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The effective dimension and quasi-Monte Carlo integration ☆

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Abstract

Quasi-Monte Carlo (QMC) methods are successfully used for high-dimensional integrals arising in many applications. To understand this success, the notion of *effective dimension* has been introduced. In this paper, we analyse certain function classes commonly used in QMC methods for empirical and theoretical investigations and show that the problem of determining their effective dimension is analytically tractable. For arbitrary square integrable functions, we propose a numerical algorithm to compute their truncation dimension. We also consider some realistic problems from finance: the pricing of options. We study the special structure of the corresponding integrands by determining their effective dimension and show how large the effective dimension can be reduced and how much the accuracy of QMC estimates can be improved by using the Brownian bridge and the principal component analysis techniques. A critical discussion of the influence of these techniques on the QMC error is presented. The connection between the effective dimension and the performance of QMC methods is demonstrated by examples.

Keywords: Effective dimension; Quasi-Monte Carlo methods; Low discrepancy sequences; Multivariate integration; Dimension reduction

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1. Introduction

There has been an increasing interest in studying high-dimensional integration due to its numerous applications in physics, statistics and finance (see [1–4,23,25]). Consider the problem of approximating the integral

$$I(f) = \int_{C^d} f(\mathbf{x})\, d\mathbf{x}, \quad \mathbf{x} = (x_1, \ldots, x_d),$$

where $C^d = [0,1)^d$ is the d-dimensional unit cube. It has long been known that multivariate integration is subject to the *curse of dimensionality*, making the classical quadratures infeasible for use in dimensions beyond say 5 or 6. Monte Carlo (MC) methods are useful tools to break the curse of dimensionality. MC methods use the sample mean

$$Q(f) = \frac{1}{n} \sum_{i=1}^{n} f(\mathbf{x}_i)$$

to approximate $I(f)$, where the points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are independent and identical distributed random draws from the uniform distribution on C^d. The MC error is of order $O(n^{-1/2})$ for square integrable functions independently of the dimension.

Quasi-Monte Carlo (QMC) methods are deterministic versions of MC methods. There are two important classes of point sets that are well suited to multivariate integration: digital nets (or sequences) and lattice rules [15,27]. The Koksma–Hlawka inequality yields the QMC error bound

$$|I(f) - Q(f)| \leqslant V_{\mathrm{HK}}(f) \mathscr{D}^*(P),$$

where $\mathscr{D}^*(P)$ is the star discrepancy of $P = \{\mathbf{x}_i\}$ and $V_{\mathrm{HK}}(f)$ is the variation in the sense of Hardy and Krause. Several digital sequences are known to have star discrepancy $O(n^{-1}(\log n)^d)$. Hence, QMC integration based on a digital sequence has a deterministic error bound in the order $O(n^{-1}(\log n)^d)$, which is *asymptotically* better than that of MC. Thus for fixed d and sufficiently large n, QMC is superior to MC. But when d is large, the factor $n^{-1}(\log n)^d$ is substantially larger than $n^{-1/2}$ unless n is huge. Similar situation occurs for lattice rules. Therefore, it was widely believed that QMC methods should not be used for high-dimensional integration, say, for $d \geqslant 15$.

However, Paskov and Traub [25] found empirically that QMC methods are superior to MC for high-dimensional integrals arising in finance (up to $d = 360$ in their examples). Many other numerical experiments also showed that the order of convergence of QMC in these problems is roughly n^{-1} independently of the dimension (see, for example, [16,24]). See [36] for a survey of the state of the art. It is a challenging problem to understand the apparent success of QMC for high-dimensional integrals. There are several ways to explain this. Paskov and Traub [25], Caflisch et al. [3] and Paskov [24] used the concept of *effective dimension* and argued that the performance of QMC integration is intimately related to the effective dimension of the problems. Sloan and Woźniakowski [31,32] used the notions of

tractability and *strong tractability* and showed that there exists QMC algorithm for which the curse of dimensionality is not present in some *weighted* function classes. Papageorgiou [20] and Owen [19] investigated some isotropic integrals and showed the superiority of QMC methods.

As can be seen from the Koksma–Hlawka inequality (and its various generalizations [5]), the efficiency of an algorithm for computing $I(f)$ depends on both the algorithm and on the integrand. In this paper, we focus on the aspect of integrand. Currently, little is known about the effective dimension of various problems. A clear understanding of how QMC error depends on the effective dimension is important. It would be interesting to know exactly the effective dimension of the problems at hand (such as these in empirical and theoretical studies of QMC methods and these in computational finance). Some attempts have been made in [6,19].

The main objectives of this paper are threefold: (1) to provide ways to analyse the effective dimension for some functions; (2) to develop numerical algorithms for determining the effective dimension of an arbitrary square integrable function; (3) to compare the performance of dimension reduction techniques, such as the *Brownian bridge* (BB) and the *principal component analysis* (PCA) techniques.

This paper is organized as follows. In Section 2, after introducing the notion of effective dimension and discussing its relationships to the integration errors and approximation errors, we propose a numerical algorithm to compute the effective dimension for arbitrary square integrable function. In Section 3, function classes commonly used in QMC methods for empirical and theoretical investigations are analysed. It is shown that the problem of determining the effective dimension for such functions is analytically tractable. In particular, it is shown that for the weighted Korobov spaces with the weights satisfying the strong tractability conditions, the corresponding effective dimension is small relative to the nominal dimension. In Section 4, we consider some realistic problems from finance: the pricing of Asian options and multi-asset options. We study the special features of these problems and compare the BB and PCA techniques from the point of view of effective dimension. The connections of effective dimension with the performance of QMC algorithms is demonstrated by examples. A critical discussion of the influence of the dimension reduction techniques on the QMC error is presented.

2. Effective dimension and its determination

2.1. The ANOVA decomposition

ANOVA decomposition is a way of decomposing a function into a sum of simpler functions and has been studied by many authors in statistics (see references in [19]) and in QMC methods [6,11,18,19,35].

Let $\mathscr{S} = \{1, \ldots, d\}$. For any subset $u \subseteq \mathscr{S}$, let $|u|$ denote its cardinality and $\mathscr{S} - u$ denote its complementary set in $\mathscr{S}$. Let $\mathbf{x}_u$ be the $|u|$-dimensional vector containing the coordinates of $\mathbf{x}$ with indices in u. Furthermore, let C^u denote the $|u|$-dimensional unit cube involving the coordinates in u (so $C^{\mathscr{S}}$ is the same as $C^d = [0,1)^d$).

Assume that $f(\mathbf{x})$ is a square integrable function. We can write $f(\mathbf{x})$ as the sum of its 2^d ANOVA terms:

$$f(\mathbf{x}) = \sum_{u \subseteq \mathscr{S}} f_u(\mathbf{x}). \tag{1}$$

The ANOVA terms $f_u(\mathbf{x})$ are defined recursively by

$$f_u(\mathbf{x}) = \int_{C^{\mathscr{S}-u}} f(\mathbf{x})\, d\mathbf{x}_{\mathscr{S}-u} - \sum_{v \subset u} f_v(\mathbf{x}), \tag{2}$$

with $f_\emptyset = \int_{C^d} f(\mathbf{x})\, d\mathbf{x} = I(f)$. The sum in (2) is over strict subsets $v \neq u$ (we use the convention $\int_{C^0} f(\mathbf{x})\, d\mathbf{x}_\emptyset = f(\mathbf{x})$). The ANOVA term $f_u(\mathbf{x})$ is the part of the function depending only on the variables x_j with $j \in u$. The ANOVA terms enjoy some interesting properties:

(1) $\int_0^1 f_u(\mathbf{x})\, dx_j = 0$ for $j \in u$.
(2) The ANOVA decomposition (1) is orthogonal in that $\int_{C^d} f_u(\mathbf{x}) f_v(\mathbf{x})\, d\mathbf{x} = 0$ whenever $u \neq v$.
(3) Let $\sigma^2(f) = \int_{C^d} f^2(\mathbf{x})\, d\mathbf{x} - [I(f)]^2$ be the variance of f, then $\sigma^2(f) = \sum_{u \subseteq \mathscr{S}} \sigma_u^2(f)$, where $\sigma_u^2(f) = \int_{C^d} [f_u(\mathbf{x})]^2\, d\mathbf{x}$ for $|u| > 0$ is the variance of f_u and $\sigma_\emptyset^2(f) = 0$.

