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Plurality preference digraphs realized by trees, II: On realization numbers

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Abstract

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A digraph D with vertex set $X = \{x_1, x_2, \ldots, x_n\}$ is realizable by a connected graph G if there exists a subset $C = \{c_1, c_2, \ldots, c_n\}$ of vertices of G so that for all distinct i and j in $\{1, 2, \ldots, n\}$, x_ix_j is an arc of D if and only if more vertices of G are closer to c_i than to c_j. For a positive integer n, let $\mathscr{F}_n$ denote the family of digraphs of order n which are realizable by trees. For a fixed $D \in \mathscr{F}_n$, the realization number of D, denoted $\alpha(D)$, is the smallest order of a tree which realizes D. Let $\alpha(\mathscr{F}_n) = \max\{\alpha(D): D \in \mathscr{F}_n\}$. In this paper $\alpha(\mathscr{F}_n)$ is determined explicitly.

1. Definitions and notations

This paper may be considered a sequel to papers [1] and [4]. In particular, [4] contains some motivational material connecting this work with user preferences (based on distances) for location of desirable facilities on tree networks. All graphs (or oriented graphs) here are graphs (or oriented graphs) without loops and multiple edges (or arcs). For a graph G (or digraph D), let $V(G)$ (or $V(D)$) denote its vertex set and let $E(G)$ (or $A(D)$) denote its edge set (or arc set). For a disconnected digraph D with components $D_1, D_2, \ldots, D_k$ we write $D = D_1 \cup D_2 \cup \cdots \cup D_k$ (see [2]). A connected digraph D is said to be *bipartitionable* if

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there exist two subdigraphs D_1 and D_2 satisfying the following conditions:

(1) $V(D) = V(D_1) \cup V(D_2)$ and $V(D_1) \cap V(D_2) = \emptyset$; and

(2) For any $v_i \in V(D_i)(i = 1, 2)$, $v_1v_2 \in A(D)$.

For bipartitionable D, write $D = D_1 \Rightarrow D_2$.

Let T be a tree. For $x \in V(T)$, the *branch weight of* x is defined by $b(x) = \max\{|V(T')|: T'$ is a subtree of $T - x\}$. The *branch weight centroid of* T (centroid of T for short), denoted $C_d(T)$, consists of all vertices x for which $b(x)$ is a minimum. Each vertex in $C_d(T)$ is called a *centroid vertex* of T (see [5]).

A digraph D with vertex set $X = \{x_1, x_2, \ldots, x_n\}$ is (p, h, n)*-realizable* if there exists a connected graph G of order p, a subset V of h vertices of G (voters), and a subset $C = \{c_1, c_2, \ldots, c_n\}$ of vertices of G (candidates) so that for all distinct i and j in $\{1, 2, \ldots, n\}$, x_ix_j is an arc of D if and only if a purlarity of the voters in V are closer to c_i than c_j in G, i.e., more vertices in V are closer to c_i than c_j in G. The terms voters and candidates arise from the connection with the location of deriable facilities on tree networks (see [4]). Note that any such D is necessarily an oriented graph. To say that D is *realizable* by G or that G *realizes* D means that $p = h$ (i.e., all vertices of G are voters). Of course, $n \leq p$. In this paper we restrict our attention to all digraphs realized only by trees.

Let T be a tree. If x is a vertex of T and w is either a vertex or edge of T, then $T[x, w]$ denotes the subtree of $T - w$ which contains x.

The following Theorem proved in [4] is useful in the next two sections:

Theorem A. *Let D be an oriented graph of order n which is (p, p, n)-realizable by a tree T of order p. Then $xy \in A(D)$ if and only if one of the following statements holds*:

(a) $d_T(x, C_d(T)) < d_T(y, C_d(T))$;

(b) *If $d_T(x, C_d(T)) = d_T(x, c) = d_T(y, c) = d_T(y, C_d(T))$ for some c in $C_d(T)$, let w be the vertex on the shortest path from x to y in T so that $d_T(x, w) = d_T(y, w)$. Then*

$$|V(T(x, w))| > |V(T(y, w))|.$$

For a positive integer n, let $\mathscr{F}_n$ be the family of oriented graphs of order n which are realizable by trees. For any $D \in \mathscr{F}_n$, D is said to be *p-realizable* if D is realizable by a tree of order p. The *realization number of* D, denoted $\alpha(D)$, is the smallest integer p for which D is p-realizable. Let $\alpha(\mathscr{F}_n) = \max\{\alpha(D): D \in \mathscr{F}_n\}$. The aim of this work is to evaluate $\alpha(\mathscr{F}_n)$.

In order to determine $\alpha(\mathscr{F}_n)$, an interesting family of oriented graphs will be introduced in the next section. In the third section, an explicit formula for $\alpha(\mathscr{F}_n)$ will be derived.

2. An example

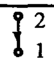
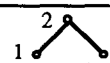
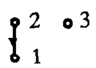
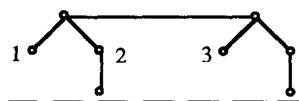
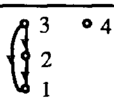
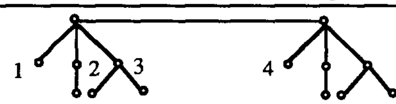
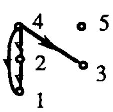
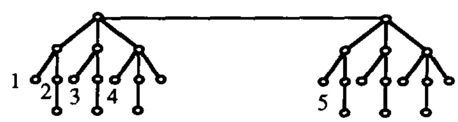
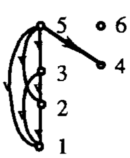
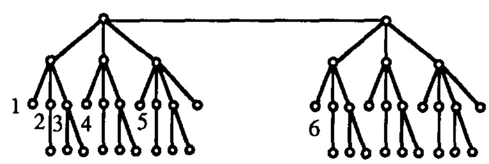
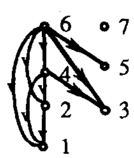
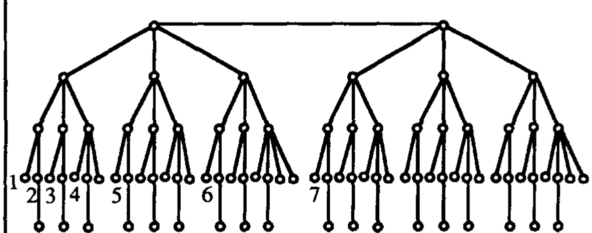
An exhaustive examination of all digraphs of small orders yields the digraphs listed in Table 1 as those with the maximum realization number for each order $n \leq 7$. The labels on the vertices in the trees are to indicate the candidate vertices corresponding to the vertices in the digraphs.

Table 1 suggests the following family of digraphs whose realization numbers attain the maximum values.

For a positive integer n, let H_n be the oriented graph defined recursively by $H_1 = K_1$, $H_2 = K_1 \Rightarrow K_1$, and $H_n = (K_1 \Rightarrow H_{n-2}) \cup K_1$.

