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Obtaining the neutrino mixing matrix with the tetrahedral group

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Abstract

We discuss various "minimalist" schemes to derive the neutrino mixing matrix using the tetrahedral group A_4.

1. Neutrino mixing matrix

The neutrino mixing matrix V relates the neutrino current eigenstates (denoted by ν_α, $\alpha = e, \mu, \tau$, and coupled by the W bosons to the corresponding charged leptons) to the neutrino mass eigenstates (denoted by ν_i, $i = 1, 2, 3$, and endowed with definite masses m_i) according to

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{1}$$

Thanks to heroic experimental efforts, the neutrino mixing angles have now been determined [1] to be given by $\sin^2\theta_{12} \sim 0.31$, $\sin^2\theta_{23} \sim 0.50$, and $\sin^2\theta_{31} \sim 0.01$, with the mixing angles defined by the standard parametrization (with $c_{23} \equiv \cos\theta_{23}$, $s_{23} \equiv \sin\theta_{23}$, and so forth)

$$V_{\text{angular}} = V_{23}V_{31}V_{12} \tag{2}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & s_{23} & -c_{23} \end{pmatrix} \begin{pmatrix} c_{31} & 0 & s_{31}e^{-i\phi} \\ 0 & 1 & 0 \\ -s_{31}e^{i\phi} & 0 & c_{31} \end{pmatrix} \begin{pmatrix} -c_{12} & s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

$$= \begin{pmatrix} -c_{31}c_{12} & c_{31}s_{12} & s_{31}e^{-i\phi} \\ s_{12}c_{23} + c_{12}s_{23}s_{31}e^{i\phi} & c_{12}c_{23} - s_{12}s_{23}s_{31}e^{i\phi} & s_{23}c_{31} \\ s_{12}s_{23} - c_{12}c_{23}s_{31}e^{i\phi} & c_{12}s_{23} + s_{12}c_{23}s_{31}e^{i\phi} & -c_{23}c_{31} \end{pmatrix}. \tag{4}$$

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(This parametrization may differ slightly from others in that we take $\det V_{23} = \det V_{12} = -1$.) The error bars are such that θ_{31} is consistent with 0, in which case the CP violating phase $e^{i\phi}$ does not enter.

We could suppose either that the entries in V represent a bunch of meaningless numbers possibly varying from village to village in the multiverse landscape as advocated by some theorists of great sophistication or that they point to some deeper structure or symmetry as some theorists with a more traditional faith in the power of theoretical physics might dare to hope for. It is natural to imagine that there is a family symmetry [2] linking the three lepton families. Starting with the standard model we assign (all fermionic fields are left handed) the lepton doublets $\psi_a = \binom{\nu_a}{l_a}$, the lepton singlets l_a^C $(a = 1, 2, 3)$, and the required Higgs fields to various representations of a family group [3] G_F.

Indeed, if we guess that $s_{12} = 1/\sqrt{3}$, $s_{23} = 1/\sqrt{2}$, and $s_{31} = 0$, we obtain the attractive mixing matrix

$$V = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{5}$$

first proposed by Harrison, Perkins and Scott [4]. Later, X.G. He and I independently arrived at the same ansatz [5]. Also, this mixing matrix (but curiously, with the first and second column interchanged) was first suggested by Wolfenstein more than 20 years ago [6] based on some considerations involving the permutation group S_3. It has subsequently been studied extensively by Harrison, Perkins and Scott [7], and by Xing [8]. Attempts to derive this mixing matrix have been discussed by Low and Volkas [9,10]. A parametrization of the experimental data in terms of deviation from V is given in [11]. Following Wolfenstein and defining $\nu_x \equiv (\nu_\mu + \nu_\tau)/\sqrt{2}$ and $\nu_y \equiv (\nu_\mu - \nu_\tau)/\sqrt{2}$, we see that (5) says that the mass eigenstates are given by

$$\nu_1 = -\sqrt{\frac{2}{3}}\nu_e + \frac{1}{\sqrt{3}}\nu_x, \qquad \nu_2 = \frac{1}{\sqrt{3}}\nu_e + \sqrt{\frac{2}{3}}\nu_x, \qquad \nu_3 = \nu_y. \tag{6}$$

The basis $\{\nu_1, \nu_2\}$ is rotated from $\{\nu_e, \nu_x\}$ through $\arcsin(1/\sqrt{3}) \sim 35°$.

