+l_{Q_1}}} \right\}, \\ E_Q &= \bigcup_{P:\ 0 \leqslant \deg P < \deg Q,\ \gcd(P,Q)=1} E_{\frac{P}{Q}}, \\ E_n &= \left\{ f \in \mathbb{L}\colon\ f \text{ satisfies (6) for some } Q_1,\ \deg Q_1 = n \right\}. \end{aligned}$$

The proof of Theorem 2.1 is based on the fact that the F_Q's were disjoint. However, the sets E_Q's may no longer be disjoint even if we fix the degree n (note that the sets $E_{\frac{P}{Q}}$'s remain disjoint for a given Q). Therefore we have to define ξ_n at a different level. We will thus have to change the summation with respect to the index m which now ranges from $N_1 \to n$ instead of $N_1 \to n - 1$. Indeed, the term corresponding to $m = n$ in the following summation will no longer vanish. Furthermore, we will have to distinguish two cases $Q_1 = Q_1'$, and $Q_1 \neq Q_1'$, in order to apply Lemma 2.3 (where Q and Q' are assumed to be coprime).

Let

$$\xi_n := \sum_{Q = Q_1^t,\ \deg Q_1 = n,\ Q_1 \text{ irr.}} \chi_{E_Q}$$

and $\eta_n = \hat{\eta}_n = \int \xi_n \, dm$. One has

$$m(E_{\frac{P}{Q}}) = \frac{1}{q^{(t+1)n+l_{Q_1}}}, \qquad m(E_Q) = \frac{q^{tn} - q^{(t-1)n}}{q^{(t+1)n+l_{Q_1}}},$$

by noting that there are $q^{tn} - q^{(t-1)n}$ polynomials P coprime with $Q = Q_1^t$, and

$$\eta_n = \sum_{Q_1:\ \deg Q_1 = n,\ Q_1\ \text{irr.}} m(E_Q) = \sum_{Q_1:\ \deg Q_1 = n,\ Q_1\ \text{irr.}} \frac{1}{q^{n+l_{Q_1}}}\left(1 - \frac{1}{q^n}\right)$$
$$\sim \sum_{Q_1:\ \deg Q_1 = n,\ Q_1\ \text{irr.}} \frac{1}{q^{n+l_{Q_1}}}.$$

Note that condition (i) in Lemma 1.1 is satisfied. To check condition (ii), we need to estimate

$$\int \left(\sum_{n=N_1}^{N_2} \xi_n - \eta_n\right)^2 dm = \sum_{n,m=N_1}^{N_2} \sum_{\substack{\deg Q_1 = n \\ Q = Q_1^t}} \sum_{\substack{\deg Q_1' = m \\ Q' = Q_1'^{\,t}}} m(E_Q \cap E_{Q'}) - m(E_Q)m(E_{Q'}).$$

It will be sufficient to show that

$$\sum_{m=N_1}^{n} \sum_{\substack{\deg Q_1 = n \\ Q = Q_1^t}} \sum_{\substack{\deg Q_1' = m \\ Q' = Q_1'^{\,t}}} m(E_Q \cap E_{Q'}) - m(E_Q)m(E_{Q'}) \ll \eta_n.$$

Note that we deduce from the fact that the sets $E_{\frac{P}{Q}}$'s are disjoint that

$$\sum_{\substack{\deg Q_1 = n \\ Q = Q_1^t}} \sum_{\substack{\deg Q_1' = m \\ Q' = Q_1'^{t}}} m(E_Q \cap E_{Q'}) - m(E_Q)m(E_{Q'})$$
$$= \sum_{Q_1} \sum_{Q_1'} \sum_{P} \sum_{P'} m(E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}}) - (E_{\frac{P}{Q}})m(E_{\frac{P'}{Q'}}).$$

We have two cases, namely $Q_1 = Q_1'$ and $Q_1 \neq Q_1'$.

For the first case ($Q_1 = Q_1'$), we have

$$\sum_{Q_1} \sum_{Q_1'} \sum_{P} \sum_{P'} m(E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}}) - m(E_{\frac{P}{Q}})m(E_{\frac{P'}{Q'}}) \leqslant \sum_{Q_1} \sum_{P} m(E_{\frac{P}{Q}}) = \sum_{Q_1} m(E_Q) = \eta_n.$$

When $Q_1 \neq Q_1'$, we again distinguish two cases.