To simplify the proof of Theorem 2.1, we label all vertices of H_n as $1, 2, \ldots, n$

Table 1

n	D	T realizing D	α (D)
1			1
2			3
3			8
4			14
5			28
6			46
7			88

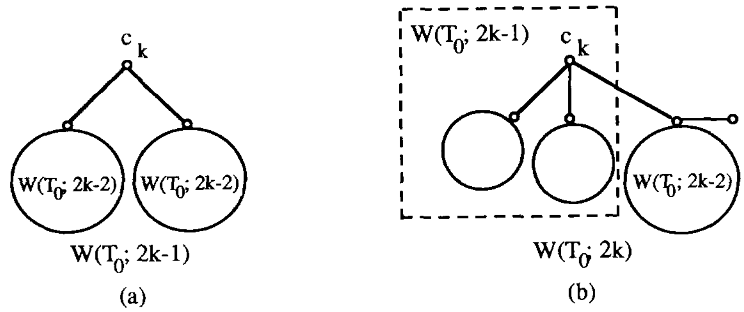


Fig. 1.

so that $A(H_n) = \{ij: i > j \text{ and } n - i \equiv 1 \pmod 2\} \cup \{21\}$. Examples of labeled H_n ($n \leq 7$) are given in the second column of Table 1.

In order to determine $\alpha(H_n)$, a tree of the smallest possible order will be constructed to realize H_n.

For a given tree T_0 and $c_0 \in V(T_0)$, define a sequence of trees $\{W(T_0; i): i \geq 0\}$ and a sequence of vertices $\{c_i: i \geq 0\}$, where $W(T_0; 0) = T_0$, according to the following rules: For $k \geq 1$,

(i) $W(T_0; 2k-1)$ is obtained from two vertex-disjoint copies of $W(T_0; 2k-2)$ by adding a new vertex c_k adjacent to exactly the two copies of c_{k-1} (see Fig. 1(a));

(ii) $W(T_0; 2k)$ is obtained from copies of $W(T_0; 2k-1)$ and $W(T_0; 2k-2)$ by adding a new vertex adjacent to the copy of the vertex c_{k-1} in $W(T_0; 2k-2)$ and by adding an edge connecting the copy of c_k in $W(T_0; 2k-1)$ and the copy of c_{k-1} in $W(T_0; 2k-2)$ (see Fig. 1(b)).

From the construction of the sequence of trees $\{W(T_0; i): i \geq 0\}$, it is easy to verify the following observations:

(1) for any $k \geq 1$, $W(T_0; k)$ has only one centroid vertex; $W(T_0; 2k-1) - c_k$ contains exactly two identical components, each a copy of $W(T_0; 2k-2)$, while $W(T_0; 2k) - c_k$ contains exactly three components, two of which are identical and a third one which has one more vertex than the other two identical components.

(2) $$|V(W(T_0; 2k))| = 3\,|V(W(T_0; 2k-2))| + 2$$

and

$$|V(W(T_0; 2k-1))| = 2\,|V(W(T_0; 2k-2))| + 1.$$

(3) $$|V(W(T_0; 2k))| = 3^k(|V(T_0)| + 1) - 1$$

and

$$|V(W(T_0; 2k-1))| = 2 \cdot 3^{k-1}(|V(T_0)| + 1) - 1.$$

Of course, as explicitly seen in (3), for any positive integer n, the order of tree $W(T_0; n)$ is a function of n and the order of the initial tree T_0.

Fig. 2.

Now consider the oriented graph H_n, which was defined above, when $n = 2k + 1$ $(k \geq 1)$. Let A denote the path of length three shown in Fig. 2(a). Let a_0, $b_0 \in V(W(A;1))$ lie in the same component of $W(A;1) - c_1$ so that $d_{W(A;1)}(a_0, c_1) = d_{W(A;1)}(b_0, c_1) = 2$. Let $a_i \in V(W(A;i))$ be any vertex in the component of $W(A;i) - c_{\lceil i/2 \rceil}$ not containing a_j $(0 \leq j \leq i-1)$ so that

$$d_{W(A;i)}(a_i, c_{\lceil i/2 \rceil}) = \left\lceil \frac{i}{2} \right\rceil + 1,$$

where $\lceil x \rceil$ is the least integer greater than or equal to x. In $W(A; 2k-1)$ consider the set $\{a_0, b_0, a_1, \ldots, a_{2k-1}\}$ as the set of candidates. Then by Theorem A, it is straightforward, but tedious, to check that $W(A; 2k-1)$ realizes a digraph isomorphic to $H_n = H_{2k+1}$. Also, by observation (3),

$$\begin{aligned}|V(W(A; 2k-1))| &= 2 \cdot 3^{(n-3)/2}(|V(A)| + 1) - 1 \\ &= 2 \cdot 3^{(n-3)/2} \cdot 5 - 1 = 10 \cdot 3^{(n-3)/2} - 1.\end{aligned}$$

So, for odd n, H_n is realizable by a tree of order $10 \cdot 3^{(n-3)/2} - 1$. By observation (1) above, the tree $W^*(A; 2k-1)$ obtained from $W(A; 2k-1)$ by replacing the path of length two which contains centroid vertex c_k as interior vertex with a single edge connecting the two ends of that path is also a tree that realizes H_n; moreover

$$|V(W^*(A; 2k-1))| = |V(W(A; 2k-1))| - 1 = 10 \cdot 3^{(n-3)/2} - 2.$$

Therefore, the following result follows.

Remark 2.1. *For any odd integer n $(n \geq 3)$, H_n is $(10 \cdot 3^{(n-3)/2} - 2)$-realizable.*

Next consider H_n when $n = 2k$ $(k \geq 2)$. Let A denote the tree of order 7 shown in Fig. 2(b). Let $a_0, b_0, d_0 \in V(W(A;1))$ be in the same component of $W(A;1) - c_1$ so that $d_{W(A;1)}(a_0, c_1) = d_{W(A;1)}(b_0, c_1) = d_{W(A;1)}(d_0, c_1) = 2$. For $i \geq 1$, let a_i be the vertex of $W(A;i)$ defined as in the case when $n = 2k + 1$. In $W(A; 2k-3)$ consider the set $\{a_0, b_0, d_0, a_1, \ldots, a_{2k-3}\}$ as the set of candidates. Then by Theorem A, it is straightforward, but tedious, to check that the tree $W(A; 2k-3)$

realizes a digraph isomorphic to $H_n = H_{2k}$. Also, by observation (3),

$$|V(W(A;2k-3))| = 2 \cdot 3^{(n-4)/2}(|V(A)|+1) - 1$$
$$= 2 \cdot 3^{(n-4)/2} \cdot 8 - 1 = 16 \cdot 3^{(n-4)/2} - 1.$$

So, H_n is $(16 \cdot 3^{(n-4)/2} - 1)$-realizable. By observation (1) above, the tree $W^*(A;2k-3)$ obtained from $W(A;2k-3)$ by replacing the path of length two which contains centroid vertex c_{k-1} as interior vertex with a single edge connecting the two ends of that path is also a tree that realizes H_n. Moreover,

$$|V(W^*(A;2k-3))| = |V(W(A;2k-3))| - 1 = 16 \cdot 3^{(n-4)/2} - 2.$$

Remark 2.2. *For any even integer n ($n \geq 4$), H_n is $(16 \cdot 3^{(n-4)/2} - 2)$-realizable.*

An obvious observation from Theorem A is the following.