In this Letter we will take the neutrinos to be Majorana [12] as seems likely, so that we have in the Lagrangian the mass term $\mathcal{L} = -\nu_\alpha M_{\alpha\beta} C \nu_\beta + \text{h.c.}$, where C denotes the charge conjugation matrix. Thus, the neutrino mass matrix M is symmetric. Also, for the sake of simplicity we will assume CP conservation so that M is real. With this simplification, the orthogonal transformation $V^{\mathrm{T}} M V$ produces a diagonal matrix with diagonal elements m_1, m_2 and m_3. We are free to multiply V on the right by some diagonal matrix whose diagonal entries are equal to ± 1. This merely multiplies each of the columns in V by an arbitrary sign. Various possible phases have been discussed in detail in the literature [13,14].

At present, we have no understanding of the neutrino masses just as we have no understanding of the charged lepton and quark masses. The well-known solar and atmospheric neutrino experiments have determined, respectively, that $\Delta m_\odot^2 = m_2^2 - m_1^2 \sim 8 \times 10^{-5}\ \text{eV}^2$ and $\Delta m_{\text{atm}}^2 = m_3^2 - m_2^2 \sim \pm 2.4 \times 10^{-3}\ \text{eV}^2$. The sign of Δm_{atm}^2 is currently unknown, while $\Delta m_\odot^2$ has to be positive in order for the Mikheyev–Smirnov–Wolfenstein resonance to occur inside the sun. We could have either the so-called normal hierarchy in which $|m_3| > |m_2| \sim |m_1|$ or the inverted hierarchy $|m_3| < |m_2| \sim |m_1|$.

2. Family symmetry and the tetrahedral group

For some years, Ma [15] has advocated choosing the discrete group A_4, namely, the symmetry group of the tetrahedron, as G_F. With various collaborators he has written a number of interesting papers [16–19] using A_4 to study the lepton sector.

For the convenience of the reader and to set the notation, we give a concise review of the relevant group theory. Evidently, A_4 is a subgroup of $SO(3)$ (which was often used in the early literature on family symmetry but which has proved to be too restrictive). Since the tetrahedron lives in 3-dimensional space, A_4 has a natural 3-dimensional representation denoted by $\underline{3}$ suggestive of the 3 families observed in nature. The tetrahedron has 4 vertices and thus A_4 is also formed by the even permutations of 4 objects so that A_4 has $4!/2 = 12$ elements which could be represented as elements of $SO(3)$. Besides the identity $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we have the 3 rotations through 180°:

$$r_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad r_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have the cyclic permutation $c = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, which together with r_1cr_1, r_2cr_2 and r_3cr_3, form an equivalence class with 4 members. Finally, we have the anticyclic permutation $a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, which together with r_1ar_1, r_2ar_2 and r_3ar_3, form another equivalence class with 4 members. Thus, the 12 elements belong to 4 equivalence classes with membership 1, 3, 4, and 4, which tells us that there are 4 irreducible representations with dimension d_j such that $\sum_j d_j^2 = 12$ which has the unique solution $d_1 = d_2 = d_3 = 1$ and $d_4 = 3$. The natural 3-dimensional representation $\underline{3}$ has just been displayed explicitly.

The multiplication of representations is easy to work out by using the following trick. Start with the familiar multiplication within $SO(3)$: $\underline{3} \times \underline{3} = \underline{1} + \underline{3} + \underline{5}$. Given two vectors $\vec{x}$ and $\vec{y}$ of $SO(3)$, the $\underline{3}$ is of course given by the cross product $\vec{x} \times \vec{y}$ while the $\underline{5}$ is composed of the symmetric combinations $x_2y_3 + x_3y_2$, $x_3y_1 + x_1y_3$, $x_1y_2 + x_2y_1$, together with the 2 diagonal traceless combinations $2x_1y_1 - x_2y_2 - x_3y_3$ and $x_2y_2 - x_3y_3$. Upon restriction of $SO(3)$ to A_4 the $\underline{5}$ evidently decompose into $\underline{5} \to \underline{3} + \underline{1}' + \underline{1}''$ with the $\underline{3}$ given by the 3 symmetric combinations just displayed. The $\underline{1}'$ and $\underline{1}''$ could be taken, respectively, as linear combinations of the 2 traceless combinations just given:

$$\underline{1}' \sim u' = x_1y_1 + \omega x_2y_2 + \omega^2 x_3y_3, \tag{7}$$

$$\underline{1}'' \sim u'' = x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3, \tag{8}$$

with $\omega \equiv e^{i2\pi/3}$ the cube root of unity so that

$$1 + \omega + \omega^2 = 0. \tag{9}$$

It is perhaps worth emphasizing the obvious, that while $\underline{1}'$ and $\underline{1}''$ furnish 1-dimensional representations of A_4 they are not invariant under A_4. For example, under the cyclic permutation c, $u' \to \omega u'$ and $u'' \to \omega^2 u''$. Evidently $\underline{1}' \times \underline{1}'' = \underline{1}$, $\underline{1}' \times \underline{1}' = \underline{1}''$, and $\underline{1}'' \times \underline{1}'' = \underline{1}'$, and also $(\underline{1}')^* = \underline{1}''$.