2.2. The definitions of effective dimension

Let u be a subset of $\mathscr{S}$, the variance corresponding to u is defined as

$$D_u := \sum_{v \subseteq u} \sigma_v^2(f).$$

The *total-effect variance* corresponding to u is defined by

$$D_u^{\text{tot}} := \sum_{v \cap u \neq \emptyset} \sigma_v^2(f) = \sigma^2(f) - D_{\mathscr{S}-u}.$$

The total-effect variance D_u^{tot} characterizes the total contribution of the variable $\mathbf{x}_u$ to the variance of f: it includes the pure effect D_u, as well as all the effects due to its interactions with others.

The relative size of D_u or D_u^{tot} with respect to $\sigma^2(f)$ indicates the relative importance of $\mathbf{x}_u$. If D_u is close to $\sigma^2(f)$, then $\mathbf{x}_u$ affects the function singularly, its interactions with other variables are negligible. If D_u is small, while D_u^{tot} is large, then $\mathbf{x}_u$ influences the function mainly through interactions. If D_u^{tot} is small, then $\mathbf{x}_u$ has a negligible effect on the function and can be fixed at some values (in this case $\mathbf{x}_{\mathscr{S}-u}$ affects the function singularly). The relative size of $\sum_{|u| \leqslant m} \sigma_u^2(f)$ indicates the impact of all interactions of order no larger than m. The following two notions of effective dimension are introduced in [3].

Definition 1. The effective dimension of f in the superposition sense (or 'superposition dimension' in short) is the smallest integer d_s, such that

$$\sum_{0<|u|\leqslant d_s} \sigma_u^2(f) \geqslant p\sigma^2(f),$$

where p is the proportion with $0<p<1$ (p is taken to be close to 1).

Definition 2. The effective dimension of f in the truncation sense (or 'truncation dimension') is the smallest integer d_t such that

$$D_{\{1,\ldots,d_t\}} \geqslant p\sigma^2(f).$$

The idea of effective dimension appears in [25]. These notions are closely related to *sensitivity indices* (see [35]). The truncation dimension and superposition dimension are appropriate for different kind of functions. The truncation dimension is roughly the number of "important" variables. It indicates on how many variables we should pay our main attention. Truncation dimension is especially appropriate for the characterization of the "weighted" functions, where some variables are more important than others. On the other hand, superposition dimension is an indicator of whether low-order ANOVA terms dominate the function. It is especially useful in the case that all variables are equally important or almost equally important (in such cases the information about the truncation dimension is less useful). For example, for the isotropic function (see [19,20])

$$f(\mathbf{x}) = g\left(\sqrt{\textstyle\sum_{j=1}^{d}[\Phi^{-1}(x_j)]^2}\right),$$

where $\Phi(\cdot)$ is the standard normal distribution function and $g: R \to R$ (in [23] the function $g(\cdot) = \cos(\cdot)$), all the variables are equally important, it may be not interesting to discuss its truncation dimension, but it is useful to know its superposition dimension [19].

The effective dimension depends on p. For the same p, we always have $d_s \leqslant d_t$. But the converse is not true. For example, the function $f(\mathbf{x}) = x_1 + \cdots + x_d$ has superposition dimension 1, but may have much larger truncation dimension. Many functions in practice have large truncation dimension, but have small superposition dimension. Especially, such a feature is typical for a number of financial problems (see [29]). In general, it is hard to compute the effective dimension for an arbitrary function. To overcome this difficulty, some interesting variants (such as *effective dimension of a function space* and *dimension distribution*) are introduced in [6,19].

2.3. QMC integration error and effective dimension

It is important to know how QMC error depends on the effective dimension. Under suitable definitions of discrepancy and variation (see [3,5,6,19]), we have

$$|Q(f) - I(f)| \leqslant \sum_{u\subseteq\mathscr{S}} \mathscr{D}_u(P_u)\|f_u\|, \tag{3}$$

where P_u is the projection of the point set P on C^u, $\mathscr{D}_u(P_u)$ is the discrepancy of P_u and $||f_u||$ is the variation of f_u. Thus, the QMC error depends on the uniformity of all the projections P_u and all the low-dimensional parts f_u.

The basic properties of low discrepancy point sets are their "better" uniformity than that of random points. But the "better" uniformity is not preserved for all dimensions and for all projections. The following facts are useful to understand the possible advantage and potential problem of QMC.

First, at least the first coordinates of low discrepancy point sets have better distribution properties than random points do. More precisely, for small l (say $l \leqslant 10$), the term $\mathscr{D}_u(P_u)$ with $u \subseteq \{1, \ldots, l\}$ for a low discrepancy point set is much smaller than that for random points. But for other subset u (e.g., when $|u|$ is large or u contains large indices), the results for a low discrepancy point set could be worse than that for random points, unless n is extremely large.

Second, the low-order projections (i.e., $|u|$ is small) of low discrepancy point sets have better distribution property "on the average" than random points do. Many low discrepancy sequences have some poor projections even for small $|u|$. In fact, two-dimensional projections with bad distribution properties have been observed for several common low discrepancy sequences (see, for example, [14]). The non-uniformity in the higher order projections is a more serious problem. This is an indication of potential problem in using QMC. However, for small $|u|$, most $|u|$-dimensional projections of low discrepancy point sets are good ones. In general, it is shown in [30] that for d in the range of 10–100, if l is small (say $l \leqslant 3$), then the *superposition discrepancy*

$$\mathscr{D}_{(l)}(P) := \left(\sum_{|u|=l} [\mathscr{D}_u(P_u)]^2 \right)^{1/2}$$

of a low discrepancy point set is smaller than that of random points. But this superiority decreases as l and d increase. For large l (say $l > 3$ and $d > 30$), the superposition discrepancies of low discrepancy point sets and random points are almost the same, unless n is huge.

Suppose that f has truncation dimension d_t. Rewrite (3) as

$$|Q(f) - I(f)| \leqslant \sum_{u \subseteq \{1,\ldots,d_t\}} \mathscr{D}_u(P_u)||f_u|| + \sum_{u \cap (\mathscr{S} - \{1,\ldots,d_t\}) \neq \emptyset} \mathscr{D}_u(P_u)||f_u||. \tag{4}$$

If d_t is small (say $d_t \leqslant 10$), then the quantities $\mathscr{D}_u(P_u)$ involved in the first sum on the right-hand side of (4) are much smaller for QMC than for MC. For the subset u involved in the second term of (4), $||f_u||$ is often small. Although it may happen that $\mathscr{D}_u(P_u)$ is larger for QMC than for MC, but it is multiplied by a small $||f_u||$. So all terms in (4) can be expected to be small for QMC. Therefore, if f has low truncation dimension, we have good reason to expect an improvement of QMC over MC. Note that if the second sum in (4) is smaller for MC than for QMC, then a "mixed" point set (with the first d_t coordinates being the leading coordinates of a low discrepancy point set and the remaining dimensions being the random numbers) has the potential to improve pure QMC.

Now suppose that f has superposition dimension d_s, then we have

$$|Q(f)-I(f)| \leqslant \sum_{|u|\leqslant d_s} \mathscr{D}_u(P_u)||f_u|| + \sum_{|u|>d_s} \mathscr{D}_u(P_u)||f_u||$$

$$\leqslant \sum_{l=1}^{d_s} \mathscr{D}_{(l)}(P)\left(\sum_{|u|=l} ||f_u||^2\right)^{1/2} + \sum_{|u|>d_s} \mathscr{D}_u(P_u)||f_u||.$$

Clearly, the QMC error has a strong dependence on the superposition dimension d_s. If d_s is small (say $d_s \leqslant 3$), then the superposition discrepancy $\mathscr{D}_{(l)}(P)$ (with $l \leqslant d_s$) of a low discrepancy point set is smaller than that of random points. Similar arguments as above lead to similar conclusion. Thus if f has small superposition dimension d_s (even if the truncation dimension is large), we still have good reason to expect that QMC will be more efficient than MC.