Remark 2.3. *Let D be a disconnected digraph. If D is realizable by a tree T, then all vertices of D in T have the same distance to $C_d(T)$. Moreover, if $C_d(T) = \{c\}$ and each component of $T - c$ contains a vertex of D, then all components of $T - c$ have the same order.*

Lemma 2.1. *If D is m-realizable by a tree T, then D is $(m+1)$-realizable by a tree which contains exactly one centroid vertex.*

Proof. If T contains a single centroid vertex c, then let T^* be the tree obtained from T by adding a new vertex adjacent to c. Then T^* still realizes D and $|V(T^*)| = |V(T)| + 1 = m + 1$. So, we may assume that T contains two centroid vertices c_1 and c_2. It is well known that c_1c_2 is an edge of T (see [3]), and that $T - c_1c_2$ contains two components of the same order. Let T' denote the tree obtained from T by deleting the edge c_1c_2 and adjoining two new edges $c'c_1$ and $c'c_2$, where c' is a new vertex. Then c' is the only centroid vertex of T', T' still realizes D (by Theorem A), and $|V(T')| = |V(T)| + 1 = m + 1$. □

Lemma 2.2. *Let D be a disconnected digraph with components $D_1, D_2, \ldots, D_k$. If T is a tree realizing D, then for each i ($1 \leq i \leq k$), there exists a centroid vertex c and a component C_i of $T - c$ containing all vertices of D_i and so that $|V(C_i)| \leq \frac{1}{2}|V(T)|$.*

Proof. Since D is disconnected, by Remark 2.3, all vertices of D in T are equal distance to $C_d(T)$. Fix i, $1 \leq i \leq k$.

Case 1: If $|C_d(T)| = 2$, let $C_d(T) = \{c_1, c_2\}$. Note that $T - c_1c_2$ contains exactly two components $T(c_1, c_1c_2)$ and $T(c_2, c_1c_2)$ with $|V(T(c_1, c_1c_2))| = |V(T(c_2, c_1c_2))|$. So, by Theorem A, the connectivity of D_i implies that $V(D_i)$ must be contained in one of $T(c_1, c_1c_2)$ and $T(c_2, c_1c_2)$, say $T(c_1, c_1c_2)$. Then

$|V(T(c_1, c_1c_2))| = \frac{1}{2}|V(T)|$. Thus, $T(c_1, c_1c_2)$ is the required component of $T - c_2$.

Case 2: If $|C_d(T)| = 1$, let $C_d(T) = \{c\}$. Pick two adjacent vertices x and y in D_i (i.e., $xy \in A(D_i)$ or $yx \in A(D_i)$). Suppose that $T(x, c) \neq T(y, c)$. Then by Theorem A,

$$|V(T(x, c))| \neq |V(T(y, c))|. \tag{1}$$

Pick a vertex z in D_j, where $j \neq i$. If $T(z, c) = T(x, c)$, it follows from (1) that $|V(T(y, c))| \neq |V(T(z, c))|$. By Theorem A again, y and z are adjacent, a contradiction to the fact that y and z are in different components of D. If $T(z, c) \neq T(x, c)$, since z and x are not adjacent in D, by Theorem A, $|V(T(x, c))| = |V(T(z, c))|$. Hence, by (1), $|V(T(y, c))| \neq |V(T(z, c))|$. That is, by Theorem A again, z and y are adjacent, again a contradiction. Therefore, $T(x, c) = T(y, c)$. Consequently, any two adjacent vertices of D_i are in $T(x, c)$. Since D_i is connected, all vertices of D_i are in $T(x, c)$.

Let C be the component of $T - c$ so that $bw(c) = |V(C)|$ (C might be $T(x, c)$). Since c is the only centroid vertex of T, by a basic result in [4] (Lemma 1.1),

$$|V(T)| - |V(C)| \geq bw(c) = |V(C)|,$$

i.e., $2|V(C)| \leq |V(T)|$. But $2|V(T(x, c))| \leq 2|V(C)|$. So, $|V(T(x, c))| \leq \frac{1}{2}|V(T)|$. Therefore, $T(x, c)$ is as required. □

Lemma 2.3. *Let D be a disconnected digraph realized by a tree T of the smallest possible order with a single centroid vertex c. Then each component of $T - c$ contains a vertex of D.*

Proof. Since D is disconnected, by Remark 2.3, for any $x, y \in V(D)$, $d_T(x, c) = d_T(y, c)$.

Let $\mathcal{S} = \{u : d_T(u, x) = d_T(u, y)$, for all $x, y \in V(D)\}$. Pick a vertex $w \in \mathcal{S}$ so that, for $x \in V(D)$,

$$d_T(x, w) = \min\{d_T(x, u) : u \in \mathcal{S}\}.$$

By the choice of w, at least two components of $T - w$ contain a vertex of D. Let $C_1, C_2, \ldots, C_k$ ($k \geq 2$) be all components of $T - w$, each of which contains a vertex of D. Note that all vertices of D are in $\bigcup_{i=1}^{k} V(C_i)$. Since D is disconnected, by Theorem A, for any $i, j \in \{1, 2, \ldots, k\}$, C_i and C_j have the same order. The subtree induced by $(\bigcup_{i=1}^{k} V(C_i)) \cup \{w\}$, denoted by T^*, has a single centroid vertex w. By Theorem A again, T^* realizes D. So, by the choice of T, $T^* = T$. Therefore, $w = c$ and hence each component of $T - c$ contains a vertex of D. □

Theorem 2.1. *For any integer n with $n \geq 3$,*

$$\alpha(H_n) = \begin{cases} 10 \cdot 3^{(n-3)/2} - 2, & \text{if } n \text{ is odd,} \\ 16 \cdot 3^{(n-4)/2} - 2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Assume that H_n is labeled as described following the definition of H_n.

(1) suppose that n is odd. Let $\mathcal{T}_1(H_n)$ denote the set of trees which realize H_n and contain exactly one centroid vertex. Pick a tree $T_n^{(1)} \in \mathcal{T}_1(H_n)$ so that

$$|V(T_n^{(1)})| = \min\{|V(T)|: T \in \mathcal{T}_1(H_n)\}.$$

Let c_n be the centroid vertex of $T_n^{(1)}$. By Lemma 2.1, to show $\alpha(H_n) \geqslant 10 \cdot 3^{(n-3)/2} - 2$, it suffices to show that

$$|V(T_n^{(1)})| \geqslant 10 \cdot 3^{(n-3)/2} - 1.$$

This is done by induction on odd n.

If $n = 3$, it is straightforward to check that there does not exist a tree of order less than 9, with a single centroid vertex, which realizes H_n. Thus, $|V(T_3^{(1)})| \geqslant 9$. Suppose that the result is true for H_{n-2}, where $n-2 \geqslant 3$. Note that H_n is disconnected with exactly two components $\{n\}$ and $\{n-1\} \Rightarrow H_{n-2}$. So, by Lemmas 2.2 and 2.3, $T_n^{(1)} - c_n$ contains exactly two components $T(n, c_n)$ and $T(n-1, c_n)$. Since there are no arcs between n and $n-1$, by Theorem A, $|V(T(n, c_n))| = |V(T(n-1, c_n))|$. Note that $T(n-1, c_n)$ is a tree realizing the digraph $H_n - n$. Let c_{n-1} be the vertex in $T(n-1, c_n)$ adjacent to c_n. Then $T(n-1, c_n) - c_{n-1}$ contains at least two components each of which contains a vertex of D. For otherwise, let w and T^* be the vertex and the subtree of $T(n-1, c_n)$ defined as in the proof of Lemma 2.3. Let T be the tree obtained from T^* and a copy of T^* by adding a new vertex v adjacent to c_{n-1} and the copy of c_{n-1}. Take a vertex in the copy of T^*, which is a copy of a vertex of D, as the vertex n. Then the resulting tree T realizes D and contains a single centroid vertex v. But $|V(T)| < |V(T_n^{(1)})|$. This contradicts the minimality of $|V(T_n^{(1)})|$.