- Let us assume that $(t+1)n + l_{Q_1} \geqslant (t+1)m + l_{Q_1'}$. Then, $m(E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}}) = \frac{1}{q^{(t+1)n+l_{Q_1}}}$, whenever $E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}} \neq \emptyset$, and the number of pairs of non-zero polynomials (P, P') such that $E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}} \neq \emptyset$ is less than $q^{tn-m-l_{Q_1'}}$ by Lemma 2.3.
- If $(t+1)n + l_{Q_1} < (t+1)m + l_{Q_1'}$, then one similarly has $m(E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}}) = \frac{1}{q^{(t+1)m+l_{Q_1'}}}$ whenever $E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}} \neq \emptyset$ and the number of pairs (P, P') such that $E_{\frac{P}{Q}} \cap E_{\frac{P'}{Q'}} \neq \emptyset$ is less than $q^{tm-n-l_{Q_1}}$.

We thus deduce that

$$\sum_{\substack{\deg Q_1=n\\ Q=Q_1^t}} \sum_{\substack{\deg Q_1'=m\\ Q'=Q_1'^t}} m(E_Q\cap E_{Q'})-m(E_Q)m(E_{Q'})$$

$$< \sum_{Q_1}\sum_{Q_1'} \frac{1}{q^{n+l_{Q_1}}}\frac{1}{q^{m+l_{Q_1'}}} - \frac{(1-\frac{1}{q^n})}{q^{n+l_{Q_1}}}\frac{(1-\frac{1}{q^m})}{q^{m+l_{Q_1'}}}$$

$$= \sum_{Q_1}\sum_{Q_1'} \frac{1}{q^{n+l_{Q_1}}}\frac{1}{q^{m+l_{Q_1'}}}\left(\frac{1}{q^n}+\frac{1}{q^m}\right).$$

Consequently,

$$\sum_{m=N_1}^{n}\sum_{\substack{\deg Q_1=n\\ Q=Q_1^t}} \sum_{\substack{\deg Q_1'=m\\ Q'=Q_1'^t}} m(E_Q\cap E_{Q'})-m(E_Q)m(E_{Q'})$$

$$< \sum_{Q_1}\sum_{m=N_1}^{n}\sum_{Q_1'} \frac{1}{q^{n+l_{Q_1}}}\cdot\frac{1}{q^{m+l_{Q_1'}}}\cdot\frac{2}{q^m} \ll \sum_{Q_1}\frac{1}{q^{n+l_{Q_1}}}\sim \eta_n,$$

which concludes the proof of Theorem 3.1. □

We similarly deduce, as for Corollary 2.4, the following application in the case where l_{Q_1} vanishes, i.e., l_{Q_1} takes infinite value if Q_1 is not monic irreducible, and zero value otherwise.

Corollary 3.2. *Let $t\geqslant 2$ be a fixed positive integer. For almost all $f\in\mathbb{L}$,*

$$\operatorname{Card}\left\{1\leqslant n\leqslant N\colon\ \exists Q_1 \text{ irr.},\ \deg Q_1=n, \exists P \text{ s.t. } \gcd(P,Q)=1 \text{ and } \left|f-\frac{P}{Q_1^{t+1}}\right|<\frac{1}{q^{(t+1)n}}\right\}$$
$$=\log N + O\big(\log^{1/2} N \log^{(3+\varepsilon)/2}\log N\big),\quad \text{for any } \varepsilon>0.$$

By (5) and the Borel–Cantelli lemma, we cannot deduce here a statement analogous to Corollary 2.5: indeed, for a.e. $f\in\mathbb{L}$, there exist finitely many convergents P_n/Q_n, with Q_n being a fixed power of an irreducible polynomial.