Thus, under A_4 we have $\underline{3} \times \underline{3} = \underline{1} + \underline{1}' + \underline{1}'' + \underline{3} + \underline{3}$. It is perhaps also worth remarking that the two $\underline{3}$'s on the right-hand side may be taken as (x_2y_3, x_3y_1, x_1y_2) and (x_3y_2, x_1y_3, x_2y_1). The existence of 3 inequivalent 1-dimensional representations also suggests the relevance of A_4 to the family problem. I cannot resist mentioning here the possibly physically irrelevant fact that [20] alone among all the alternating groups A_n's the group A_4 is not simple.

3. A minimalist framework

Given these attractive features of A_4, there has been, perhaps not surprisingly, a number of recent attempts [15,21,22] to derive V using A_4. In our opinion, they all appear to involve a rather elaborate framework, for example, supersymmetry, higher-dimensional spacetime, and so on. Within this recent literature Ma [27] has produced a particularly interesting and relatively economical scheme in which the neutrino mixing matrix depends on a parameter such that when that parameter takes on "reasonable" values the matrix V as given in (5) is recovered approximately.

The guiding philosophy of this Letter is that we would like to have as minimal a theoretical framework as possible.

Within a minimalist framework, charged lepton masses are generated by the dimension-4 operator

$$O_4 = \varphi^\dagger l^C \psi. \tag{10}$$

Here φ denotes generically the standard Higgs doublet, of which we may have more than one. According to a general low energy effective field theory analysis [23–25] neutrino masses are generated by the dimension-5 operator

$$O_5 = (\xi \tau_2 \psi) C (\xi' \tau_2 \psi) \tag{11}$$

in the Lagrangian. Here ξ and ξ' denote various Higgs doublets that may or may not be the same as the φ's. We will suppress the charge conjugation matrix C and the Pauli matrix τ_2 in what follows. It is important to emphasize that the analysis leading up to (11) is completely general and depends only on $SU(2) \times U(1)$, and not on which dynamical model you believe in, be it the seesaw mechanism or some other mechanism (such as the model in [26]).

We suppose that the family symmetry remains unbroken down to the scale of $SU(2) \times U(1)$ breaking, so that the operators O_4 and O_5 have to be singlets under G_F. As is completely standard, when φ, ξ, and ξ' acquire vacuum expectation values, $SU(2) \times U(1)$ and G_F are broken and the neutrinos acquire masses given by the mass matrix $M_\nu \propto \langle\xi\rangle\langle\xi'\rangle$ as well as the charged leptons. (Henceforth, for a Higgs doublet ξ we use $\langle\xi\rangle$ to denote the vacuum expectation of the lower electrically neutral component of ξ.)

Let M_ν be diagonalized by $U_\nu^{\mathrm{T}} M_\nu U_\nu = D_\nu$ so that the 3 neutrino fields that appear in ψ_a are related to the neutrino fields ν^m with definite masses by $\nu = U_\nu \nu^m$. Similarly, let the 3 charged left handed lepton fields l that appear in ψ_a be related to the physical charged lepton fields l^m by $l = U_l l^m$. Then $\psi_a = \binom{(U_\nu)_{ab}\nu_b^m}{(U_l)_{ab} l_b^m}$ so that the neutrino mixing matrix as defined in (1) is given by $V = U_l^\dagger U_\nu$. One difficulty in constructing a theory for V is that it arises from the "mismatch" between two rotations U_l and U_ν.

As in turns out, in our model building efforts, we often have to forbid the φ's that appear in O_4 from appearing in O_5. This could easily be implemented by imposing a discrete symmetry under which $\varphi \to e^{i\varkappa}\varphi$, $l^C \to e^{i\varkappa} l^C$ (where $e^{i\varkappa} \neq -1$ is some appropriate phase factor), with all other fields unaffected. We will leave this implicit in what follows.