Note that effective dimension only provide partial information about the difficulty in approximating integrals. Small effective dimension *does not suffice* to guarantee the effectiveness of QMC. For example, since bad two-dimensional projections are not rare for several common low discrepancy sequences (especially for relative small n and large d), so integrating functions with strong dependence on just the two dimensions may lead to bad result (such functions could have superposition dimension only 2 or 1). We could also have function $f(\mathbf{x})$ with small superposition dimension and the points $\{\mathbf{x}_i\}$ with good low dimensional projections, but still get a poor QMC result. Effective dimension has a strong influence on QMC error. However, other factors, such as the regularity of the integrand (and the regularity of its low-dimensional part), the dimension d, the point set and the sample size, may play a very important role in the QMC error.

The arguments in this subsection are based on the error bound (3). In some cases an entirely different approach is possible. For example, it is shown in [20] that the QMC error for some isotropic problems depends on the uniformity of the norms of the sample points (this is referred to as *radial discrepancy* in [19]) and a fast QMC convergence order $O(\sqrt{\log n}/n)$ is proved. These ideas have been extended in [22].

2.4. Effective dimension and the approximation errors

Effective dimension has close relation to the approximation error. Let $h(\mathbf{x})$ be an approximation to $f(\mathbf{x})$. The normalized approximation error is defined by

$$\mathrm{Err}(f,h) = \frac{1}{\sigma^2(f)} \int_{C^d} [f(\mathbf{x}) - h(\mathbf{x})]^2 \, d\mathbf{x}.$$

Two simple ways for approximating $f(\mathbf{x})$ are:

(1) Freezing the nonessential variables (see [34]): $f(\mathbf{x}) \approx f(\mathbf{x}_u, \mathbf{z}_0)$, where $\mathbf{z}_0$ is some fixed point in $C^{\mathscr{S}-u}$.

(2) Deleting the high-order ANOVA terms in the decomposition (1):

$$f(\mathbf{x}) \approx \sum_{|v|\leqslant l} f_v, \quad \text{for some } 0 \leqslant l \leqslant d.$$

The theorems below indicate that the effective dimension of a function is related to the ability of approximating the function by a low-dimensional function or a sum of low-dimensional functions. If f has truncation dimension d_t, then $D_{\{1,\dots,d_t\}}$ is close to $\sigma^2(f)$ and $D^{\text{tot}}_{\{d_t+1,\dots,d\}}$ is close to zero. In this case $f(\mathbf{x})$ can be well approximated by a d_t-dimensional function by freezing the variables $x_{d_t+1}, \dots, x_d$ to some value within their range. The next theorem is a direct consequence of the result of Sobol [34].

Theorem 1. *Assume that the function $f(\mathbf{x})$ has truncation dimension d_t in proportion p and let $u_0 = \{1, \dots, d_t\}$. Let $h(\mathbf{x}) = f(\mathbf{x}_{u_0}, \mathbf{z})$ with the freezing variable $\mathbf{z}$ uniformly distributed in the $(d - d_t)$-dimensional unit cube $C^{\{d_t+1,\dots,d\}}$. Then for arbitrary $\varepsilon > 0$, with probability $\geqslant 1 - \varepsilon$ we have*

$$\text{Err}(f, h) < (1 + \varepsilon^{-1})(1 - p).$$

If $f(\mathbf{x})$ has superposition dimension d_s, then $f(\mathbf{x})$ can be well approximated by a sum of d_s-dimensional functions as shown below.

Theorem 2. *Assume that $f(\mathbf{x})$ has superposition dimension d_s in proportion p. Let $h(\mathbf{x}) = \sum_{|u| \leqslant d_s} f_u(\mathbf{x})$. Then*

$$\text{Err}(f, h) \leqslant (1 - p).$$

Proof. First, since $f(\mathbf{x})$ has superposition dimension d_s in proportion p, then

$$\sum_{|u| \leqslant d_s} \sigma_u^2(f) \geqslant p\sigma^2(f). \tag{5}$$

Second, from the ANOVA decomposition (1), we have

$$f(\mathbf{x}) - h(\mathbf{x}) = \sum_{|u| > d_s} f_u(\mathbf{x}).$$

By squaring and integrating over C^d, using the orthogonality of the ANOVA terms, we obtain

$$\int_{C^d} [f(\mathbf{x}) - h(\mathbf{x})]^2 \, d\mathbf{x} = \sum_{|u| > d_s} \sigma_u^2(f) = \sigma^2(f) - \sum_{|u| \leqslant d_s} \sigma_u^2(f) \leqslant (1 - p)\sigma^2(f),$$

where the inequality follows from (5). □

2.5. The algorithm for determining effective dimension

The calculation of effective dimension is related to the computation of sensitivity indices [34,35]. The key is to compute the variance D_u corresponding to any subset u of $\mathscr{S}$. For any fixed $u \subseteq \mathscr{S}$, let $\mathbf{x} = (\mathbf{x}_u, \mathbf{x}_{\mathscr{S}-u})$ and $\mathbf{y} = (\mathbf{y}_u, \mathbf{y}_{\mathscr{S}-u})$. The result of

Sobol [34,35] leads to the following relation

$$D_u = \int f(\mathbf{x}) f(\mathbf{x}_u, \mathbf{y}_{\mathscr{S}-u})\, d\mathbf{x}\, d\mathbf{y}_{\mathscr{S}-u} - [I(f)]^2, \tag{6}$$

where the integration is over the $(2d - |u|)$-dimensional unit cube (integrals below without an explicit domain is considered to be over appropriate unit cube). To gain some insight and for completeness, the proof is given here briefly. In fact,

$$\begin{aligned}
&\int f(\mathbf{x}) f(\mathbf{x}_u, \mathbf{y}_{\mathscr{S}-u})\, d\mathbf{x}\, d\mathbf{y}_{\mathscr{S}-u} \\
&\quad = \int d\mathbf{x}_u \int f(\mathbf{x}_u, \mathbf{x}_{\mathscr{S}-u})\, d\mathbf{x}_{\mathscr{S}-u} \int f(\mathbf{x}_u, \mathbf{y}_{\mathscr{S}-u})\, d\mathbf{y}_{\mathscr{S}-u} \\
&\quad = \int d\mathbf{x}_u \left[\int f(\mathbf{x}_u, \mathbf{y}_{\mathscr{S}-u})\, d\mathbf{y}_{\mathscr{S}-u} \right]^2.
\end{aligned}$$

From the ANOVA decomposition (1) and the property that $\int_0^1 f_u(\mathbf{x})\, dx_j = 0$ for $j \in u$, we have

$$\int f(\mathbf{x}_u, \mathbf{y}_{\mathscr{S}-u})\, d\mathbf{y}_{\mathscr{S}-u} = \sum_{v \subseteq u} f_v(\mathbf{x}).$$

By squaring and integrating over $d\mathbf{x}_u$, we have

$$\int f(\mathbf{x}) f(\mathbf{x}_u, \mathbf{y}_{\mathscr{S}-u})\, d\mathbf{x}\, d\mathbf{y}_{\mathscr{S}-u} = f_\emptyset^2 + \sum_{v \subseteq u} \sigma_v^2 = f_\emptyset^2 + D_u,$$

which is equivalent to (6).

Thus for computing D_u and the variance $\sigma^2(f)$ one needs to estimate the following three types of integrals:

$$\int f(\mathbf{x})\, d\mathbf{x}, \quad \int f^2(\mathbf{x})\, d\mathbf{x}, \quad \int f(\mathbf{x}) f(\mathbf{x}_u, \mathbf{y}_{\mathscr{S}-u})\, d\mathbf{x}\, d\mathbf{y}_{\mathscr{S}-u}.$$

All these integrals can be computed by QMC (or MC). Let $(\mathbf{x}_i, \mathbf{y}_i)$, $i = 1, 2, \ldots$, be a low discrepancy sequence of points in the $2d$-dimensional unit cube C^{2d}. For $u \subseteq \mathscr{S}$, write

$$\mathbf{x}_i = ((\mathbf{x}_i)_u, (\mathbf{x}_i)_{\mathscr{S}-u}) \quad \text{and} \quad \mathbf{y}_i = ((\mathbf{y}_i)_u, (\mathbf{y}_i)_{\mathscr{S}-u}).$$

We have the following approximations based on the QMC method:

$$\hat{f}_\emptyset = \frac{1}{n} \sum_{i=1}^n f(\mathbf{x}_i), \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n f^2(\mathbf{x}_i) - \hat{f}_\emptyset^2,$$

$$\hat{D}_u = \frac{1}{n} \sum_{i=1}^n f((\mathbf{x}_i)_u, (\mathbf{x}_i)_{\mathscr{S}-u}) f((\mathbf{x}_i)_u, (\mathbf{y}_i)_{\mathscr{S}-u}) - \hat{f}_\emptyset^2.$$

Thus the truncation dimension can be determined by computing D_u with $u = \{1, \ldots, l\}$ for $l = 1, 2, \ldots$, until the inequality in Definition 2 is satisfied. We need to compute one integral for the mean, one for the variance, plus at most $(d - 1)$ integrals for $\hat{D}_u$.