4. Variable powers of irreducible denominators

In Section 3, we have considered denominators that are powers of an irreducible polynomial, i.e., $Q=Q_1^t$, for a fixed $t\geqslant 2$. Let us generalize this situation to the case where the power t is variable. Thus we consider the following inequality;

$$\left|f-\frac{P}{Q}\right|<\frac{1}{|Q_1|^{t+1}q^{l_{Q_1}+l_t}},\quad t\geqslant 1,\quad Q=Q_1^t,\quad Q_1 \text{ irreducible},\tag{7}$$

where l_{Q_1} takes nonnegative integer values if Q_1 is a monic irreducible polynomial, and infinite value otherwise, and l_t takes nonnegative integer values. As before, we also assume that P and Q are coprime.

Theorem 4.1. *We assume that the series* $\sum_{t\geqslant 1}\frac{1}{q^{l_t}}$ *is convergent. Then, for almost all* $f\in\mathbb{L}$, *the number of solutions of* (7) *with* $\deg Q\leqslant N$ *is*

$$\Xi(N)+O\big(\Xi^{1/2}(N)\log^{\frac{3+\varepsilon}{2}}\Xi(N)\big)\quad \textit{for any } \varepsilon>0$$

with

$$\Xi(N)=\sum_{n=1}^{N}\sum_{(k,t):\ kt=n}\sum_{\substack{Q=Q_1^t\\ Q_1\ \text{irr.}\\ \deg Q_1=k}}\frac{1}{q^{k+l_t+l_{Q_1}}}\left(1-\frac{1}{q^k}\right).$$

Proof. For $Q=Q_1^t$, $k=\deg Q_1$, and $n=\deg Q$, we put

$$\begin{cases} G_{\frac{P}{Q}}=\big\{f\in\mathbb{L}:\ \big|f-\frac{P}{Q}\big|<\frac{1}{q^{(t+1)k+l_{Q_1}+l_t}}\big\},\\ G_Q=\bigcup_{P:\ 0\leqslant\deg P<\deg Q,\ \gcd(P,Q)=1}G_{\frac{P}{Q}}.\end{cases}$$

Then we define ξ_n, η_n, $\hat\eta_n$ for all n as

$$\xi_n:=\sum_{(k,t):\ kt=n}\ \sum_{Q_1:\ \text{irr.},\deg Q_1=k}\chi_{G_{Q_1^t}}\quad\text{and}\quad \eta_n=\hat\eta_n=\int\xi_n\,dm.$$

One has

$$m(G_{\frac{P}{Q}})=\frac{1}{q^{(t+1)k+l_{Q_1}+l_t}},\qquad m(G_Q)=\frac{1}{q^{(t+1)k+l_{Q_1}+l_t}}\big(q^{tk}-q^{(t-1)k}\big).$$

We first estimate η_n:

$$\begin{aligned}\eta_n&=\sum_{(k,t):\ kt=n}\ \sum_{Q_1:\ \text{irr.},\deg Q_1=k}m(G_{Q_1^t})\\ &=\sum_{(k,t):\ kt=n}\ \sum_{Q_1:\ \text{irr.},\deg Q_1=k}\frac{1}{q^{(t+1)k+l_{Q_1}+l_t}}\big(q^{tk}-q^{(t-1)k}\big)\\ &=\sum_{(k,t):\ kt=n}\ \sum_{Q_1:\ \text{irr.},\deg Q_1=k}\frac{1}{q^{k+l_{Q_1}+l_t}}\left(1-\frac{1}{q^k}\right).\end{aligned}$$

Consequently, there exists $M > 0$ such that $\eta_n \leqslant M$ for any $n \geqslant 1$. Hence we can apply Lemma 1.1 to $(\frac{1}{M}\xi_n\colon\ n \geqslant 1)$ which satisfies condition (i) of Lemma 1.1. Note, furthermore, that we have

$$\sum_{(k,t)\colon\ kt=n}\ \sum_{Q_1\colon \text{ irr., deg } Q_1=k} \frac{1}{q^{k+l_{Q_1}+l_t}} \ll \eta_n.$$