Within the minimalist framework outlined here we offer some possible schemes. None of these could be said to be terribly compelling but at least we keep within the usual rules of the model building literature. The various schemes, depending on what representations of A_4 we choose for the various fields ψ, l^C, and φ, could be listed systematically.

4. Model A

We first try the assignment $\psi \sim \underline{3}$, $l^C \sim \underline{1}, \underline{1}'$, and $\underline{1}''$, and $\varphi \sim \underline{3}$. The Lagrangian then contains the terms

$$h_1 l_1^C (\varphi_1^\dagger\psi_1 + \varphi_2^\dagger\psi_2 + \varphi_3^\dagger\psi_3) + h_2 l_2^C (\omega\varphi_1^\dagger\psi_1 + \varphi_2^\dagger\psi_2 + \omega^2\varphi_3^\dagger\psi_3) + h_3 l_3^C (\omega^2\varphi_1^\dagger\psi_1 + \varphi_2^\dagger\psi_2 + \omega\varphi_3^\dagger\psi_3) \tag{12}$$

$$= \begin{pmatrix} l_1^C & l_2^C & l_3^C \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix} \begin{pmatrix} \varphi_1^\dagger & 0 & 0 \\ 0 & \varphi_2^\dagger & 0 \\ 0 & 0 & \varphi_3^\dagger \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \tag{13}$$

It is natural for the 3 $\langle\varphi_\alpha\rangle = v_\alpha$'s to be equal since A_4 requires that the coefficients of $\varphi_\alpha^\dagger\varphi_\alpha$ and of $(\varphi_\alpha^\dagger\varphi_\alpha)^2$ in the potential be independent of $\alpha = 1, 2, 3$. (See Appendix A for a more detailed analysis.) If so, then upon spontaneous gauge symmetry breaking we obtain

$$\begin{pmatrix} l_1^C & l_2^C & l_3^C \end{pmatrix} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{14}$$

with $m_e = h_1 v$ and so on. It is useful to define the "magic" matrix[1]

$$A = \begin{pmatrix} 1 & 1 & 1 \\ \omega & 1 & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix}. \tag{15}$$

Then $l^m = \frac{1}{\sqrt{3}} A l$ or $l = \sqrt{3} A^{-1} l^m$ so that $U_l^\dagger = (\sqrt{3} A^{-1})^\dagger = \frac{1}{\sqrt{3}} A$.

The crucial observation at this point is that the sum of the first and third columns in A gives $\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ and that the difference of the first and third columns in A gives $\sqrt{3} i \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, which up to some overall factors are precisely the first and third column, respectively, in the desired V in (5). In other words, if

$$U_\nu = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}, \tag{16}$$

then $U_l^\dagger U_\nu = V\Phi$ with V the desired mixing matrix in (5) and the diagonal phase matrix Φ with the diagonal elements -1, 1, and $-i$. Thus, if we could obtain U_ν we would achieve our goal of deriving V.

We recognize that U_ν is just a rotation through 45° in the (1–3) plane. Recalling that U_ν is determined by requiring $U_\nu^T M_\nu U_\nu = D_\nu$ be diagonal we see that if we could obtain an M_ν of the form

$$M_\nu = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \beta & 0 & \alpha \end{pmatrix} \tag{17}$$

(note that the 2×2 matrix in the (1–3) sector has equal diagonal elements) then we are done. Our discussion here overlaps with that given recently by Babu and He [22]; however, their discussion is given in the context of a much more elaborate scheme involving supersymmetry.

Referring to (11) we see that by imposing a discrete symmetry K_2 under which $\psi_2 \to -\psi_2$, $\varphi_2 \to -\varphi_2$, with all other fields unaffected, or equivalently a discrete symmetry K_{13} under which $\psi_1 \to -\psi_1$, $\psi_3 \to -\psi_3$, $\varphi_1 \to -\varphi_1$, $\varphi_3 \to -\varphi_3$, with all other fields unaffected, we can obtain the texture zeroes in (17), but unfortunately this does not imply that $(M_\nu)_{11} = (M_\nu)_{33}$. Furthermore, K_{13} is just the element r_2 of A_4 and so it does not commute with A_4. Note that upon the φ's acquiring equal vacuum expectation values, A_4 is broken down to a Z_3 generated by $\{I, c, a\}$ and unfortunately r_2 does not belong in Z_3. Perhaps, there is a more attractive scheme in which a reflection symmetry like K_2 could emerge effectively.