The numerical computation of the superposition dimension for an arbitrary function is more complicated. In this case, we have to compute the variance term $\sigma_u^2(f)$ for every subset u under consideration. Moreover, direct use of (6) often lead to a loss of accuracy in computing $\sigma_u^2(f)$ when $|u|$ is large (say $|u|>3$). The variance fraction of two-dimensional structure in some problems from finance is estimated by quasi-regression [11]. Some other attempts have been made in [29].

3. Effective dimension of multiplicative functions

The purpose of this section is threefold: to analyse the effective dimension for functions with multiplicative structure; to test the numerical algorithm given above and to analyse the effective dimension of the weighted Korobov spaces.

3.1. Test functions: analytical and numerical results

Consider a class of test functions:

$$f(\mathbf{x}) = \prod_{k=1}^{d} \frac{|4x_k - 2| + a_k}{1 + a_k}, \tag{7}$$

where a_k are parameters. Such functions allow an automatic tuning of the relative importance of the variables, as well as of their interactions, by appropriate choices of a_k. QMC algorithms for such functions have quite different performance [26,39]. The reason will be clear soon.

The value of the integral and the variance of f can be computed analytically: $I(f) = 1$, $\sigma^2(f) = \prod_{k=1}^{d}[1 + \frac{1}{3(1+a_k)^2}] - 1$. The ANOVA terms and the corresponding variances are:

$$f_\emptyset = 1, \quad f_u = \prod_{k\in u} \frac{|4x_k - 2| - 1}{1 + a_k}, \ |u|>0,$$

$$\sigma_\emptyset(f) = 0, \quad \sigma_u^2(f) = \prod_{k\in u} \frac{1}{3(1 + a_k)^2}, \ |u|>0. \tag{8}$$

The quantities involved in the definitions of effective dimension can be computed analytically. First, based on (6), we have

$$D_u = \int f(\mathbf{x}) f(\mathbf{x}_u, \mathbf{y}_{\mathscr{S}-u})\, d\mathbf{x}\, d\mathbf{y}_{\mathscr{S}-u} - f_\emptyset^2 = \prod_{k\in u}\left[1 + \frac{1}{3(1 + a_k)^2}\right] - 1.$$

Second, for $l = 1, 2, \ldots, d$, from (8) it follows that

$$\sum_{0<|u|\leqslant l} \sigma_u^2(f) = \sum_{0<|u|\leqslant l} \prod_{k\in u} \frac{1}{3(1 + a_k)^2} = \sum_{m=1}^{l} \sum_{|u|=m} \prod_{k\in u} \frac{1}{3(1 + a_k)^2}. \tag{9}$$

Such quantities can be computed recursively. To show this, consider a sequence of numbers $c_1, c_2, \ldots$. Define

$$T(i,m) = \sum_{u\subseteq\{1,\ldots,i\},|u|=m} \prod_{j\in u} c_j, \text{ for } i = 1,2,\ldots,d \text{ and } m = 1,2,\ldots,i.$$

We view T as a $d\times d$ lower triangular matrix. Obviously, $T(i,1) = \sum_{j=1}^{i} c_j$ and $T(i,i) = \prod_{j=1}^{i} c_j$ for $i = 1,2,\ldots,d$. It is easy to prove the following recursive relation:

$$T(i,m) = T(i-1,m) + c_i T(i-1,m-1), \quad \text{for } i\geqslant 3, m\geqslant 2. \tag{10}$$

Using this recursive relation and the initial values of $T(i,1)$ and $T(i,i)$, we can easily compute the elements of the last row of the matrix T: $T(d,1), T(d,2), \ldots, T(d,d)$, as well as the sums $\sum_{m=1}^{l} T(d,m)$ for $l = 1,2,\ldots,d$. The sums in (9) can be computed in this way.

Three choices of the parameters will be considered:

$$\text{(a)}\quad a_1 = \cdots = a_d = 1;\ \text{(b)}\quad a_k = k,\ 1\leqslant k\leqslant d;\ \text{(c)}\quad a_k = k^2,\ 1\leqslant k\leqslant d.$$

The effective dimension is computed by using the analytical formulas and the numerical algorithm in the previous section (the purpose of using numerical algorithm is to test its accuracy). In the latter case, we use the Sobol sequence [33] (or mixed sequence, if the dimension is larger than 100) with $n = 2^{10}$ for the computations of the integrals involved. The results are given in Table 1. We see that the numerical algorithm is quite accurate.

Observe that for (a), all variables are equally important, the truncation dimension is approximately the same as the nominal dimension, this is the most difficult case for numerical integration. QMC works badly for such integrands in dimension $d>20$, see [39]. For (b), the importance of the successive variables is decreasing. The truncation dimension is smaller than the nominal dimension and the superposition dimension is only 2. QMC works better than MC (note that the truncation dimension is not small). For (c), the importance of the successive variables is decreasing quickly. The effective dimension in both senses is very small. One can expect that QMC will be much more efficient than MC. The computational

Table 1
The effective dimension (with $p = 0.99$) for the test function (7). Values in the parentheses are the results obtained by numerical algorithm

Dimension	$a_k = 1$		$a_k = k$		$a_k = k^2$	
d	d_t	d_s	d_t	d_s	d_t	d_s
10	10 (10)	3	10 (10)	2	5 (5)	2
20	20 (20)	5	18 (19)	2	5 (5)	2
40	40 (39)	8	33 (34)	2	5 (5)	2
80	80 (79)	12	55 (56)	2	5 (5)	2
100	100 (100)	14	64 (66)	2	5 (5)	2
200	200 (198)	25	93 (95)	2	5 (5)	2

experiments in [39] confirm this. See also [26]. Thus the efficiency of QMC strongly depends on the effective dimension.

3.2. General multiplicative functions

The method presented above can be extended to more general functions with multiplicative structure. Suppose that

$$f(\mathbf{x}) = \prod_{k=1}^{d} g_k(x_k),$$

and

$$\mu_k := \int g_k(x)\,dx, \quad \lambda_k^2 := \int (g_k(x) - \mu_k)^2\,dx < \infty.$$

It is obvious that $I(f) = \prod_{k=1}^{d} \mu_k$ and

$$\sigma^2(f) = [I(f)]^2 \left[\prod_{k=1}^{d} \left(1 + \frac{\lambda_k^2}{\mu_k^2}\right) - 1 \right].$$

As shown in [19], the ANOVA terms and the corresponding variances are (for $u \neq \emptyset$)

$$f_u = \prod_{k \in u} (g_k(x_k) - \mu_k) \cdot \prod_{k \notin u} \mu_k,$$

$$\sigma_u^2(f) = \prod_{k \in u} \lambda_k^2 \prod_{k \notin u} \mu_k^2 = [I(f)]^2 \prod_{k \in u} \frac{\lambda_k^2}{\mu_k^2}.$$

Therefore, the quantities involved in the definitions of effective dimension can be computed by the formulas:

$$D_u = \sum_{v \subseteq u} \sigma_v^2(f) = [I(f)]^2 \left[\prod_{k \in u} \left(1 + \frac{\lambda_k^2}{\mu_k^2}\right) - 1 \right],$$

$$\sum_{|u| \leqslant l} \sigma_u^2(f) = [I(f)]^2 \sum_{m=1}^{l} \sum_{|u|=m} \prod_{k \in u} \frac{\lambda_k^2}{\mu_k^2}, \quad l = 1, \ldots, d.$$

The latter one can be computed recursively. Many test functions used in QMC for empirical studies have the multiplicative structure (see [19]). Some functions from finance also have multiplicative structure [29].

