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Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential and integral equations

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Abstract

Some new weakly singular integral inequalities of Gronwall–Bellman type are established, which generalized some known weakly singular inequalities and can be used in the analysis of various problems in the theory of certain classes of differential equations, integral equations and evolution equations. Some applications to fractional differential and integral equations are also indicated.

Keywords: Integral inequality; Weakly singular; Boundedness; Fractional differential and integral equations

1. Introduction

It is well known that Gronwall type integral inequalities play a dominant role in the study quantitative properties of solutions of differential and integral equations. The literature on such inequalities and their applications is vast; see [1–4] and the references given therein. Usually, the integrals concerning this type inequalities have regular or continuous kernels, but some problems of theory and practicality require us to solve integral inequalities with singular kernels. For example, D. Henry [5] used this type integral inequalities to prove a global existence and an exponential decay result for a parabolic Cauchy problem; Sano and Kunimatsu [6] gave a sufficient condition for stabilization of semilinear parabolic distributed systems by making use of a modification of Henry's type inequality. Very recently, Ye, Gao and Ding [7] also proved a generalized this type inequality and used it to study the dependence of the solution on the order and the initial condition of a fractional differential equation. All this type inequalities are proved by an iteration argument and the estimation formulas are expressed by a complicated power series which are sometimes not very convenient for applications. To avoid the weakness, Medveď [8] presented an new method to solve Henry's type inequalities and got the explicit bounds with a quite simple formulas which are similar to the classic Gronwall–Bellman inequalities.

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In this paper, we use the modification of Medveď's method to study a certain class of nonlinear inequalities of Henry's type, which generalizes some known results and can be used as handy and effective tools in the study of differential equations and integral equations. To illustrate this, applications of our result to fractional differential and integral equations are also indicated.

2. Main result

In what follows, R denotes the set of real numbers, $R_+ = [0, +\infty)$; $C^i(M, S)$ denotes the class of all i-times continuously differentiable defined on set M with range in the set S ($i = 1, 2, \ldots$) and $C^0(M, S) = C(M, S)$.

For convenience, before giving our main results, we cite some useful lemmas and definitions in the discussion of our proof as follows:

Lemma 2.1. *(See [9].) Let $a \geqslant 0$, $p \geqslant q \geqslant 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}$$

for any $K > 0$.

Definition 2.2. (See [10].) Let $[x, y, z]$ be an ordered parameter group of nonnegative real numbers. The group is called belong to the first class distribution and denoted by $[x, y, z] \in I$ if conditions $x \in (0, 1]$, $y \in (\frac{1}{2}, 1)$ and $z \geqslant \frac{3}{2} - y$ are satisfied; The group is called belong to the second class distribution and denoted by $[x, y, z] \in II$ if conditions $x \in (0, 1]$, $y \in (0, \frac{1}{2}]$ and $z > (1 - 2y^2)/(1 - y^2)$ are satisfied.

Lemma 2.3. *(See [11, p. 296].) Let α, β, γ and p be positive constants. Then*

$$\int_0^t \left(t^\alpha - s^\alpha\right)^{p(\beta-1)} s^{p(\gamma-1)}\, ds = \frac{t^\theta}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in R_+,$$

where $B[\xi, \eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}\, ds$ ($\Re\xi > 0$, $\Re\eta > 0$) is the well-known B-function and $\theta = p[\alpha(\beta-1) + \gamma - 1] + 1$.

Lemma 2.4. *(See [10].) Suppose that the positive constants $\alpha, \beta, \gamma, p_1$ and p_2 satisfy conditions:*

(a) *if $[\alpha, \beta, \gamma] \in I$, $p_1 = \frac{1}{\beta}$;*
(b) *if $[\alpha, \beta, \gamma] \in II$, $p_2 = \frac{1+4\beta}{1+3\beta}$, then*

$$B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right] \in (0, +\infty)$$

and

$$\theta_i = p_i\left[\alpha(\beta-1) + \gamma - 1\right] + 1 \geqslant 0$$

are valid for $i = 1, 2$.

