Diffusion induced by bounded noise in a two-dimensional coupled memory system

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Abstract The diffusion behavior driven by bounded noise under the influence of a coupled harmonic potential is investigated in a two-dimensional coupled-damped model. With the help of the Laplace analysis we obtain exact descriptions for a particle’s two-time dynamics which is subjected to a coupled harmonic potential and a coupled damping. The time lag is used to describe the velocity autocorrelation function and mean square displacement of the diffusing particle. The diffusion behavior for the time lag is also discussed with respect to the coupled items and the amplitude of bounded noise.


Keywords generalized Langevin equation, bounded noise, memory kernel

The information about the stochastic dynamical behavior of particles is usually extracted from the moments of response, such as the mean square displacement (MSD), velocity autocorrelation function (VACF), or stationary moments. For example, stochastic resonance can be investigated by measuring the moments. Moreover, anomalous diffusion is quantified in complex or disordered media by measuring the MSD, if a general Mittag–Leffler noise and a broadband noise are introduced respectively in the generalized Langevin equation (GLE). Furthermore, the response moments about the multiple-dimensional system are significantly interesting. Kumar studied the orbital magnetic moment via the second fluctuation-dissipation relation in the two-dimensional coupled-damped dissipative model, which involved the rarely reported investigation of the diffusion behavior in a magnetic environment.

Considering a two-dimensional coupled-damped model in the static magnetic environment described by the following GLE

\[\ddot{x}_1(t) = -\omega_1^2 x_1(t) - \lambda \omega_1 \omega_2 x_2(t) - \int_0^t \beta_1(t - t') \dot{x}_1(t') \, dt' - c \dot{x}_2(t) + F_1(t), \]  
\[\ddot{x}_2(t) = -\omega_2^2 x_2(t) - \lambda \omega_1 \omega_2 x_1(t) - \int_0^t \beta_2(t - t') \dot{x}_2(t') \, dt' + c \dot{x}_1(t) + F_2(t), \]

where \(\beta_i(t)\) is memory kernel and \(F_i(t)\) is a random force. In addition, \(\omega_i\) is the frequency of oscillators in \(x_i\) coordinate \((i = 1, 2)\). \(\lambda\) is the coupled strength of oscillators. \(c\) is the coupled-

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damped coefficient. Moreover, the simpler expressions of Eqs. (1a) and (1b) can be written as

$$\dot{X}(t) + \int_0^t \beta(t-t')X(t') \, dt' + C X(t) + W X(t) = F(t),$$

(2)

where $X(t) = [x_1(t), x_2(t)]^T$, $F(t) = [F_1(t), F_2(t)]^T$, $\beta(t) = \begin{bmatrix} \beta_1(t) & 0 \\ 0 & \beta_2(t) \end{bmatrix}$, $C = \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}$, and $W = \begin{bmatrix} \omega_1^2 & \lambda \omega_1 \omega_2 \\ \lambda \omega_1 \omega_2 & \omega_2^2 \end{bmatrix}$. With a requirement that $W$ should be positive, the $\lambda$ satisfies $|\lambda| < 1$. Both $F(t)$ and $\beta(t)$ are related to each other via the second fluctuation-dissipation theorem

$$\langle F(t_1) F(t_2) \rangle = \rho^{-1} \beta (|t_1 - t_2|),$$

(3)

where $\rho = 1 / (k_B T_a)$ is calculated with the Boltzmann constant $k_B$ and the absolute temperature $T_a$ of the environment. Here, $\langle \cdots \rangle$ denotes the ensemble average. The bounded noise $F_1(t)$ is described by

$$F_1(t) = A \cos (\Omega t + \gamma B(t)), $$

where $\gamma$ is the intensity added to the unit Wiener process $B(t)$. $A$ and $\Omega$ are the amplitude and frequency of $F_1(t)$, respectively. In addition, $F_2(t)$ is a zero-centered Gaussian white noise with the noise intensity $\alpha$.