3.3. The effective dimension of weighted Korobov spaces

To understand the success of QMC methods for high-dimensional integration, Sloan and Woźniakowski [31,32] used the notions of tractability and strong tractability. They introduced the weighted Sobolev spaces and the weighted Korobov spaces, in which the importance of the successive variables is increasingly limited. This dependence is controlled by a sequences of weights. They showed that

there exist QMC algorithms for which the curse of dimensionality is not present under certain conditions on the weights. More precisely, they established the necessary and sufficient conditions of tractability and strong tractability. In [7,37] it is shown that QMC algorithms based on some low discrepancy sequences achieve the optimal convergence order $O(n^{-1+\delta})$ for any $\delta>0$ in weighted Sobolev spaces independently of the dimension under appropriate conditions. We are interested in the question of how large the effective dimension of the weighted spaces of function is, especially in the case when the weights satisfy the tractability or strong tractability conditions.

Consider the d-dimensional weighted Korobov spaces, which are reproducing kernel Hilbert spaces having the reproducing kernels given by

$$K_{d,\alpha}(\mathbf{x},\mathbf{y})=\prod_{k=1}^{d}\left[\beta_k+\gamma_k\sum_{\substack{h=-\infty\\ h\neq 0}}^{\infty}\frac{e^{2\pi i h(x_k-y_k)}}{|h|^{\alpha}}\right],\quad \alpha>1,\ \ i=\sqrt{-1}, \tag{11}$$

where $\{\beta_k\}$ and $\{\gamma_k\}$ are two sequences of positive numbers. The smoothness parameter α characterizes the rate of decay of the Fourier coefficients.

Consider the case of $\alpha=2$. This case is important, since the weighted Sobolev spaces studied in [28,31,37] are related to the weighted Korobov spaces with $\alpha=2$ by using the tool of the shift-invariant kernels (see [6,8]). The complexity of integration over the weighted Sobolev spaces is related to that over certain weighted Korobov spaces with $\alpha=2$. Since the Fourier expansion of the Bernoulli polynomial $B_2(x)=x^2-x+1/6$ is given by

$$B_2(x)=\frac{1}{2\pi^2}\sum_{h\neq 0}\frac{e^{2\pi i h x}}{h^2},\quad x\in[0,1],$$

the kernel $K_{d,\alpha}(\mathbf{x},\mathbf{y})$ in (11) with $\alpha=2$ can be written as

$$K_{d,2}(\mathbf{x},\mathbf{y})=\prod_{k=1}^{d}[\beta_k+2\pi^2\gamma_k B_2(\{x_k-y_k\})],$$

where the notation $\{x\}$ means the fractional part of x.

Now consider the "typical functions" in the weighted Korobov space associated to the kernel $K_{d,2}(\mathbf{x},\mathbf{y})$. For any fixed $\mathbf{y}^*=(y_1^*,\ldots,y_d^*)\in C^d$, define a function of $\mathbf{x}$

$$f_{\mathbf{y}^*}(\mathbf{x}):=\prod_{k=1}^{d}[\beta_k+2\pi^2\gamma_k B_2(\{x_k-y_k^*\})].$$

Such functions have multiplicative structure. By direct computation we find that

$$\int_0^1 B_2(\{x_k-y_k^*\})\,dx_k=0\quad\text{and}\quad\int_0^1 B_2^2(\{x_k-y_k^*\})\,dx_k=\frac{1}{180},$$

Table 2
The effective dimension of the weighted Korobov space with $\alpha = 2$

d	$\gamma_j = 1$		$\gamma_j = 1/j$		$\gamma_j = 1/j^2$		$\gamma_j = 1/2^{(j-1)}$	
	d_t	d_s	d_t	d_s	d_t	d_s	d_t	d_s
10	10	10	10	4	4	2	5	3
20	20	18	19	4	5	2	5	3
50	50	42	42	5	5	2	5	3
100	100	79	71	5	5	2	5	3
200	200	152	109	5	5	2	5	3
500	500	366	161	5	5	2	5	3

for all $y_k^* \in [0, 1)$. Thus the corresponding parameters μ_k, λ_k defined in the previous subsection are

$$\mu_k = \int_0^1 [\beta_k + 2\pi^2 \gamma_k B_2(\{x_k - y_k^*\})]\, dx_k = \beta_k$$

and

$$\lambda_k^2 = \int_0^1 4\pi^4 \gamma_k^2 B_2^2(\{x_k - y_k^*\})\, dx_k = \frac{\pi^4}{45} \gamma_k^2.$$

Therefore, the effective dimension in both senses of the function $f_{\mathbf{y}^*}(\mathbf{x})$ is independent of $\mathbf{y}^*$ and can be computed using the analytical method given in the previous subsection. In our computations, we put

$$\text{(A)} \quad \gamma_j = 1, \quad \text{(B)} \quad \gamma_j = \frac{1}{j}, \quad \text{(C)} \quad \gamma_j = \frac{1}{j^2}, \quad \text{(D)} \quad \gamma_j = \frac{1}{2^{j-1}}$$

and $\beta_j = 1$ for all $j = 1, 2, \ldots, d$. Based on the theoretical results in [32], case (A) corresponds to the intractability of the problem, while (B) corresponds to tractability but not strong tractability of the problem. Both (C) and (D) correspond to the strong tractability of the problem. The effective dimension is given in Table 2 (again $p = 0.99$).

Choice (A) corresponds to the classical Korobov space, in which the multivariate integration is subject to the curse of dimensionality. We cannot expect that QMC works better than MC if d is large. In this case, both the truncation dimension and the superposition dimension are large.

The situation for case (B) is better than for (A). The truncation dimension is still large, but the superposition dimension is "moderate". This indicates that the high-order interactions are negligible. In this case the efficiency of QMC algorithms depends mainly on the uniformity of the lower-order projections (with order no larger than 5).

In cases (C) and (D), the effective dimension in both senses is very small and is independent of the nominal dimension. Thus, the faster convergence rate of QMC than MC should be expected. Note that case (D) has higher superposition dimension than case (C), also the truncation dimension for case (D) is never smaller than for (C)

and in one case is larger. This may seem surprising, since the weights in case (D) decrease more quickly than case (C). The explanation is the following. Note that the weights in these cases decay very fast, thus the effective dimension is mainly determined by the initial weights. The leading weights in case (C) are 1, 1/4, 1/9, 1/16, 1/25, 1/36, 1/49, …, while in case (D) are 1, 1/2, 1/4, 1/8, 1/16, 1/32, 1/64, … . Only from the 7-th weight, the weights in (D) are smaller than these in case (C).

4. Option pricing: the effective dimension

In this section we study the option pricing problems from the point of view of effective dimension. We try to answer the question of how large the effective dimension of the option pricing problems is and in what extent the BB and PCA techniques can reduce the effective dimension and improve QMC estimates. A critical discussion of the dimension reduction techniques is presented.

4.1. The pricing of Asian options

Consider the problem of pricing an Asian option on the discrete arithmetic average. The terminal payoff of a European-style Asian call option is

$$\max(S_{\text{ave}} - K, 0),$$

where K is the strike price at the expiration date T and $S_{\text{ave}} = \frac{1}{d}\sum_{j=1}^{d} S_{t_j}$ is the arithmetic average of the underlying asset at equally spaced times $t_0 = 0$, $t_j = t_{j-1} + \Delta t,\ j = 1, \ldots, d, \Delta t = T/d$. We assume the Black-Scholes model for the evolution of the underlying:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t, \tag{12}$$

where μ is the expected rate of return of the underlying, σ is the volatility and B_t is the standard Brownian motion. Based on the risk-neutral valuation principle (see [9]), the value of the option at time 0 is given by

$$C_A = E_Q[e^{-rT}\max(S_{\text{ave}} - K, 0)],$$

where $E_Q[\cdot]$ is the expectation under the risk-neutral measure Q. Since we are interested in the expectations under the risk-neutral measure, we take $\mu = r$ (r is the risk-free interest rate). With $\mu = r$, the analytical solution to (12) is

$$S_t = S_0 \exp((r - \tfrac{1}{2}\sigma^2)t + \sigma B_t).$$

Thus to price Asian options by simulation, it suffices to simulate the path of Brownian motion. The standard approach generate the Brownian motion sequentially in time: given $B_0 = 0$,

$$B_{t_j} = B_{t_{j-1}} + \sqrt{\Delta t} Z_j, \quad j = 1, \ldots, d, \tag{13}$$

where $Z_1, \ldots, Z_d$ are independent standard normal random variates. Note that under the standard construction (13), the price of the Asian option can be written as a

d-dimensional integral (see also [10])

$$C_A = \int_{C^d} e^{-rT} \max\left(0, \frac{1}{d}\sum_{j=1}^{d} S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)t_j + \sigma\sqrt{\Delta t}\sum_{i=1}^{j}\Phi^{-1}(x_i)\right] - K\right) d\mathbf{x}, \tag{14}$$

where $\Phi(\cdot)$ is the standard normal distribution.