In another attempt to obtain an M_ν of the form in (17) we introduce Higgs doublets χ and ξ transforming as $\underline{1}$ and $\underline{3}$, respectively. We then have three types of O_5 operators, namely, $(\chi\psi)^2$, $(\chi\psi)(\xi\psi)$, and $(\xi\psi)^2$. As mentioned earlier, we impose a discrete symmetry to forbid φ from participating in O_5. As discussed in Appendix A, we could naturally suppose that the vacuum expectation value of ξ points in the 2-direction, that is, $\langle\xi_2\rangle \neq 0$ with $\langle\xi_1\rangle = \langle\xi_3\rangle = 0$. Let us now list how the different O_5 operators contribute to M_ν upon χ and ξ_2 acquiring a vacuum expectation value. The operator $(\chi\psi)^2$ contributes a term proportional to the identity matrix. Next, $(\chi\psi)(\xi\psi)$, which is formed by $\underline{3} \times \underline{3} \times \underline{3}$, consists of two terms, corresponding to the two ways of obtaining a $\underline{3}$ upon multiplying $\underline{3} \times \underline{3}$. One term has the form $\chi(\psi_1\xi_2\psi_3 + \psi_2\xi_3\psi_1 + \psi_3\xi_1\psi_2)$, with the other term having an analogous

[1] Note that $A^4 = 9\omega I$, so that A is up to an overall factor the matrix 4th root of the identity.

form. Thus, the operator $(\chi\psi)(\xi\psi)$ contributes the term denoted by β in (17). Finally, the operator $(\xi\psi)^2$ actually denotes schematically 4 different operators since it is formed by $(\underline{3}\times\underline{3})\times(\underline{3}\times\underline{3})$ and this contains $\underline{1}\times\underline{1}$, $\underline{1}'\times\underline{1}''$, $\underline{3}\times\underline{3}$, $\underline{3}\times\underline{3}$, and $\underline{3}\times\underline{3}$, corresponding, respectively, to the operators

$$(\xi_1\psi_1+\xi_2\psi_2+\xi_3\psi_3)^2, \tag{18}$$

$$\big(\xi_1\psi_1+\omega\xi_2\psi_2+\omega^2\xi_3\psi_3\big)\big(\xi_1\psi_1+\omega^2\xi_2\psi_2+\omega\xi_3\psi_3\big), \tag{19}$$

$$(\xi_2\psi_3,\xi_3\psi_1,\xi_1\psi_2)\cdot(\xi_3\psi_2,\xi_1\psi_3,\xi_2\psi_1)=\xi_1\psi_2\xi_2\psi_1+\xi_2\psi_3\xi_3\psi_2+\xi_3\psi_1\xi_1\psi_3, \tag{20}$$

$$(\xi_3\psi_2,\xi_1\psi_3,\xi_2\psi_1)\cdot(\xi_3\psi_2,\xi_1\psi_3,\xi_2\psi_1)=(\xi_3\psi_2)^2+(\xi_1\psi_3)^2+(\xi_2\psi_1)^2, \tag{21}$$

$$(\xi_2\psi_3,\xi_3\psi_1,\xi_1\psi_2)\cdot(\xi_2\psi_3,\xi_3\psi_1,\xi_1\psi_2)=(\xi_1\psi_2)^2+(\xi_2\psi_3)^2+(\xi_3\psi_1)^2. \tag{22}$$

(This is essentially the same as the analysis of an A_4 invariant Higgs potential given in Appendix A.) Upon ξ_2 acquiring a vacuum expectation value, we obtain, respectively, $\psi_2\psi_2$, $\psi_2\psi_2$, 0, $\psi_1\psi_1$ and $\psi_3\psi_3$. Unfortunately, the effective coupling constants in front of the operator in (21) and (22) are in general not equal to each other and thus we obtain an M_ν of the form

$$M_\nu=\begin{pmatrix}\alpha-\varepsilon & 0 & \beta\\ 0 & \gamma & 0\\ \beta & 0 & \alpha+\varepsilon\end{pmatrix} \tag{23}$$

rather than the M_ν in (17). To set ε to 0 we would have impose a discrete interchange symmetry P_{13} which interchanges the indices 1 and 3 but unfortunately, just as before for K_{13}, P_{13} does not commute with A_4.