It remains to show (and this will be sufficient) that the sequence (ξ_n) satisfies condition (ii) of Lemma 1.1. One has

$$\begin{aligned}
&\int \Bigg(\sum_{n=N_1}^{N_2} (\xi_n - \eta_n)\Bigg)^2 dm \\
&\quad = \int \Bigg(\sum_{n=N_1}^{N_2} \Bigg(\sum_{(k,t)\colon\ kt=n}\ \sum_{Q_1\colon \text{ irr., deg } Q_1=k} \big(\chi_{G_{Q_1^t}} - m(G_{Q_1^t})\big)\Bigg)\Bigg)^2 dm \\
&\quad = \int \Bigg[\sum_{n=N_1}^{N_2} \Bigg(\sum_{(k_1,t_1)\colon\ k_1t_1=n}\ \sum_{Q_1\colon \text{ irr., deg } Q_1=k_1} \big(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}})\big)\Bigg)\Bigg] \\
&\qquad \times \Bigg[\sum_{m=N_1}^{N_2} \Bigg(\sum_{(k_2,t_2)\colon\ k_2t_2=m}\ \sum_{Q_2\colon \text{ irr., deg } Q_2=k_2} \big(\chi_{G_{Q_2^{t_2}}} - m(G_{Q_2^{t_2}})\big)\Bigg)\Bigg] dm \\
&\quad = \int \Bigg[\sum_{k_1=1}^{N_2}\ \sum_{t_1=\lceil \frac{N_1}{k_1}\rceil}^{\lfloor \frac{N_2}{k_1}\rfloor}\ \sum_{Q_1} \big(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}})\big)\Bigg] \\
&\qquad \times \Bigg[\sum_{k_2=1}^{N_2}\ \sum_{t_2=\lceil \frac{N_1}{k_2}\rceil}^{\lfloor \frac{N_2}{k_2}\rfloor}\ \sum_{Q_2} \big(\chi_{G_{Q_2^{t_2}}} - m(G_{Q_2^{t_2}})\big)\Bigg] dm \\
&\quad \leqslant 2\int \sum_{k_1=1}^{N_2} \sum_{k_2=1}^{k_1} \sum_{t_1=\lceil \frac{N_1}{k_1}\rceil}^{\lfloor \frac{N_2}{k_1}\rfloor} \sum_{Q_1} \sum_{t_2=\lceil \frac{N_1}{k_2}\rceil}^{\lfloor \frac{N_2}{k_2}\rfloor} \sum_{Q_2\neq Q_1} \big(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}})\big)\big(\chi_{G_{Q_2^{t_2}}} - m(G_{Q_2^{t_2}})\big)\, dm \\
&\qquad + \int \sum_{k_1=1}^{N_2} \sum_{t_1=\lceil \frac{N_1}{k_1}\rceil}^{\lfloor \frac{N_2}{k_1}\rfloor} \sum_{Q_1} \sum_{t_2=\lceil \frac{N_1}{k_1}\rceil}^{\lfloor \frac{N_2}{k_1}\rfloor} \big(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}})\big)\big(\chi_{G_{Q_1^{t_2}}} - m(G_{Q_1^{t_2}})\big)\, dm \\
&\quad =: 2[\mathrm{A}] + [\mathrm{B}] \\
&\quad =: 2[\mathrm{A}] + [\mathrm{B1}\colon\ t_1 = t_2] + 2[\mathrm{B2}\colon\ t_1 > t_2],
\end{aligned}$$

by setting

$$[\mathrm{A}] := \int \sum_{k_1=1}^{N_2} \sum_{k_2=1}^{k_1} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \sum_{t_2=\lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \sum_{Q_2 \neq Q_1} \left(\chi G_{Q_1^{t_1}} - m(G_{Q_1^{t_1}})\right)\left(\chi G_{Q_2^{t_2}} - m(G_{Q_2^{t_2}})\right) dm,$$

$$[\mathrm{B1}] = \int \sum_{k_1=1}^{N_2} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \left(\chi G_{Q_1^{t_1}} - m(G_{Q_1^{t_1}})\right)^2 dm,$$

$$[\mathrm{B2}] = \int \sum_{k_1=1}^{N_2} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \sum_{t_2:\, t_2<t_1,\, t_2=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \left(\chi G_{Q_1^{t_1}} - m(G_{Q_1^{t_1}})\right)\left(\chi G_{Q_1^{t_1}} - m(G_{Q_1^{t_2}})\right) dm.$$

We will follow the same scheme of proof as previously, but for the estimate of [B2], we shall need an extended version of Lemma 2.3.