To reduce the effective dimension, we use the BB technique (see [3,12,13]). Let the number of time steps $d = 2^m$ (m is a nonnegative integer). Given $B_0 = 0$, the Brownian motion is generated at times in order $T, T/2, T/4, 3T/4, \ldots$

$$B_T = \sqrt{T}\Phi^{-1}(x_{i,1}),$$

$$B_{T/2} = \tfrac{1}{2}(B_0 + B_T) + \sqrt{T/4}\Phi^{-1}(x_{i,2}),$$

$$\vdots$$

$$B_{(d-1)T/d} = \tfrac{1}{2}(B_{(d-2)T/d} + B_T) + \sqrt{T/2d}\Phi^{-1}(x_{i,d}),$$

where $(x_{i,1}, x_{i,2}, \ldots, x_{i,d})$ is the ith point of certain low discrepancy sequence. In MC, it is a random vector.

Another way to reduce the effective dimension is to use PCA (see [1]). Let V be the covariance matrix of $B_{t_1}, \ldots, B_{t_d}$, the ij-element of V is $V_{i,j} = \min(t_i, t_j)$, $i,j = 1, \ldots, d$. To introduce the construction by PCA, we first write the standard construction of Brownian motion (13) as

$$\begin{pmatrix} B_{t_1} \\ \vdots \\ B_{t_d} \end{pmatrix} = L \begin{pmatrix} Z_1 \\ \vdots \\ Z_d \end{pmatrix}, \tag{15}$$

where L is a $d \times d$ lower triangular matrix with nonzero entries $\sqrt{\Delta t}$. The matrix L is the Cholesky matrix of the covariance matrix V: $LL' = V$. Note that the BB construction corresponds to replacing the L by a certain matrix B with $BB' = V$.

In the PCA construction, we replace the matrix L in (15) by another one:

$$M := (\sqrt{\lambda_1}\ \mathbf{v}_1, \sqrt{\lambda_2}\ \mathbf{v}_2, \ldots, \sqrt{\lambda_d}\ \mathbf{v}_d), \tag{16}$$

where $\lambda_1 \geqslant \cdots \geqslant \lambda_d$ are the eigenvalues of V in decreasing order and $\mathbf{v}_1, \ldots, \mathbf{v}_d$ are the corresponding unit-length column eigenvectors of V.

In our calculations, we use the following parameters: $S_0 = 100$, $\sigma = 0.2$, $r = 0.1$, $T = 1$ year, $K = 100$. Table 3 shows the effective dimension (with $p = 0.99$) of the corresponding functions under the standard, BB and PCA constructions. In all cases the truncation dimension is smaller than the nominal dimension, but the functions are *not* determined by just a small number of leading variables (the truncation dimension is approximately 0.8 times the nominal dimension d). So judged by truncation dimension, the option pricing problems are still high dimensional. The BB and PCA constructions reduce the effective dimension remarkably, especially the

Table 3
The truncation dimension and the cumulative variance (in percent) from the first two variables for the problem of pricing Asian option: under the standard, BB and PCA constructions, respectively

d	Effective dimension			Cumulative variance (in percent)		
	Standard	BB	PCA	Standard	BB	PCA
8	7	5	2	26.59 49.31	77.64 94.57	98.73 99.81
16	14	7	2	13.06 25.91	75.15 93.49	98.74 99.79
32	27	7	2	3.56 10.70	73.10 92.76	98.62 99.75
64	53	8	2	0.10 4.05	72.43 92.66	98.56 99.78
128	105	8	2	0.03 1.41	72.78 92.66	98.68 99.76
256	205	8	2	0.01 0.05	72.06 92.61	98.65 99.81

PCA construction. The truncation dimensions in these cases is only 7–8 or 2, respectively, and is *insensitive* to the nominal dimension d.

For more clear comparison, we include in Table 3 the *cumulative variance* captured by the first two variables for the Asian option pricing problem under these three constructions. For example, on the column under "Standard", the first number is the variance (in percentage) captured by the first variable x_1 (i.e., $100\sigma^2_{\{1\}}(f)/\sigma^2(f)$ with f being the integrand in (14)), the second number is the cumulative variance captured by the first two variables x_1, x_2. It is clear that under the standard construction the variance captured by the first two variables decreases rapidly as d increases, but it remains almost the same under BB and PCA constructions. Note that the 'cumulative variance' is different from the *variability explained* by the first k normals used in [1] (the latter is defined as the sum of the squared norms of the first k columns of the matrix L in Brownian motion construction (15) with $LL' = V$).

The BB and PCA constructions change the structure of the integrand. The total variance remains the same as in standard construction, but much of the variance is allocated to the first few dimensions. Thus BB and PCA can reduce the effective dimension of the problem and would seem to make a better use of the leading components of a low discrepancy sequence.

Tables 4 and 5 show the relative efficiency of using BB and PCA in MC and QMC (for 16, 64 time intervals, respectively). Note that the relative efficiency ratio of two estimates is computed as the inverse ratio of their sample variance. In QMC, we use the digit-scrambling Sobol sequence. See [39] for such a technique. It is a version of Owen's scrambling [17]. We observe the following:

- The relative efficiency ratios of the standard QMC (with respect to crude MC) are about 15 for $d = 16$ and about 10 for $d = 64$ (decreases slightly as d increases). The dimension effect is not very serious.
- QMC combining with BB or PCA improve QMC with efficiency ratio approximately 10 or 20, respectively. The efficiency ratio is rather *insensitive* to the dimension d, but has a clear increasing trend with the increase of n. QMC combining with PCA is more efficient than with BB, this is consistent with the fact that PCA reduces the effective dimension more remarkably than BB does.

Table 4
The estimated standard deviation and relative efficiency to crude MC (in parentheses) with 50 repetitions

n	MC methods			QMC methods		
	MC	MC + BB	MC + PCA	QMC	QMC + BB	QMC + PCA
2^6	1.60e-1 (1.00)	1.63e-1 (0.97)	1.61e-1 (0.99)	5.48e-2 (9)	3.76e-2 (18)	3.09e-2 (27)
2^8	7.91e-2 (1.00)	7.88e-2 (1.01)	7.87e-2 (1.01)	1.91e-2 (17)	1.05e-2 (57)	9.74e-3 (66)
2^{10}	3.96e-2 (1.00)	3.97e-2 (1.00)	3.98e-2 (0.99)	1.02e-2 (15)	3.96e-3 (100)	2.83e-3 (196)
2^{12}	1.98e-2 (1.00)	1.98e-2 (1.00)	1.99e-2 (0.99)	5.21e-3 (14)	9.50e-4 (433)	7.50e-4 (694)

Asian option: the number of time step is $d = 16$.

Table 5
The same as Table 4 (Asian option), but with $d = 64$

n	MC methods			QMC methods		
	MC	MC + BB	MC + PCA	QMC	QMC + BB	QMC + PCA
2^6	1.53e-1 (1.00)	1.54e-1 (0.99)	1.54e-1 (0.99)	6.69e-2 (5)	3.33e-2 (21)	2.90e-2 (28)
2^8	7.77e-2 (1.00)	7.73e-2 (1.01)	7.73e-2 (1.01)	2.87e-2 (7)	1.01e-2 (59)	9.01e-3 (74)
2^{10}	3.81e-2 (1.00)	3.83e-2 (0.99)	3.84e-2 (0.99)	1.14e-2 (11)	3.36e-3 (129)	2.55e-3 (223)
2^{12}	1.91e-2 (1.00)	1.92e-2 (0.99)	1.92e-2 (0.99)	6.02e-3 (10)	1.12e-3 (293)	7.27e-4 (689)

- BB and PCA are useless to improve MC (the reason is that the MC error depends on the variance of the integrand and the variance remains the same under the BB or PCA construction).