From Eq. (3), the $\beta(t)$ are obtained by the statistical properties of $F_1(t)^2$ and $F_2(t)$ as

$$\beta(t) = \begin{bmatrix} \rho A^2 \frac{\exp(-D \tau) \cos (\Omega t)}{2} & 0 \\ 0 & 2 \rho \alpha \delta(t) \end{bmatrix},$$

where $\gamma$ equals to $\sqrt{2D}$ with $D$ being the strength of Gaussian white noise, and $\delta(t)$ denotes Dirac function.

The two-time dynamics can be calculated by imposing the Laplace transformation on Eq. (2). We have

$$\hat{\dot{X}}(s) X(s')^T = \hat{K}(s) X(0) X(0)^T \hat{K}(s')^T + \hat{K}(s) X(0) V(0)^T \hat{G}(s')^T + \hat{G}(s) V(0) X(0)^T \hat{K}(s')^T + \hat{G}(s) V(0) V(0)^T \hat{G}(s')^T + \hat{G}(s) \hat{F}(s') \hat{G}(s')^T, $$

(4)

$$\hat{\dot{V}}(s) V(s')^T = \hat{\dot{g}}(s) V(0) V(0)^T \hat{g}(s')^T - \hat{\dot{g}}(s) V(0) X(0)^T W^T \hat{G}(s')^T - \hat{\dot{g}}(s) \hat{W} X(0) V(0)^T \hat{g}(s')^T + \hat{\dot{g}}(s) \hat{F}(s') \hat{g}(s')^T, $$

(5)

where $X(0)$ and $V(0)$ are the initial position and the initial velocity of the particle. The expressions $K(s) = I/s - \hat{R}(s) W$, $\hat{R}(s) = \hat{G}(s)/s$, $\hat{G}(s) = (s^2 I + s \hat{\dot{g}}(s) + s C + W)^{-1}$, and $\hat{g}(s) = s \hat{G}(s)$ are corresponding to the Laplace transform of $K(t)$, $R(t)$, $G(t)$, and $g(t)$.

Utilizing the double Laplace inversion to Eqs. (4) and (5), we obtain the two-time dynamical
expressions

\[
\begin{align*}
\langle X(t)X(t')^T \rangle &= K(t)X(0)X(0)^TK(t')^T + K(t)X(0)V(0)^TG(t')^T + \\
\rho^{-1} \left( R(t) + R(t')^T - R(t-t') - R(t)WW(t')^T \right) + G(t)V(0)X(0)^TK(t')^T + \\
G(t) \left( V(0)V(0)^T - \rho^{-1}I \right) G(t')^T,
\end{align*}
\]

(6)

\[
\begin{align*}
\langle V(t)V(t')^T \rangle &= G(t)W \left( X(0)X(0)^T W^T - \rho^{-1}I \right) G(t')^T + \rho^{-1} \left( t-t' \right) + \\
g(t) \left( V(0)V(0)^T - \rho^{-1}I \right) g(t')^T - g(t)V(0)X(0)^T W^T G(t')^T - \\
G(t)WX(0)V(0)^T g(t')^T.
\end{align*}
\]

(7)

From an experimental point of view, the information about the diffusion behavior of particles is extracted from the MSD and VACF, which can be expressed, respectively, as

\[
\begin{align*}
\rho(t) &= \lim_{t \to \infty} \langle (X(t+\tau) - X(t))(X(t+\tau) - X(t))^T \rangle, \quad (8) \\
C_v(\tau) &= \lim_{t \to \infty} \frac{\langle V(t+\tau)V(t)^T \rangle}{\langle V(t)V(t)^T \rangle}, \quad (9)
\end{align*}
\]

where \( \tau \) is the time lag.