Let $C(i)$ be the component of $T(n-1, c_n) - c_{n-1}$ containing the vertex i of $H_n - n$. Since $T(n-1, c_n) - c_{n-1}$ contains at least two components each of which contains a vertex of D, there is a vertex $j \in \{1, 2, \ldots, n-2\}$ so that $C(j) \neq C(n-1)$. Note that, for $j < n-2$, there is no arc in H_n between $n-2$ and j, but there is an arc from $n-1$ to j. So, by Theorem A and the fact that $C(j)$ is not equal to $C(n-1)$, $C(n-2)$ is not equal to $C(n-1)$.

Case 1: If $C(n-3) = C(n-2)$, then since for any $i \in \{1, 2, \ldots, n-4\}$, $(n-3)i \in A(H_n)$, but neither $(n-2)i$ nor $i(n-2)$ is in $A(H_n)$, $C(i) = C(n-2)$. Thus, $T(n-1, c_n) - c_{n-1}$ contains exactly two components $C(n-1)$ and $C(n-2)$ (see Fig. 3(a)). And, $C(n-2)$ realizes H_{n-2}. By the minimality of $|V(T_n^{(1)})|$,

$$|V(C(n-1))| = |V(C(n-2))| + 1.$$

By Lemma 2.1, $\alpha(H_{n-2}) \geqslant |V(T_{n-2}^{(1)})| - 1$, where $T_{n-2}^{(1)}$ is a tree of the smallest order in $\mathcal{T}_1(H_{n-2})$. So,

$$\begin{aligned}|V(T_n^{(1)})| &= 2\,|V(T(n-1, c_n))| + 1 \geqslant 2(2\,|V(C(n-2))| + 2) + 1 \\ &\geqslant 4\alpha(H_{n-2}) + 5 \geqslant 4(|V(T_{n-2}^{(1)})| - 1) + 5 \\ &\geqslant 3\,|V(T_{n-2}^{(1)})| + 2.\end{aligned}$$

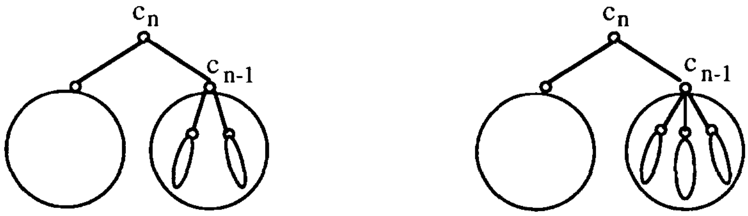


Fig. 3.

Case 2: If $n-3 \notin V(C(n-2))$, then neither $(n-2)(n-3)$ nor $(n-3)(n-2)$ in $A(H_n)$ implies that $|V(C(n-2))| = |V(C(n-3))|$. Since $|V(C(n-1))| > |V(C(n-2))|$, $n-3$ is not in $C(n-1)$. Note that the vertex $n-2$ is not adjacent to i, but $(n-3)i \in A(H_n)$, for any $i \in \{1, 2, \ldots, n-4\}$. So, by Theorem A, $i \in C(n-3)$, for any $i \in \{1, 2, \ldots, n-4\}$. It follows that $T(n-1, c_n) - c_{n-1}$ contains exactly three components $C(n-1)$, $C(n-2)$, and $C(n-3)$ (see Fig. 3(b)). By the minimality of $|V(T_n^{(1)})|$,

$$|V(C(n-1))| = |V(C(n-2))| + 1 = |V(C(n-3))| + 1.$$

Note that the tree T'_{n-2} obtained from $T(n-1, c_n)$ by deleting $C(n-1)$ is in $\mathscr{T}_1(H_{n-2})$. It is easy to see that

$$|V(T_n^{(1)})| \geq 2[|V(T'_{n-2})| + \tfrac{1}{2}(|V(T'_{n-2})| + 1)] + 1$$
$$= 3\,|V(T'_{n-2})| + 2 \geq 3\,|V(T_{n-2}^{(1)})| + 2,$$

where $T_{n-2}^{(1)}$ is a tree of smallest order in $\mathscr{T}_1(H_{n-2})$.

Hence, each case yields $|V(T_n^{(1)})| \geq 3\,|V(T_{n-2}^{(1)})| + 2$. By the induction hypothesis,

$$|V(T_n^{(1)})| \geq (3(10 \cdot 3^{((n-2)-3)/2} - 1) + 2$$
$$= 10 \cdot 3^{(n-3)/2} - 3 + 2 = 10 \cdot 3^{(n-3)/2} - 1.$$

Therefore, $\alpha(H_n) \geq 10 \cdot 3^{(n-3)/2} - 2$. By Remark 2.1,

$$\alpha(H_n) = 10 \cdot 3^{(n-3)/2} - 2.$$

(2) A similar analysis can be applied for the case when n is even. □

3. Main results

Lemma 3.1. *Let $D = D_1 \cup D_2$ be realizable by a tree. Then*

$$\alpha(D) \leqslant \begin{cases} \max\limits_{1\leqslant i\leqslant 2} \{3\alpha(D_i)+3\}, & \text{if } D_i \text{ is disconnected } (i=1,2),\\ \max\limits_{1\leqslant i\leqslant 2} \{\tfrac{3}{2}\alpha(D_i \cup \{x_i\})\}, & \text{if } D_i \text{ is connected } (i=1,2),\\ \max\{3\alpha(D_i)+3,\ \tfrac{3}{2}\alpha(D_j\cup\{x_j\})\}, & \text{if } D_i \text{ is disconnected and } D_j \text{ is connected, } 1\leqslant i,j\leqslant 2,\ i\neq j,\end{cases}$$

where x_i is a vertex not in $V(D)$, for $i = 1, 2$.

Proof. *Case* 1: Suppose that D_i is disconnected for $i = 1, 2$. Let T_i be a tree of order $\alpha(D_i) + 1$ with a single centroid vertex c_i so that T_i realizes D_i. By Remark 2.3, all vertices of D_i are at equal distance to c_i. Denote this distance by d_i.

For $i = 1, 2$, let $k = 2$, if $i = 1$, and $k = 1$, if $i = 2$. Let T be the tree obtained from T_1 and T_2 by joining c_1 to c_2 by a path of length $|d_2 - d_1| + 1$ and adding α vertices and β vertices at c_i and c_k, respectively, where

$$\alpha = \begin{cases} \lceil \tfrac{3}{2}|V(T_k)|\rceil - |V(T_i)| - (d_k - d_i), & \text{if } d_i \leqslant d_k \text{ and } |V(T_i)| \leqslant |V(T_k)|,\\ \lceil \tfrac{1}{2}|V(T_k)|\rceil - (d_k - d_i), & \text{if } d_i \leqslant d_k \text{ and } |V(T_i)| \geqslant |V(T_k)| \end{cases}$$

and

$$\beta = \begin{cases} \lceil \tfrac{1}{2}|V(T_k)|\rceil, & \text{if } d_i \leqslant d_k \text{ and } |V(T_i)| \leqslant |V(T_k)|,\\ |V(T_i)| - \lfloor \tfrac{1}{2}|V(T_k)|\rfloor, & \text{if } d_i \leqslant d_k \text{ and } |V(T_i)| \geqslant |V(T_k)|. \end{cases}$$

The number $\lceil x\rceil$ (or $\lfloor x\rfloor$) is the least (respectively, greatest) integer greater (respectively, smaller) than or equal to x. The tree shown in Fig. 4 illustrates the case when $d_1 \leqslant d_2$ and $|V(T_1)| \leqslant |V(T_2)|$.