Although the results are not included here, a further large efficiency improvement in QMC can be achieved by combining BB or PCA with variance reduction techniques (such as the antithetic variables and control variates). Note that BB and PCA have strong impact on the use of variance reduction techniques in QMC. In fact, without the use of BB or PCA, variance reduction techniques may not lead to efficiency improvement or the efficiency improvement is small in QMC (see [38]).

4.2. The pricing of multi-asset option

Consider a European-style multi-asset call option on the arithmetic average over d assets. Its terminal payoff at time T is

$$\phi(S_T^1, \ldots, S_T^d) = \max\left(0, \frac{1}{d}\sum_{j=1}^{d} S_T^j - K\right),$$

where S_T^j is the jth asset price at T. Assume that the prices of the assets satisfy

$$dS_t^j = \mu_j S_t^j\, dt + \sigma_j S_t^j\, dB_t^j,$$

for some mean return parameters $\mu_1, \ldots, \mu_d$ and volatility parameters $\sigma_1, \ldots, \sigma_d$, and $dB_t^1, \ldots, dB_t^d$ are correlated Brownian motion with correlations ρ_{ij}. With each $\mu_j = r$, the solutions to the stochastic differential equation are

$$S_t^j = S_0^j \exp((r - \tfrac{1}{2}\sigma_j^2)t + \sigma_j B_t^j). \tag{17}$$

Denote the covariance matrix of $B_T^1, \ldots, B_T^d$ by Σ. Then the ij-element of Σ is $\Sigma_{ij} = \rho_{ij} T$. The standard method to generate the Brownian motions is

$$\begin{pmatrix} B_T^1 \\ \vdots \\ B_T^d \end{pmatrix} = A \begin{pmatrix} Z_1 \\ \vdots \\ Z_d \end{pmatrix},$$

where A is the Cholesky matrix obtained from Σ, i.e., a lower triangular matrix satisfying $AA' = \Sigma$, and $Z_1, \ldots, Z_d$ are the same as in (13).

To construct the Brownian motion by PCA, it suffices to replace the matrix A in the standard construction by a matrix M of the form (16), but now the $\lambda_1 \geqslant \cdots \geqslant \lambda_d$ are the eigenvalues of Σ and $\mathbf{v}_1, \ldots, \mathbf{v}_d$ are the corresponding unit-length column eigenvectors.

Let $x_t^j = \log S_t^j$, then $(x_T^1, \ldots, x_T^d)$ is normally distributed with mean

$$\mathbf{v} := (v_1, \ldots, v_d) = (\log S_0^1 + (\mu_1 - \tfrac{1}{2}\sigma_1^2)T, \ldots, \log S_0^d + (\mu_d - \tfrac{1}{2}\sigma_d^2)T)$$

and covariance matrix $\Gamma := (\rho_{ij}\sigma_i\sigma_j T)_{i,j=1}^d$. The price of the option is given by

$$C_M = \frac{e^{-rT}}{(2\pi)^{d/2}\sqrt{\det \Gamma}} \int_{R^d} \phi(e^{x_1}, \ldots, e^{x_d}) \exp\left(-\frac{1}{2}(\mathbf{x} - v)'\Gamma^{-1}(\mathbf{x} - v)\right) d\mathbf{x},$$

which can be transformed into integral on C^d.

In our numerical experiments, we set the following parameters: $S_0^j = 100$, $\sigma_j = 0.2$, $\rho_{ij} = 0.3$, $r = 0.1$, $T = 1$ year, $K = 100$. The effective dimension and the cumulative variance from the first two variables are given in Table 6. The PCA construction reduce the truncation dimension significantly. In fact, the truncation dimension in the PCA construction is only 1 (thus the superposition is also 1), and is rather insensitive to d. We observe that in the standard construction the variance captured by the first variable decreases as d increases, but in PCA construction, it increases

Table 6
The truncation dimension and the cumulative variance (in percentage) from the first two variables for the problem of pricing multi-asset option: under standard and PCA

d	Effective dimension		Cumulative variance (in percent)	
	Standard	PCA	Standard	PCA
8	8	1	36.69 57.20	99.6627 99.6933
16	15	1	31.22 48.88	99.7625 99.7796
32	29	1	27.17 44.06	99.8324 99.8333
64	52	1	25.97 42.32	99.9194 99.9212
128	87	1	26.44 41.89	99.9664 99.9665
256	136	1	25.67 41.35	99.9856 99.9859

Table 7
The same as Table 4, but for the pricing of multi-asset option with the number of assets $d = 16$

n	MC methods		QMC methods	
	MC	MC + PCA	QMC	QMC + PCA
2^6	1.81e-1 (1.00)	1.81e-1 (1.00)	5.62e-2 (10)	3.15e-2 (33)
2^8	8.90e-2 (1.00)	8.89e-2 (1.00)	1.56e-2 (32)	1.01e-2 (78)
2^{10}	4.46e-2 (1.00)	4.49e-2 (0.99)	7.19e-3 (38)	2.74e-3 (265)
2^{12}	2.23e-2 (1.00)	2.24e-2 (0.99)	3.20e-3 (49)	8.36e-4 (710)

slightly. Thus the corresponding function in PCA construction is getting even more one dimensional as d increases.

The comparison of the accuracy and the relative efficiency ratios are given in Tables 7 and 8 (again the digit-scrambling Sobol sequence is used in QMC). The conclusions are similar with the case of Asian options.

Remark. We have shown that the BB and PCA constructions achieve a huge dimension reduction (from 256 to 2 or 1 for PCA). But in terms of standard deviation they do not seem to have a corresponding huge advantage over the standard construction (the efficiency ratio over standard QMC is 10–20). To explain this, take an extreme example. Consider the d-dimensional integral $\int_{C^d}[x_1 + \cdots + x_d]\,d\mathbf{x}$ (suppose d is large). It can be transformed into one-dimensional integral $d \cdot \int_0^1 x\,dx$. By doing this, a huge dimension reduction is achieved! But, it can be easily verified that if one takes the first-component of a d-dimensional QMC rule for approximating the one-dimensional integral, the error reduction will be small or even

Table 8
The same as Table 7, but for the number of assets $d = 64$

n	MC methods		QMC methods	
	MC	MC + PCA	QMC	QMC + PCA
2^6	1.73e-1 (1.00)	1.73e-1 (1.00)	5.22e-2 (11)	3.04e-2 (32)
2^8	8.72e-2 (1.00)	8.68e-2 (1.01)	1.69e-2 (26)	9.09e-3 (92)
2^{10}	4.29e-2 (1.00)	4.31e-2 (0.99)	8.20e-3 (27)	2.47e-3 (302)
2^{12}	2.15e-2 (1.00)	2.15e-2 (1.00)	4.29e-3 (25)	7.17e-4 (899)

there will be no error reduction. For example, take a d-dimensional rank-1 good lattice rule with prime n, then there will be no error reduction. The important reason here is that the function $f(\mathbf{x}) = x_1 + \cdots + x_d$ has superposition dimension 1 and a QMC algorithm (with all perfect one-dimensional projections) is *already very efficient* for such a function due to its low superposition dimension.

The answer to the question above will be clear, *if the original function corresponding to the standard construction has low superposition dimension*. A recent research shows that it is exactly the case: the superposition dimension in the standard construction is very small (only about 2), even the truncation dimension is *not* much smaller than d (see [29]). Therefore, for the option pricing problems it seems that we have a similar situation as for the extreme example. The QMC algorithm is already very efficient under the standard construction because QMC has already taken the advantage of the special feature of the original function, e.g., low superposition dimension. Moreover, since the superposition dimension in the standard construction is already very small, there is no much room to reduce it further by BB and PCA (though BB and PCA reduce the truncation dimension significantly). This is why when reducing the truncation dimension considerably will probably result in a moderate QMC error reduction.