At this point, we could only suppose that ε is small compared to β, in which case U_ν is perturbed from the desired U_ν in (16) to

$$U_\nu=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\\ 0 & \sqrt{2} & 0\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -\frac{\varepsilon}{2\beta}\\ 0 & 1 & 0\\ \frac{\varepsilon}{2\beta} & 0 & 1\end{pmatrix}. \tag{24}$$

The resulting deviation from the V in (5) may be interesting phenomenologically. In particular, $V_{e3}\simeq-\frac{\varepsilon}{\sqrt{6}\beta}$ is no longer identically 0. In [11] it was advocated that experimental data be parametrized as a deviation from V in (5) as discussed in Section III there.

In this scheme, the neutrino masses come out to be $\alpha-\sqrt{\beta^2+\varepsilon^2}$, γ, and $\alpha+\sqrt{\beta^2+\varepsilon^2}$ and thus both the normal hierarchy and the inverse hierarchy could be accommodated by suitable tuning, but there is no true understanding of neutrino masses as remarked earlier.

5. Model B

Following Ma [27], we take $\psi\sim\underline{3}$, $l^C\sim\underline{3}$, and $\varphi\sim\underline{1}$, $\underline{1}'$, and $\underline{1}''$. In other words, we have 3 Higgs doublets φ each transforming as a singlet under A_4. The Lagrangian then contains the terms

$$h_1\varphi_1^\dagger\big(l_1^C\psi_1+l_2^C\psi_2+l_3^C\psi_3\big)+h_2\varphi_2^\dagger\big(l_1^C\psi_1+\omega^2l_2^C\psi_2+\omega l_3^C\psi_3\big)+h_3\varphi_3^\dagger\big(l_1^C\psi_1+\omega l_2^C\psi_2+\omega^2l_3^C\psi_3\big). \tag{25}$$

Upon the φ's acquiring vacuum expectation values v we obtain a diagonal charged lepton mass matrix, with the charged lepton masses given by the absolute values of $h_1v_1+h_2v_2+h_3v_3$, $h_1v_1+\omega^2h_2v_2+\omega h_3v_3$, and $h_1v_1+\omega h_2v_2+\omega^2h_3v_3$. All that matters here for our purposes is that we have enough freedom to match the observed masses m_e, m_μ, and m_τ. The salient point here is that $U_l=I$, so that we only have to worry about getting the desired U_ν.

As is obvious and as was discussed in [11] and in [5], in a basis in which the charged lepton mass matrix is already diagonal, the neutrino mass matrix M_ν is of course determined in terms of the three neutrino masses and the neutrino mixing matrix V. Call the three column vectors in the mixing matrix $\vec{v}_i$. Then M_ν is given by

$$M_\nu = \sum_{i=1}^{3} m_i \vec{v}_i (\vec{v}_i)^{\mathrm{T}}. \tag{26}$$

In particular, if we believe in the V in (5) we have

$$M_\nu = \frac{m_1}{6}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} + \frac{m_2}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_3}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \tag{27}$$

With A_4 it is natural to obtain the matrix $M_D \equiv \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and the identity matrix. In particular, if we introduce a Higgs doublets ξ transforming as $\underline{3}$ under A_4 and arrange the Higgs potential such that the 3 vacuum expectation values $\langle \xi_1 \rangle = \langle \xi_2 \rangle = \langle \xi_3 \rangle$ are equal, we then see from the list of operators of the form $(\xi\varphi)(\xi\varphi)$ given in (18)–(21) at the end of the last section that we obtain for M_ν an arbitrary linear combination of M_D and the identity matrix, which is not what we want.

In [5], in discussing the neutrino mass matrix, we proposed a basis of 3 matrices other than those that appear in (27). First, the 3 column-vectors in V are the eigenvectors of the matrix

$$M_0 = a\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix} \tag{28}$$

with eigenvalues $m_1 = m_2 = 2a$, and $m_3 = -4a$. (The parameter a merely sets the overall scale.) Thus, with M_0 as the mass matrix $\Delta m_{21}^2 = 0$ and this pattern reproduces the data $|\Delta m_{21}^2|/|\Delta m_{32}^2| \ll 1$ to first approximation. Because of the degeneracy in the eigenvalue spectrum, V is not uniquely determined. To determine V, and at the same time to split the degeneracy between m_1 and m_2, we perturb M_0 to $M = M_0 + \varepsilon a M_D$. The matrix M_D is evidently a projection matrix that projects the first and third columns in V to zero. Thus, the eigenvalues are given by $m_1 = 2a$, $m_2 = 2a(1 + 3\varepsilon/2)$, and $m_3 = -4a$, where to the lowest order $\varepsilon = \Delta m_{21}^2/\Delta m_{32}^2$ and $a^2 = \Delta m_{32}^2/12$. Finally, to break the relation $|m_3| = 2|m_1| \simeq 2|m_2|$ we can always add to M a term proportional to the identity matrix. But it seems difficult to get the matrix in (28) using A_4 alone.