Let c be the vertex on the path joining c_1 and c_2, which is adjacent to c_k. Then by the construction of T, c and c_k are centroid vertices of T. Also, all vertices of $D = D_1 \cup D_2$ are at equal distance to $C_d(T)$. By Theorem A, it can be verified

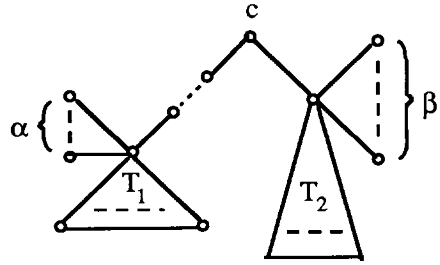


Fig. 4.

that T realizes D. Moreover, by calculating $|V(T)|$ in each case,

$$\alpha(D) \leq |V(T)| \leq \max_{1\leq i\leq 2} \{3\,|V(T_i)|\}$$

$$= \max_{1\leq i\leq 2} \{3(\alpha(D_i)+1)\} = \max_{1\leq i\leq 2} \{3\alpha(D_i)+3\}.$$

Case 2: Suppose that D_i is connected for $i = 1, 2$. Let T_i^* be a tree of order $\alpha(D_i \cup \{x_i\})$ so that T_i^* realizes $D_i \cup \{x_i\}$, where $x_i \notin V(D)$. By Lemma 2.2 applied to $D_i \cup \{x_i\}$, there exists a component T_i of $T_i^* - c_i^*$ containing all vertices of D_i, where $c_i^* \in C_d(T_i^*)$. Moreover,

$$|V(T_i)| \leq \tfrac{1}{2}\,|V(T_i^*)|\,. \tag{2}$$

Let $c_i \in V(T_i)$ be adjacent to c_i^* in T_i^*. Let T be the tree constructed as in Case 1. Then T realizes D. By (2),

$$\alpha(D) \leq |V(T)| \leq \max_{1\leq i\leq 2} \{3\,|V(T_i)|\} \leq \max_{1\leq i\leq 2} \{\tfrac{3}{2}\,|V(T_i^*)|\}$$

$$= \max_{1\leq i\leq 2} \{\tfrac{3}{2}\alpha(D_i \cup \{x_i\})\}.$$

Case 3: Suppose that only one of D_1 and D_2 is connected. Combining the two cases above, we can obtain the required result.

This completes the proof. □

Note that the tree T' obtained from the tree T constructed above by adding a new vertex adjacent to c_1 realizes the digraph $D = D_1 \Rightarrow D_2$, and T' contains a single centroid vertex. Thus, the next Remark 3.1 follows immediately.

Remark 3.1. *Let $D = D_1 \Rightarrow D_2$ be realizable by a tree. Then*

$$\alpha(D) \leq \begin{cases} \max\limits_{1\leq i\leq 2} \{3\alpha(D_i)+4\}, & \text{if } D_i \text{ is disconnected } (i=1,2),\\ \max\limits_{1\leq i\leq 2} \{\tfrac{3}{2}\alpha(D_i \cup \{x_i\})+1\}, & \text{if } D_i \text{ is connected } (i=1,2),\\ \max\{3\alpha(D_i)+4,\ \tfrac{3}{2}\alpha(D_j \cup \{x_j\})+1\}, & \text{if } D_i \text{ is disconnected}\\ & \text{and } D_j \text{ is connected},\\ & 1\leq i,j\leq 2,\ i\neq j, \end{cases}$$

where x_i is a vertex not in $V(D)$, for $i = 1, 2$.

Lemma 3.2. *Let $D = \{u\} \cup D_2$ (respectively $D = \{u\} \Rightarrow D_2$, $D = D_2 \Rightarrow \{u\}$), where D_2 is a disconnected digraph. If T_2 is a tree with a single centroid vertex c, which realizes D_2 and is of the smallest order, then there exists a tree T with a single* *centroid vertex which realizes D so that*

$$|V(T)| \leqslant \frac{k+1}{k}|V(T_2)| - \frac{1}{k}$$

$\Big($*respectively,*

$$|V(T)| \leqslant \frac{k+1}{k}|V(T_2)| - \frac{1}{k} + 1\Big),$$

where k is the number of components of $T_2 - c$.

Proof. Let $C_1, C_2, \ldots, C_k$ be the components of $T_2 - c$ and let c_i be the vertex of C_i adjacent to c $(1 \leqslant i \leqslant k)$. By Remark 2.3, all vertices of D_2 in T_2 have the same distance to c, and each component of $T_2 - c$ contains at least one vertex of D_2. Moreover, the disconnectedness of D_2 implies, by Theorem A, that any two components of $T_2 - c$ have the same order. Hence,

$$|V(C_i)| = \frac{1}{k}(|V(T_2)| - 1) \qquad (1 \leqslant i \leqslant k).$$

Let T be the tree obtained from T_2 and a copy of C_1 by adding an edge connecting c to the copy of c_1. This second copy of C_1 in T is denoted by C_{k+1} (see Fig. 5). Denote a vertex of C_{k+1} which is a copy of a vertex of D_2 by u. Consider the vertices of T_2 which represent vertices of D_2, together with u, as the set of candidates, and consider $V(T)$ as the set of voters. Then by Theorem A again, T realizes $D = \{u\} \cup D_2$. Clearly,

$$|V(T)| \leqslant |V(T_2)| + \frac{1}{k}(|V(T_2)| - 1) \leqslant \frac{k+1}{k}|V(T_2)| - \frac{1}{k}.$$

If $D = \{u\} \Rightarrow D_2$ (respectively, $D = D_2 \Rightarrow \{u\}$), let T be the tree obtained from the tree constructed above by adding a new vertex adjacent to the vertex of C_{k+1} which is a copy of c_1 (respectively, deleting an end vertex of C_{k+1}). Hence, T is as required. □

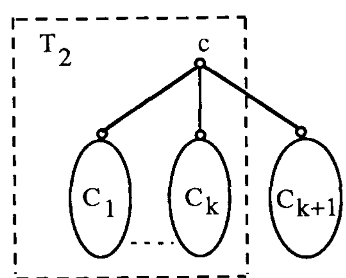


Fig. 5.

Theorem 3.1. *For any* $D \in \mathscr{F}_n$,

$$\alpha(D) \leqslant \begin{cases} 16 \cdot 3^{(n-4)/2} - 2, & \text{if } n \text{ is even,} \\ 10 \cdot 3^{(n-3)/2} - 2, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The proof is by induction on n.

It is easy to check Table 1 in Section 2 to see that the inequality holds for $n = 1, 2, 3, 4$. Suppose that the result is true for any $D \in \mathscr{F}_k$ $(4 \leqslant k \leqslant n-1)$. Let $D \in \mathscr{F}_n$. It is known (see Theorems 2.1 and 2.3 in [4]) that D is transitive and contains no anti-directed path of length 3.

First of all, consider the case when n is even. Note that $n \geqslant 6$.

Case A: suppose that D is connected. Now by Lemma 2.5 in [4], $D = D_2 \Rightarrow D_1$ for some subdigraphs D_1 and D_2 of D. Let $n_i = |V(D_i)|$, $i = 1, 2$.