The error reduction by using BB or PCA depends both on the algorithm and on the problem. There is no guarantee that dimension reduction will result in error reduction. As shown in [21], BB and PCA do not offer any improvement in QMC applications for some problems. The relationship of QMC error to effective dimension is not so simple. It is hard to draw general conclusions just based on several examples.

Note that for some problems in computational finance, the ideas in [20] have been extended in [22] to show the superiority of QMC.

5. Conclusion

Effective dimension characterizes in some degree the complexity of multivariate integration. Knowing the effective dimension helps us to understand the difficulty of the problems and to predict the performance of QMC algorithms. Such knowledge can be useful in gaining insight into the design of more efficient QMC algorithms.

The superiority of QMC over MC has been reported mainly for two classes of functions. One is the class of functions with small truncation dimension (the superposition dimension is also small); another is the class of functions with large truncation dimension, but with very small superposition dimension. The functions of weighted Sobolev and weighted Korobov spaces with fast decayed weights are in the first class. Some specific isotropic functions on which QMC was seen to work well (for example, the one considered in [23]) are shown to belong to the second class [19] (but this does not mean that all isotropic functions have small superposition dimension). Many financial-related functions also belong to the second class [3,29]. We do not know the result about the superiority of QMC algorithms for functions of large truncation dimension and large superposition dimension (say $d_s > 10$ and $d_t > 100$) for practical n. On the other hand, it is important to be aware of the fact that small effective dimension does not suffice to guarantee the effectiveness of QMC and dimension reduction does not necessary lead to error reduction. Further research is worth doing to a clear understanding of the dependence of QMC error on effective dimension.

We analyse and numerically compute the effective dimension for product functions, functions of weighted Korobov spaces and some financial-related functions. The effective dimension of some other problems can also be analysed or computed in the similar ways. For example, the effective dimension of the Mortgage-Based Securities problem can be estimated.

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References

[1] P. Acworth, M. Broadie, P. Glasserman, A comparison of some Monte Carlo and quasi-Monte Carlo techniques for option pricing, in: P. Hellekallek, H. Niederreiter (Eds.), Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing, Springer, Berlin, 1997, pp. 1–18.

[2] P. Boyle, M. Broadie, P. Glasserman, Monte Carlo methods for security pricing, J. Econom. Dynam. Control 21 (1997) 1267–1321.

[3] R.E. Caflisch, W. Morokoff, A. Owen, Valuation of Mortgage backed securities using Brownian bridges to reduce effective dimension, J. Comp. Finance 1 (1997) 27–46.
[4] K.T. Fang, Y. Wang, Number-Theoretic Methods in Statistics, Chapman & Hall, London, 1994.
[5] F.J. Hickernell, A generalized discrepancy and quadrature error bound, Math. Comput. 67 (1998) 299–322.
[6] F.J. Hickernell, Lattice rules: how well do they measure up?, in: P. Hellekalek, G. Larcher (Eds.), Random and Quasi-Random Point Sets, Springer, New York, 1998, pp. 109–168.
[7] F.J. Hickernell, X. Wang, The error bounds and tractability of quasi-Monte Carlo algorithms in infinite dimension, Math. Comput. 71 (2002) 1641–1661.
[8] F.J. Hickernell, H. Woźniakowski, Integration and approximation in arbitrary dimension, Adv. Comput. Math. 12 (2000) 25–58.
[9] J.C. Hull, Options, Futures, and Other Derivatives, 4th Edition, Prentice-Hall, Englewood Cliffs, NJ, 1999.
[10] P. L'Ecuyer, C. Lemieux, Variance reduction via lattice rules, Management Sci. 46 (2000) 1214–1235.
[11] C. Lemieux, A.B. Owen, Quasi-regression and the relative importance of the ANOVA components of a function, in: K.T. Fang, F.J. Hickernel, H. Niederreiter (Eds.), Monte Carlo and Quasi-Monte Carlo Methods 2002, Springer, Berlin, 2002, pp. 331–344.
[12] B. Moskowitz, R.E. Caflisch, Smoothness and dimension reduction in quasi-Monte Carlo methods, Math. Comput. Modelling 23 (1996) 37–54.
[13] W.J. Morokoff, Generating quasi-random paths for stochastic processes, SIAM Rev. 40 (1998) 765–788.
[14] W.J. Morokoff, R.E. Caflisch, Quasi-random sequences and their discrepancies, SIAM J. Sci. Comput. 15 (1994) 1251–1279.
[15] H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods, SIAM, Philadelphia, 1992.
[16] S. Ninomiya, S. Tezuka, Toward real time pricing of complex financial derivatives, Appl. Math. Finance 3 (1996) 1–20.
[17] A.B. Owen, Randomly permuted (t,m,s)-nets and (t,s)-sequences, in: H. Niederreter, Shiue (Eds.), Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing, Springer, Berlin, 1995, pp. 299–317.
[18] A.B. Owen, Latin supercube sampling for very high dimensional simulations, ACM Trans. Model. Comput. Simul. 8 (1998) 71–102.
[19] A.B. Owen, The dimension distribution, and quadrature test functions, Technical Report, Stanford University, 2001.
[20] A. Papageorgiou, Fast convergence of quasi-Monte Carlo for a class of isotropic integrals, Math. Comput. 70 (2001) 297–306.
[21] A. Papageorgiou, The Brownian bridge does not offer a consistent advantage in quasi-Monte Carlo integration, J. Complexity 18 (2002) 171–186.
[22] A. Papageorgiou, Sufficient conditions for fast quasi-Monte Carlo convergence, J. Complexity, to appear.
[23] A. Papageorgiou, J.F. Traub, Faster valuation of multi-dimensional integrals, Comput. Phys. (1997) 474–578.
[24] S.H. Paskov, New methodologies for valuing derivatives, in: S. Pliska, M. Dempster (Eds.), Mathematics of Derivative Securities, Cambridge University Press, Cambridge, 1997, pp. 545–582.
[25] S.H. Paskov, J.F. Traub, Faster valuation of financial derivatives, J. Portfolio Management 22 (1995) 113–120.
[26] I. Radović, I.M. Sobol', R.F. Tichy, Quasi-Monte Carlo methods for numerical integration: comparison of different low discrepancy sequences, Monte Carlo Methods Appl. 2 (1996) 1–14.
[27] I.H. Sloan, S. Joe, Lattice Methods for Multiple Integration, Oxford University Press, Oxford, 1994.
[28] I.H. Sloan, F.Y. Kuo, S. Joe, On the step-by-step construction of quasi-Monte Carlo integration rules that achieves strong tractability error bounds in weighted Sobolev spaces, Math. Comput. 71 (2002) 1609–1640.

[29] I.H. Sloan, X. Wang, Why are high-dimensional financial problems often 'low-dimensional'?, 2003, submitted.
[30] I.H. Sloan, X. Wang, Projections of low discrepancy sequences: how well are they distributed? Submitted.
[31] I.H. Sloan, H. Woźniakowski, When are quasi-Monte Carlo algorithms efficient for high dimensional integrals, J. Complexity 14 (1998) 1–33.
[32] I.H. Sloan, H. Woźniakowski, Tractability of multivariate integration for weighted Korobov classes, J. Complexity 17 (2001) 697–721.
[33] I.M. Sobol, On the distribution of points in a cube and the approximate evaluation of integrals, Zh. Vychisli. Mat. i Mat. Fiz. 7 (1967) 784–802.
[34] I.M. Sobol, Sensitivity estimates for nonlinear mathematical model, Math. Modeling Comput. Exp. 1 (1993) 407–414.
[35] I.M. Sobol, Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates, Math. Comput. Simul. 55 (2001) 271–280.
[36] J.F. Traub, A.G. Werschulz, Complexity and Information, Cambridge University Press, Cambridge, 1998.
[37] X. Wang, A constructive approach to strong tractability using quasi-Monte Carlo algorithms, J. Complexity 18 (2002) 683–701.
[38] X. Wang, F.J. Hickernell, K.T. Fang, Improving quasi-Monte Carlo integration via variance reduction techniques, submitted.
[39] X. Wang, F.J. Hickernell, Randomized Halton sequences, Math. Comput. Modelling 32 (2000) 887–899.