6. Other possibilities and conclusion

Given that A_4 has only 4 distinct representations we could, of course, systematically go through all possibilities. Thus, next we could take $\varphi \sim \underline{3}$, $l^C \sim \underline{3}$, and $\psi \sim \underline{1}, \underline{1}'$, and $\underline{1}''$. The charged lepton mass term would have a form analogous to that given in (12). But clearly, if we now assume the $\langle \varphi_\alpha \rangle$'s to be equal, we once again get the matrix A but now acting on l^C instead of on ψ. Note that if we assign $\psi_2 \sim \underline{1}$ and ψ_1, ψ_3 to $\underline{1}'$ and $\underline{1}''$, respectively, and introduce a Higgs doublets χ transforming as $\underline{1}$, we get via the operator O_5 a neutrino mass matrix M_ν of the form in (17) but with $\alpha = 0$.

Another possibility is to assign ψ, φ, and l^C all to the $\underline{3}$ in which case the charged lepton mass matrix is generated by two terms, $h(\varphi_1^\dagger l_2^C \psi_3 + \varphi_2^\dagger l_3^C \psi_1 + \varphi_3^\dagger l_1^C \psi_2)$ and $h'(\varphi_1^\dagger l_3^C \psi_2 + \varphi_2^\dagger l_1^C \psi_3 + \varphi_3^\dagger l_2^C \psi_1)$. If we assume the $\langle \varphi_\alpha \rangle$'s to be equal, then the three charged lepton masses are given in terms of only two parameters.

In conclusion, we have discussed various schemes to obtain a particularly attractive neutrino mixing matrix that closely approximates the data. Instead of detailed models, we use a low energy effective field theory approach, allowing only Higgs doublets to survive down to the electroweak scale. We have also explicitly made the restrictive assumption that A_4 survives down to the $SU(2) \times U(1)$ breaking scale. Of course, if Higgs triplets could also be used, as, for example, in [16], or if A_4 is broken at higher scale (for example, by the coupling of the φ's to the singlet scalar field h in the model in [26]), then many more possibilities open up and one could go beyond the discussion given here. We have been intentionally restrictive here. Ultimately, of course, any discussion of neutrino mixing should be given in a grand unified framework (for recent attempts, see, for example, [28,29] in which neutrino masses, as well as quark masses and mixing, are also "explained"). We do not attempt this more ambitious program in this Letter.

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Appendix A

We need to study the Higgs potential for several $SU(2) \times U(1)$ Higgs doublet φ's which transform according to various representations under A_4. For the sake of simplicity, here we restrict ourselves to the Higgs potential for a single $SU(2) \times U(1)$ Higgs doublet φ which transform like a $\underline{3}$ under A_4. Hopefully, the conclusions reached with this restricted analysis continue to hold when the couplings between different Higgs doublets are small. The multiplication $\underline{3} \times \underline{3} = \underline{1} + \underline{1}' + \underline{1}'' + \underline{3} + \underline{3}$ tells us that there is only one quadratic invariant $s = \varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2 + \varphi_3^\dagger \varphi_3$.