Subcase A.1: Suppose that $n_i \geqslant 2$ $(i = 1, 2)$. Let T be a tree realizing D so that $|V(T)| = \alpha(D)$. By Remark 3.1,

$$\alpha(D) \leqslant \begin{cases} \max\limits_{1 \leqslant i \leqslant 2} \{3\alpha(D_i) + 4\}, & \text{if } D_i \text{ is disconnected } (i = 1, 2), \\ \max\limits_{1 \leqslant i \leqslant 2} \{\tfrac{3}{2}\alpha(D_i \cup \{x_i\}) + 1\}, & \text{if } D_i \text{ is connected } (i = 1, 2), \\ \max\{3\alpha(D_i) + 4, \tfrac{3}{2}\alpha(D_j \cup \{x_j\}) + 1\}, & \text{if } D_i \text{ is disconnected} \\ & \text{and } D_j \text{ is connected,} \\ & 1 \leqslant i, j \leqslant 2,\ i \neq j, \end{cases}$$

where x_i is a vertex not in $V(D)$, for $i = 1, 2$. Note that the function $f(x) = 3^x$ (or $g(x) = 3^{-x}$) is increasing (respectively, decreasing) and that if n_i is odd, then $n_i \leqslant n - 3$ since n is even. By the induction hypothesis, for $i = 1, 2$,

$$\begin{aligned} 3\alpha(D_i) + 4 &\leqslant \begin{cases} 3(16 \cdot 3^{(n_i-4)/2} - 2) + 4, & \text{if } n_i \text{ is even,} \\ 3(10 \cdot 3^{(n_i-3)/2} - 2) + 4, & \text{if } n_i \text{ is odd,} \end{cases} \\ &\leqslant \begin{cases} 3 \cdot 16 \cdot 3^{(n-2-4)/2} - 6 + 4, & \text{if } n_i \text{ is even,} \\ 3 \cdot 10 \cdot 3^{(n-3-3)/2} - 6 + 4, & \text{if } n_i \text{ is odd,} \end{cases} \\ &\leqslant 16 \cdot 3^{(n-4)/2} - 2. \end{aligned}$$

and

$$\begin{aligned} \tfrac{3}{2}\alpha(D_i \cup \{x_i\}) + 1 &\leqslant \begin{cases} \tfrac{3}{2}(16 \cdot 3^{(n_i+1-4)/2} - 2) + 1, & \text{if } n_i + 1 \text{ is even,} \\ \tfrac{3}{2}(10 \cdot 3^{(n_i+1-3)/2} - 2) + 1, & \text{if } n_i + 1 \text{ is odd,} \end{cases} \\ &\leqslant \begin{cases} \tfrac{3}{2} \cdot 16 \cdot 3^{(n-2+1-4)/2} - 3 + 1, & \text{if } n_i + 1 \text{ is even,} \\ 3 \cdot 5 \cdot 3^{(n-2+1-3)/2} - 3 + 1, & \text{if } n_i + 1 \text{ is odd,} \end{cases} \\ &< 16 \cdot 3^{(n-4)/2} - 2. \end{aligned}$$

Thus, $|V(T)| \leqslant 16 \cdot 3^{(n-4)/2} - 2$.

Subcase A.2: Suppose that $n_1 = 1$. Then $n - n_1 = n - 1$ is odd. By Lemma 2.1, D_2 is $(\alpha(D_2) + 1)$-realizable by a tree, say T_2, which contains a single centroid vertex c. Let T be the tree obtained from T_2 by adding two vertices u and v adjacent to c and to a vertex x furthest away from c, respectively. Consider the set $V(D_2)\cup\{v\}$ as the set of candidates, and consider the set $V(T)$ as the set of voters, then T realizes D. So, by the induction hypothesis,

$$|V(T)|=2+|V(T_2)|\leqslant 2+10\cdot 3^{(n-1-3)/2}-1<16\cdot 3^{(n-4)/2}-2.$$

Subcase A.3: Suppose that $n_2=1$. Then $n_1=n-1$ is odd. By Lemma 2.1, D_1 is $(\alpha(D_1)+1)$-realizable by a tree, say T_1, which contains a single centroid vertex c. Without loss of generality, assume that c is not used as a candidate, for otherwise c dominates every vertex in D_1 and $D=(D_2\cup\{c\})\Rightarrow(D_1-\{c\})$, so that this case has been treated in Subcase A.1. So, let $T=T_1$ and consider the centroid vertex of T_1 as the only vertex of D_2 (a candidate). Then T realizes D and

$$|V(T)|=|V(T_1)|\leqslant 10\cdot 3^{(n-1-3)/2}-2+1<16\cdot 3^{(n-4)/2}-2.$$

So, if n is even and D is connected, then $\alpha(D)<16\cdot 3^{(n-4)/2}-2$.

Now consider the case when D is disconnected.

Case B: Suppose that D is disconnected. Assume that $D=D_1\cup D_2$, where D_1 is connected. Let $n_i=|V(D_i)|$ $(i=1,2)$.

Subcase B.1: Suppose that $n_i\geqslant 2$ $(i=1,2)$. Let T be a tree realizing D so that $||V(T)|=\alpha(D)$. By Lemma 3.1,

$$\alpha(D)\leqslant\begin{cases}\max\limits_{1\leqslant i\leqslant 2}\{\frac{3}{2}\alpha(D_i\cup\{x_i\})\}, & \text{if } D_2 \text{ is connected,}\\ \max\{3\alpha(D_2)+3,\frac{3}{2}\alpha(D_1\cup\{x_1\})\}, & \text{if } D_2 \text{ is disconnected}\end{cases}$$

where x_i is a vertex not in $V(D)$, for $i=1,2$. A computation very similar to that done in Subcase A.1 yields $|V(T)|\leqslant 16\cdot 3^{(n-4)/2}-2$.

Subcase B.2: Suppose that $n_1=1$.

If D_2 is disconnected, let T_2 be a tree realizing D_2 so that T_2 contains a single centroid vertex and is of the smallest order. Then by Lemma 3.2, there exists a tree T realizing D and for some integer $k\geqslant 2$,

$$|V(T)|\leqslant\frac{k+1}{k}|V(T_2)|-\frac{1}{k}.$$

By the induction hypothesis,

$$\begin{aligned}|V(T)|&\leqslant\frac{k+1}{k}(10\cdot 3^{(n-1-3)/2}-1)-\frac{1}{k}\\ &\leqslant\frac{k+1}{k}\cdot 10\cdot 3^{(n-4)/2}-1-\frac{2}{k}\\ &\leqslant 16\cdot 3^{(n-4)/2}-2.\end{aligned}$$

If D_2 is connected, then by Lemma 2.5 in [4], $D_2=D_{22}\Rightarrow D_{21}$ for some subdigraphs D_{21} and D_{22} of D_2. Let $\alpha_i=|V(D_{2i})|$. Note that $\alpha_2=n-1-\alpha_1$.