Estimate for [A]. To estimate [A], we distinguish two cases as in the proofs of Theorems 2.1 and 3.1. We first suppose that $(t_1+1)k_1 + l_{Q_1} + l_{t_1} \geqslant (t_2+1)k_2 + l_{Q_2} + l_{t_2}$, i.e., $m(G_{\frac{P}{Q_1^{t_1}}}) \leqslant m(G_{\frac{P}{Q_2^{t_2}}})$. We then decompose [A] by introducing a further summation over P and P'. If $G_{Q_1^{t_1}} \cap G_{Q_2^{t_2}} \neq \emptyset$, then there exist P and P' such that $G_{\frac{P}{Q_1^{t_1}}} \cap G_{\frac{P'}{Q_2^{t_2}}} \neq \emptyset$. One has $m(G_{\frac{P}{Q_1^{t_1}}} \cap G_{\frac{P'}{Q_2^{t_2}}}) = \frac{1}{q^{(t_1+1)k_1+l_{Q_1}+l_{t_1}}}$ and similarly as in the previous proofs, $|\frac{P}{Q} - \frac{P'}{Q'}| < \frac{1}{q^{(t_2+1)k_2+l_{Q_2}+l_{t_2}}}$. From Lemma 2.3, there exist at most $k_1t_1 - (k_2 + l_{Q_2} + l_{t_2})$ such pairs of polynomials (P, P'). Thus we get

$$m(G_{Q_1^{t_1}} \cap G_{Q_2^{t_2}}) - m(G_{Q_1^{t_1}})m(G_{Q_2^{t_2}}) \leqslant \frac{1}{q^{k_1+l_{Q_1}+l_{t_1}}} \frac{1}{q^{k_2+l_{Q_2}+l_{t_2}}} \frac{2}{q^{k_2}}$$

since we are assuming $k_1 \geqslant k_2$. The same holds when assuming $(t_1+1)k_1 + l_{Q_1} + l_{t_1} < (t_2+1)k_2 + l_{Q_2} + l_{t_2}$. Now we have

$$\sum_{k_1=1}^{N_2} \sum_{k_2=1}^{k_1} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \sum_{t_2=\lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \sum_{Q_2 \neq Q_1} \frac{1}{q^{k_1+l_{Q_1}+l_{t_1}}} \frac{1}{q^{k_2+l_{Q_2}+l_{t_2}}} \frac{1}{q^{k_2}}$$

$$= \sum_{k_1=1}^{N_2} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \frac{1}{q^{k_1+l_{Q_1}+l_{t_1}}} \sum_{k_2=1}^{k_1} \sum_{t_2=\lceil \frac{N_1}{k_2} \rceil}^{\lfloor \frac{N_2}{k_2} \rfloor} \sum_{Q_2 \neq Q_1} \frac{1}{q^{k_2+l_{Q_2}+l_{t_2}}} \frac{1}{q^{k_2}}.$$

We then use the fact that $\sum \frac{1}{q^{l_t}} < \infty$ to deduce that

$$[\mathrm{A}] \ll \sum_{k_1=1}^{N_2} \sum_{t_1=\lceil \frac{N_1}{k_1} \rceil}^{\lfloor \frac{N_2}{k_1} \rfloor} \sum_{Q_1} \frac{1}{q^{k_1+l_{Q_1}+l_{t_1}}} \ll \sum_{n=N_1}^{N_2} \eta_n.$$

Estimates for [B1] *and* [B2]. We now estimate [B1]:

$$\int \sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \bigl(\chi_{G_{Q_1^{t_1}}} - m(G_{Q_1^{t_1}})\bigr)^2 \, dm \leqslant \sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} m(G_{Q_1^{t_1}}) = \sum_{n=N_1}^{N_2} \eta_n.$$