Since $(\underline{3} \times \underline{3}) \times (\underline{3} \times \underline{3})$ contains $\underline{1}$ four times, corresponding to $\underline{1} \times \underline{1}$, $\underline{1}' \times \underline{1}''$, $\underline{3} \times \underline{3}$, $\underline{3} \times \underline{3}$, and $\underline{3} \times \underline{3}$, we should have 5 quartic invariants. The obvious quartic invariant is $q = s^2 = (\varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2 + \varphi_3^\dagger \varphi_3)^2$. Corresponding to $\underline{1}' \times \underline{1}''$, we have $(\varphi_1^\dagger \varphi_1 + \omega \varphi_2^\dagger \varphi_2 + \omega^2 \varphi_3^\dagger \varphi_3)(\varphi_1^\dagger \varphi_1 + \omega^2 \varphi_2^\dagger \varphi_2 + \omega \varphi_3^\dagger \varphi_3)$ giving rise to q and the quartic invariant $q' = \varphi_1^\dagger \varphi_1 \varphi_2^\dagger \varphi_2 + \varphi_2^\dagger \varphi_2 \varphi_3^\dagger \varphi_3 + \varphi_3^\dagger \varphi_3 \varphi_1^\dagger \varphi_1$. Next, corresponding to $\underline{3} \times \underline{3}$ and $\underline{3} \times \underline{3}$ we have $q'' = (\varphi_1^\dagger \varphi_2, \varphi_2^\dagger \varphi_3, \varphi_3^\dagger \varphi_1) \cdot (\varphi_2^\dagger \varphi_1, \varphi_3^\dagger \varphi_2, \varphi_1^\dagger \varphi_3) = |\varphi_1^\dagger \varphi_2|^2 + |\varphi_2^\dagger \varphi_3|^2 + |\varphi_3^\dagger \varphi_1|^2$ and $q''' = (\varphi_1^\dagger \varphi_2, \varphi_2^\dagger \varphi_3, \varphi_3^\dagger \varphi_1) \cdot (\varphi_1^\dagger \varphi_2, \varphi_2^\dagger \varphi_3, \varphi_3^\dagger \varphi_1) = (\varphi_1^\dagger \varphi_2)^2 + (\varphi_2^\dagger \varphi_3)^2 + (\varphi_3^\dagger \varphi_1)^2$. The 5th invariant is the complex conjugate of q'''.

Thus, the most general Higgs potential is given by $V = -\mu^2 s + \lambda q + \lambda' q' + \lambda'' q'' + \frac{1}{2}(\lambda''' q''' + \text{h.c.})$. Assuming that the 3 φ's all point in the same direction within $SU(2)$, then we have $V = -\mu^2(v_1^2 + v_2^2 + v_3^2) + \lambda(v_1^4 + v_2^4 + v_3^4) + \tilde{\lambda}(v_1^2 v_2^2 + v_2^2 v_3^2 + v_3^2 v_1^2)$, where $\tilde{\lambda} \equiv 2\lambda + \lambda' + \lambda'' + \lambda'''$. For the sake of simplicity, we will take λ''' and the various v's to be real, since our focus here is not on CP violation.

It is then straightforward though tedious to calculate the value of V and the eigenvalues Ω of the second derivative matrix $\frac{\partial^2 V}{\partial v_\alpha \partial v_\beta}$ evaluated at the three mimina of interest: E: $\{v_1 = v_2 = v_3 = v\}$, U: $\{v_1 = v, v_2 = v_3 = 0\}$, and P: $\{v_1 = v_2 = v, v_3 = 0\}$. We find

$$\begin{aligned}
E:&\quad v^2 = \frac{\mu^2}{2(\lambda + \tilde{\lambda})}, \quad V\big|_E = -\frac{3\mu^4}{4(\lambda + \tilde{\lambda})}, \quad \Omega = \left[4\mu^2, \frac{2\mu^2(2\lambda - \tilde{\lambda})}{\lambda + \tilde{\lambda}}, \frac{2\mu^2(2\lambda - \tilde{\lambda})}{\lambda + \tilde{\lambda}}\right], \\
U:&\quad v^2 = \frac{\mu^2}{2\lambda}, \quad V\big|_U = -\frac{\mu^4}{4\lambda}, \quad \Omega = \left[4\mu^2, \frac{\mu^2(\tilde{\lambda} - 2\lambda)}{\lambda}, \frac{\mu^2(\tilde{\lambda} - 2\lambda)}{\lambda}\right], \\
P:&\quad v^2 = \frac{\mu^2}{2\lambda + \tilde{\lambda}}, \quad V\big|_P = -\frac{\mu^4}{2\lambda + \tilde{\lambda}}, \quad \Omega = \left[4\mu^2, \frac{2\mu^2(\tilde{\lambda} - 2\lambda)}{2\lambda + \tilde{\lambda}}, \frac{2\mu^2(\tilde{\lambda} - 2\lambda)}{2\lambda + \tilde{\lambda}}\right].
\end{aligned}$$

We note that by choosing $\tilde{\lambda} < 0$ and sufficiently close to $-\lambda$ or by not doing this we could set $V|_E$ much lower than $V|_U$ or vice versa. On the other hand, for P to be a minimum, we need $\tilde{\lambda} - 2\lambda > 0$, which would make $V|_U$ lower than $V|_P$. It appears that in this simple one Higgs doublet case, P is never the true minimum. Of course, in all the models we discussed, we have to introduce more than one Higgs doublets and so presumably almost anything is possible by coupling the various doublets together.

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