Subsubcase B.2.1: Suppose that $\alpha_i \geqslant 3$ $(i = 1, 2)$. Let T_2 be a tree of order $\alpha(D_2)$ which realizes D_2. Then by Remark 3.1, $\alpha(D_2)$ is less than or equal to

$$\begin{cases} \max\limits_{1\leqslant i\leqslant 2} \{3\alpha(D_{2i}) + 4\}, & \text{if } D_{2i} \text{ is disconnected } (i = 1, 2), \\ \max\limits_{1\leqslant i\leqslant 2} \{\tfrac{3}{2}\alpha(D_{2i} \cup \{x_i\}) + 1\}, & \text{if } D_{2i} \text{ is connected } (i = 1, 2), \\ \max\{3\alpha(D_{2i}) + 4,\ \tfrac{3}{2}\alpha(D_{2j} \cup \{x_j\}) + 1\}, & \text{if } D_{2i} \text{ is disconnected} \\ & \text{and } D_{2j} \text{ is connected,} \\ & 1 \leqslant i, j \leqslant 2,\ i \neq j, \end{cases}$$

where $x_i \notin V(D)$, for $i = 1, 2$.

Note that $3 \leqslant \alpha_i = |V(D_{2i})| \leqslant n - 4$. So, a computation very similar to that done in Subcase A.1 gives the following inequalities:

$$3\alpha(D_{2i}) + 4 \leqslant \begin{cases} 3 \cdot 16 \cdot 3^{(n-8)/2} - 2, & \text{if } \alpha_i \text{ is even,} \\ 3 \cdot 10 \cdot 3^{(n-8)/2} - 2, & \text{if } \alpha_i \text{ is odd,} \end{cases}$$
$$= \begin{cases} \tfrac{1}{3} \cdot 16 \cdot 3^{(n-4)/2} - 2, & \text{if } \alpha_i \text{ is even,} \\ \tfrac{1}{3} \cdot 10 \cdot 3^{(n-4)/2} - 2, & \text{if } \alpha_i \text{ is odd,} \end{cases}$$

and

$$\tfrac{3}{2}\alpha(D_{2i} \cup \{x_i\}) + 1 \leqslant \begin{cases} \tfrac{3}{2} \cdot 16 \cdot 3^{(n-8)/2} - 2, & \text{if } \alpha_i \text{ is odd,} \\ \tfrac{3}{2} \cdot 10 \cdot 3^{(n-6)/2} - 2, & \text{if } \alpha_i \text{ is even,} \end{cases}$$
$$= \begin{cases} \tfrac{1}{6} \cdot 16 \cdot 3^{(n-4)/2} - 2, & \text{if } \alpha_i \text{ is odd,} \\ \tfrac{1}{2} \cdot 10 \cdot 3^{(n-4)/2} - 2, & \text{if } \alpha_i \text{ is even.} \end{cases}$$

Hence,

$$|V(T_2)| < \tfrac{1}{3}(16 \cdot 3^{(n-4)/2} - 2). \tag{3}$$

It may be assumed that T_2 is a tree T' described immediately prior to Remark 3.1. So, T_2 contains a single centroid vertex, denoted c_0, and arose from the tree T of Lemma 3.1. By the construction in the proof of Lemma 3.1, it can be assumed that all vertices of D_{22} are closer to c_0 than all vertices of D_{21} by exactly distance 1. Let w be the vertex adjacent to c_0 so that the distance between w and some vertex of D_{21} is the same as the distance between c_0 and some vertex of D_{22}. Let T_2' be the tree obtained from T_2 by deleting the edge wc_0 and adding a new vertex v adjacent to both w and c_0. Let T be the tree obtained from two copies of T_2' by adding an edge connecting two copies of v. Note that all vertices of D_2 are equi-distance from v. Then take all vertices of D_2 in one copy of T_2' and a vertex of D_2 in another copy of T_2' as the set of candidates in T, and take the vertex set of T as the set of voters. Then by Theorem A, T realizes D. Also $|V(T)| = 2\,|V(T_2')| = 2(|V(T_2)| + 1)$. By (3),

$$|V(T)| < \tfrac{2}{3}16 \cdot 3^{(n-4)/2} - \tfrac{4}{3} + 2 \leqslant 16 \cdot 3^{(n-4)/2} - 2.$$

Subsubcase B.2.2: Suppose that $\alpha_1 = 2$ (The case $\alpha_2 = 2$ can be treated similarly.) Then $\alpha_2 = n - 1 - \alpha_1 = n - 3$ is odd and greater than or equal to 3.

Let $D_{22}^* = D_{22} \cup \{x\}$, where $x \notin V(D)$. Let T_{22}^* be a tree, of order $\alpha(D_{22}^*)$, which realizes D_{22}^*. By Lemma 2.2, there exists a component T_{22} of $T_{22}^* - c$, for some $c \in C_d(T_{22}^*)$, so that all vertices of D_{22} are in T_{22} and $|V(T_{22})| \leqslant \frac{1}{2}|V(T_{22}^*)|$. Let v be the vertex in $V(T_{22})$ adjacent to c. Remark 2.3 implies that all vertices of D_{22} in T_{22} are at equal distance to v. Let T_2 be the tree obtained from two vertex-disjoint copies of T_{22} by adding a new vertex u adjacent to both copies of the vertex v and adding another new vertex w adjacent to one of the copies of v. Note that u and the copy of v adjacent to w are both centroid vertices of T_2. As the set of candidates in T_2, choose two vertices of D_{22} in the copy of T_{22} to which w is not added, together with a copy of $V(D_{22})$ in the other copy of T_{22}. The former two vertices are chosen as follows: if D_{21} has no arc and there are two independent vertices in D_{22}, choose copies of those two; if D_{21} consists of a single arc and there is an arc in D_{22}, choose copies of those two vertices connected by that arc. This can be done unless

(a) D_{21} consists of two independent vertices and D_{22} is complete, hence the transitive $(n-3)$-tournament, or

(b) D_{21} consists of a single arc and D_{22} consists of $n-3$ independent vertices.

These two special cases will be handled separately below. Note that the latter $n-3$ candidates chosen above are closer to $C_d(T_2)$ than the former two candidates by exactly distance one. Take $V(T_2)$ as the set of voters in T_2. Then by Theorem A, T_2 realizes D_2. Let T be the tree obtained from two vertex-disjoint copies of T_2 by adding a new edge connecting two copies of u, each of which becomes a centroid vertex of T. As the set of candidates in T, choose a copy of a vertex of D_2 in one copy of T_2 together with all vertices of D_2 in the other copy of T_2. Take $V(T)$ as the set of voters in T. Then T realizes D. Moreover, by the induction hypothesis,

$$\begin{aligned}|V(T)| &= 2|V(T_2)| = 2(2|V(T_{22})| + 2) \leqslant 2|V(T_{22}^*)| + 4 \\ &\leqslant 2(16 \cdot 3^{(n-3+1-4)/2} - 2) + 4 < 16 \cdot 3^{(n-4)/2} - 2.\end{aligned}$$

In special case (a), define trees H_k, $k \geqslant 1$, as follows: H_1 is K_2, and for all $k \geqslant 1$, H_{k+1} is obtained from vertex-disjoint copies of H_k and $K_{1,k}$ by adding one new edge between a vertex in $C_d(H_k)$ and vertex in $C_d(K_{1,k})$. So, in H_{n-3}, if those $n-3$ vertices which are adjacent to one fixed centroid vertex are considered as candidates and if all vertices are considered as voters, then H_{n-3} realizes D_{22}, the transitive $(n-3)$-tournament. Also, $|V(H_{n-3})| = \frac{1}{2}(n^2 - 5n + 8)$. Let T_2 be the tree obtained from two vertex-disjoint copies of H_{n-3}, denoted $H_{n-3}^{(1)}$ and $H_{n-3}^{(2)}$, by adding a path of length two to a vertex v_1 in $C_d(H_{n-3}^{(1)})$, a new end vertex adjacent to a vertex v_2 in $C_d(H_{n-3}^{(2)})$, and a new vertex u adjacent to both v_1 and v_2. Note that $C_d(T_2) = \{u, v_1\}$. Now, in T_2 consider candidate vertices to be the copy of $V(D_{22})$ in $H_{n-3}^{(1)}$ together with the two endvertices of T_2 adjacent to v_2. Take $V(T_2)$ as the set of voters. Then T_2 realizes D_2. Form T as above. Then T realizes D, and

$$|V(T)| = 2\,|V(T_2)| = 2(2\,|V(H_{n-3})| + 4)$$
$$= 2n^2 - 10n + 24 < 16 \cdot 3^{(n-4)/2} - 2.$$