Concerning [B2], we need to estimate

$$\sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \sum_{t_2} m(G_{Q_1^{t_1}} \cap G_{Q_1^{t_2}}) - m(G_{Q_1^{t_1}}) m(G_{Q_1^{t_2}})$$

with $t_1 > t_2$. Again we decompose [B2] by introducing a further summation over P and P' and by comparing $(t_1+1)k_1 + l_{Q_1} + l_{t_1}$ with $(t_2+1)k_1 + l_{Q_1} + l_{t_2}$. We thus assume that $(t_1+1)k_1 + l_{Q_1} + l_{t_1} \geqslant (t_2+1)k_1 + l_{Q_1} + l_{t_2}$, with the other case being handled similarly. We have to extend Lemma 2.3 in the following way: we prove that there exist at most $q^{t_1 k_1 - (k_1 + l_{Q_1} + l_{t_2})}$ pairs of non-zero polynomials (P, P') with $\deg P < t_1 \deg Q_1$ and $\deg P' < t_2 \deg Q_1$ that satisfy

$$\left| \frac{P}{Q_1^{t_1}} - \frac{P'}{Q_1^{t_2}} \right| < \frac{1}{q^{(t_2+1)k_1 + l_{Q_1} + l_{t_2}}}.$$

Consider indeed such a pair (P, P'). Then

$$k_1 t_1 - \deg\bigl(P - P' Q_1^{t_1 - t_2}\bigr) > (t_2+1)k_1 + l_{Q_1} + l_{t_2}.$$

Thus we see that

$$\deg\bigl(P - P' Q_1^{t_1 - t_2}\bigr) < (t_1 - t_2)k_1 - (k_1 + l_{Q_1} + l_{t_2}).$$

Hence there exist at most $q^{(t_1-t_2)k_1 - (k_1 + l_{Q_1} + l_{t_2})}$ polynomials of the form $P - P' Q_1^{t_1-t_2}$. Let us now fix (P, P'). If

$$P - P' Q_1^{t_1-t_2} = P_1 - P_1' Q_1^{t_1-t_2},$$

then

$$(P - P_1) = Q_1^{t_1-t_2}\bigl(P' - P_1'\bigr).$$

This implies

$$(t_1 - t_2)k_1 + \deg\bigl(P' - P_1'\bigr) = \deg(P - P_1) < t_1 k_1,$$

which in turn implies

$$\deg\left(P' - P_1'\right) < t_2 k_1.$$

Hence there exist at most $q^{t_2 k_1}$ pairs of polynomials (P_1, P_1') that satisfy $P - P' Q_1^{t_1 - t_2} = P_1 - P_1' Q_1^{t_1 - t_2}$. Consequently, the possible numbers of pairs (P, P') is at most $q^{t_1 k_1 - (k_1 + l_{Q_1} + l_{t_2})}$.

We deduce that

$$m(G_Q \cap G_{Q'}) \leqslant \frac{1}{q^{(t_1+1)k_1 + l_{Q_1} + l_{t_2}}} q^{t_1 k_1 - (k_1 + l_{Q_1} + l_{t_2})} = \frac{1}{q^{k_1 + l_{Q_1} + l_{t_1}}} \frac{1}{q^{k_1 + l_{Q_1} + l_{t_2}}}.$$

Therefore we have, by using the fact that $\sum \frac{1}{q^{l_t}}$ converges

$$\begin{aligned}
&\sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \sum_{t_2} m(G_{Q_1^{t_1}} \cap G_{Q_1^{t_2}}) - m(G_{Q_1^{t_1}}) m(G_{Q_1^{t_2}}) \\
&\quad \ll \sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \frac{1}{q^{k_1 + l_{Q_1} + l_{t_1}}} \frac{1}{q^{k_1 + l_{Q_1}}} \\
&\quad \ll \sum_{k_1=1}^{N_2} \sum_{t_1} \sum_{Q_1} \frac{1}{q^{k_1 + l_{Q_1} + l_{t_1}}} \ll \sum_{n=N_1}^{N_2} \eta_n,
\end{aligned}$$

which ends the proof of Theorem 4.1. $\square$

By (5) we cannot deduce here again a statement analogous to Corollary 2.5 concerning convergents: indeed, for a.e. $f \in \mathbb{L}$, there exist finitely many convergents P_n/Q_n, with Q_n being some power of an irreducible polynomial.

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