It is worth noting that \( R(t) \), as the relaxation function in Eq. (6), has the Laplace transform as

\[
\hat{R}(s) = \frac{1}{s} \left( s^2 I + s\hat{\beta}(s) + s\hat{C} + \hat{W} \right)^{-1}.
\]

(10)

Applying the final value theorem in Eq. (10), one gets

\[
\lim_{t \to \infty} R(t) = \lim_{s \to 0} s\hat{R}(s) = \frac{1}{(1-\lambda^2)} \begin{bmatrix}
1/\omega_1^2 & -\lambda/\omega_1 \omega_2 \\
-\lambda/\omega_1 \omega_2 & 1/\omega_2^2
\end{bmatrix}.
\]

(11)

Similarly, the relaxation functions \( G(t) \) and \( g(t) \) in Eq. (7) satisfy

\[
G(\infty) = g(\infty) = 0.
\]

(12)

Applying these conditions in Eqs. (8) and (9), the simpler expressions are finally obtained as

\[
\begin{align*}
\rho(\tau) &= 2\rho^{-1}R(\tau), \quad (13) \\
C_v(\tau) &= g(\tau), \quad (14)
\end{align*}
\]

where \( \rho(\tau) = \begin{bmatrix} \rho_{11}(\tau) & \rho_{12}(\tau) \\ \rho_{21}(\tau) & \rho_{22}(\tau) \end{bmatrix} \) and \( C_v(\tau) = \begin{bmatrix} C_{v_{11}}(\tau) & C_{v_{12}}(\tau) \\ C_{v_{21}}(\tau) & C_{v_{22}}(\tau) \end{bmatrix} \).

The analytical expressions of Eqs. (13) and (14) can be obtained by the residue theorem. As
expected, the equilibrium values of the MSD and VACF are given by

\[ \rho_{\infty} = \frac{2}{\rho (1 - \lambda^2)} \begin{bmatrix} 1/\omega_1^2 & -\lambda/\omega_1 \omega_2 \\ -\lambda/\omega_1 \omega_2 & 1/\omega_2^2 \end{bmatrix}, \]

\[ C_v(\infty) = 0. \]

To maintain generality, we presume that \( X(0) = 0 \) and the thermal equilibrium condition \( V(0)V(0)^T = \rho^{-1} I \). In addition, we set the parameters \( \rho = 1/3, \alpha = 1, \omega_1 = \omega_2 = 1, \) and \( \gamma = 1.5 \) in Figs. 1–4.

The MSD \( \rho_{11}(\tau) = 2\rho^{-1}R_{11}(\tau) \) of \( x_1 \) is shown in Figs. 1 and 2. The oscillations of the MSD \( \rho_{11}(\tau) \) become more pronounced as the coupled-damped coefficient \( c \) is increased whether or not the coupled strength \( \lambda \) exists, as shown in Fig. 1, while the oscillations disappear gradually with the increase of the amplitude \( A \) as shown in Fig. 2.
In Fig. 3, we have plotted the VACF $\langle C_{v_{11}}(\tau) \rangle$ of $x_1$ versus time lag. The amplitude of oscillations of VACF decreases when the coupled item $\lambda$ exists. Further, the VACF decays rapidly to zero when the coupled item $c$ also exists (see Fig. 3). Moreover, it is realized that the VACF shows more zero crossings from the figure that indicate transitions between a positive velocity correlation and velocity anti-correlations.

The cross diffusion $\langle \rho_{12}(\tau) \rangle = 2\rho^{-1}R_{12}(\tau)$ as a function of the time lag for different values of $\lambda$ is displayed in Fig. 4. The cross diffusion exhibits less oscillations with increasing $\lambda$. Moreover, the cross-velocity autocorrelation function $C_{v_{12}}(\tau)$ approaches its equilibrium value quickly with an increase of $\lambda$ (see Fig. 5).

The dynamics of a particle governed by a two-dimensional coupled GLE (2) are investigated. We derive analytical descriptions of a particle’s two-time dynamics which is subjected to a coupled GLE. The MSD and VACF of the diffusing particle are given with respect to the time lag. The explicit expressions of the two-time dynamics, MSD, and VACF can be applied to an arbitrary memory kernel with respect to the internal noise. The MSD for long-time lag depends on the coupled strength between the oscillators, while the MSD with the behavior of oscillation for
intermediate-time lag is independent of $\lambda$ but depends on the coupled-damped item $c$. Meanwhile, the MSD presents a monotonic behavior with increasing amplitude of bounded noise. The VACF decays rapidly to zero due to the existence of the couplings between $\lambda$ and $c$. Moreover, a cross diffusion and a cross-velocity autocorrelation function generated by the couplings are observed.

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