Special case (b) can be treated similarly, where H_{n-3} is taken to be $K_{1,n-3}$. If the $n-3$ endvertices of H_{n-3} are considered as candidates and all $n-2$ vertices of H_{n-3} are considered as voters, then H_{n-3} realizes D_{22}. The same construction of T_2 and T as in (a) yields a tree which realizes D, and

$$|V(T)| \leq 4\,|V(H_{n-3})| + 8 = 4(n-2) + 8 < 16 \cdot 3^{(n-4)/2} - 2.$$

Subsubcase B.2.3: Suppose that $\alpha_2 = 1$ (The case $\alpha_1 = 1$ can be treated similarly.) Then $\alpha_1 = n - 1 - \alpha_2 = n-2$ is even. Let $V(D_{22}) = \{z\}$. If D_{21} is connected, then by Lemma 2.5 in [4], $D_{21} = Y \Rightarrow X$, for some subdigraphs X and Y. So, $D_2 = (Y \cup \{z\}) \Rightarrow X$, and then this case is contained in previous cases unless $|V(X)| = 1$. If $|V(X)| = 1$ and Y is connected, then again by Lemma 2.5 in [4], $Y = Y_2 \Rightarrow Y_1$ for some subdigraphs Y_1 and Y_2 of Y, and $D_2 = (Y_1 \cup \{z\}) \Rightarrow (Y_1 \cup X)$ can be treated as in previous cases. There remains the special instance where $|V(X)| = 1$ and Y is disconnected. Note that $|V(Y)| = n-3$. If $Y = \{y_1\} \cup \{y_2\} \cup \cdots \cup \{y_{n-3}\}$, then the tree T_2 illustrated in Fig. 6 realizes D_2.

Let T be the tree obtained from two copies of T_2 by adding an edge connecting the two copies of c. Then T realizes D and

$$|V(T)| = 2(|V(T_2)|) = 4n - 2 \leq 16 \cdot 3^{(n-4/2)} - 2.$$

So, we may assume that Y contains a component with at least two vertices. Let $T(Y)$ be a tree with a single centroid vertex c which realizes Y and is of the smallest order. Let $C_1, C_2, \ldots, C_k$ be the components of $T(Y) - c$. As in the proof of Lemma 3.2,

$$|V(C_i)| = \frac{1}{k}(|V(Y)| - 1) \qquad (1 \leq i \leq k).$$

Note that $|V(C_i)| \geq 3$. Construct T as in that proof, except that not only one copy C_{k+1}, of C_1 is added, but two copies of C_1 are added, denoted C_{k+1} and C_{k+2}. To complete the construction of T delete one endvertex of C_{k+1} and add a new

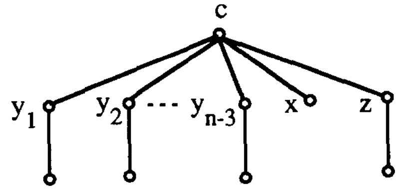


Fig. 6.

vertex adjacent to the vertex of C_{k+2} which is a copy of c_1. Choose a vertex of C_{k+1} (respectively C_{k+2}) which is a copy of a vertex of Y to be x (respectively to be z). Consider the vertices $T(Y)$ in T which represent vertices of Y, together with x and z as the set of candidates, and consider $V(T)$ as the set of voters in T. Then by Theorem A, T realizes D_2, and T has a single centroid vertex c, and all vertices of D_2 are at the same distance from c. Now, let T' be the tree obtained from two copies of T by adding an edge connecting the two copies of c. Clearly T' realizes D, and by the induction hypothesis (since $|V(Y)| = n-3$ is odd),

$$\begin{aligned} |V(T')| &= 2\,|V(T)| \leqslant 2(|V(T(Y))| + \frac{2}{k}(|V(T(Y))| - 1) \\ &= \frac{2(k+2)}{k}\,|V(T(Y))| - \frac{4}{k} \leqslant \frac{2(k+2)}{k}(10 \cdot 3^{(n-6)/2} - 1) - \frac{4}{k} \\ &= \frac{2(k+2)}{3k}\, 10 \cdot 3^{(n-4)/2} - 2 - \frac{8}{k} \leqslant 16 \cdot 3^{(n-4)/2} - 2. \end{aligned}$$

Therefore, without loss of generality, assume that D_{21} is disconnected. Let T_{21} be a tree, of the smallest order with a single centroid vertex, which realizes D_{21}. By Lemma 3.2, there exists a tree T_2 with a single centroid vertex, say c, realizing D_2 so that for some integer $k \geqslant 2$,

$$|V(T_2)| \leqslant \frac{k+1}{k}\,|V(T_{21})| - \frac{1}{k} + 1. \tag{4}$$

Let T be the tree obtained from T_2 and a copy of T_2 by adding an edge joining c and its copy. As the set of candidates in T, choose one vertex of D_2 in one copy of T_2, together with a copy of $V(D_2)$ in the other copy of T_2. Take $V(T)$ as the set of voters. Then by Theorem A, T realizes D. By the induction hypothesis and (4),

$$\begin{aligned} |V(T)| &= 2\,|V(T_2)| \leqslant \frac{2(k+1)}{k}\,|V(T_{21})| - \frac{2(1-k)}{k} \\ &\leqslant \frac{2(k+1)}{k}(16 \cdot 3^{(n-2-4)/2} - 1) - \frac{2(1-k)}{k} \\ &\leqslant \frac{2(k+1)}{3k} \cdot 16 \cdot 3^{(n-4)/2} - \frac{4}{k} \leqslant 16 \cdot 3^{(n-4)/2} - 2. \end{aligned}$$

This completes the proof that $\alpha(D) \leqslant 16 \cdot 3^{(n-4)/2} - 2$ if n is even.

Arguments similar to those in the case when n is even can be used in the case when n is odd. That is, if n is odd, then

$$\alpha(D) \leqslant 10 \cdot 3^{(n-3)/2} - 2.$$

The proof is complete. □

Theorem 3.2. *For any positive integer n,*

$$\alpha(\mathscr{F}_n) = \begin{cases} 1, & \text{if } n = 1, \\ 3, & \text{if } n = 2, \\ 10 \cdot 3^{(n-3)/2} - 2, & \text{if } n \text{ is odd and } n \geqslant 3, \\ 16 \cdot 3^{(n-4)/2} - 2, & \text{if } n \text{ is even and } n \geqslant 4. \end{cases}$$

Proof. It is easy to check that $\alpha(\mathscr{F}_1) = 1$ and $\alpha(\mathscr{F}_2) = 3$. For $n \geqslant 3$, the result follows from Remarks 2.1, 2.2, and Theorem 3.1. □

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