Lemma 2.5. *(See [12].) Let $u(t)$, $f(t)$, $g(t)$ and $h(t)$ be nonnegative continuous functions on R_+, and let $r \geqslant 1$ be a real number. If*

$$u(t) \leqslant u_0(t) + w(t)\left[\int_0^t v(s) u^r(s)\, ds\right]^{1/r}, \quad t \in R_+,$$

then

$$\int_0^t v(s) u^r(s)\, ds \leqslant \left[1 - \left(1 - W(t)\right)^{1/r}\right]^{-r} \int_0^t v(s) u_0^r(s) W(s)\, ds, \quad t \in R_+,$$

where

$$W(t)=\exp\left(-\int_0^t v(s)w^r(s)\,ds\right).$$

Theorem 2.6. *Let $u(t)$, $a(t)$, $b(t)$ and $f(t)$ be nonnegative continuous functions for $t\in R_+$. Let p and q be constants with $p\geqslant q\geqslant 0$. If $u(t)$ satisfies*

$$u^p(t)\leqslant a(t)+b(t)\int_0^t \left(t^\alpha-s^\alpha\right)^{\beta-1}s^{\gamma-1}f(s)u^q(s)\,ds,\quad t\in R_+, \tag{2.1}$$

then for any $K>0$ we have

(i) *if $[\alpha,\beta,\gamma]\in I$,*

$$u(t)\leqslant\left\{a(t)+M_1^\beta t^{(\alpha+1)(\beta-1)+\gamma}b(t)\left[\mathcal{A}_1^{1-\beta}(t)+K^{\frac{q-p}{p}}M_1^\beta\left[1-\left(1-V_1(t)\right)^{1-\beta}\right]^{-1}\right.\right.$$
$$\left.\left.\times\left(\int_0^t s^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}}f^{\frac{1}{1-\beta}}(s)b^{\frac{1}{1-\beta}}(s)\mathcal{A}_1(s)V_1(s)\,ds\right)^{1-\beta}\right]\right\}^{\frac{1}{p}}, \tag{2.2}$$

where

$$M_1=\frac{1}{\alpha}B\left[\frac{\beta+\gamma-1}{\alpha\beta},\frac{2\beta-1}{\beta}\right],\qquad A(t)=\frac{q}{p}K^{\frac{q-p}{p}}a(t)+\frac{p-q}{p}K^{\frac{q}{p}},$$

$$\mathcal{A}_1(t)=\int_0^t f^{\frac{1}{1-\beta}}(s)A^{\frac{1}{1-\beta}}(s)\,ds$$

and

$$V_1(t)=\exp\left(-K^{\frac{p-q}{p(1-\beta)}}M_1^{\frac{\beta}{1-\beta}}\int_0^t s^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}}f^{\frac{1}{1-\beta}}(s)b^{\frac{1}{1-\beta}}(s)\,ds\right);$$

(ii) *if $[\alpha,\beta,\gamma]\in II$,*

$$u(t)\leqslant\left\{a(t)+M_2^{\frac{1+3\beta}{1+4\beta}}t^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{1+4\beta}}b(t)\left[\mathcal{A}_2^{\frac{\beta}{1+4\beta}}(t)+K^{\frac{q-p}{p}}M_2^{\frac{1+3\beta}{1+4\beta}}\left[1-\left(1-V_2(t)\right)^{\frac{\beta}{1+4\beta}}\right]^{-1}\right.\right.$$
$$\left.\left.\times\left(\int_0^t s^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{\beta}}f^{\frac{1+4\beta}{\beta}}(s)b^{\frac{1+4\beta}{\beta}}(s)\mathcal{A}_2(s)V_2(s)\,ds\right)^{\frac{\beta}{1+4\beta}}\right]\right\}^{\frac{1}{p}}, \tag{2.3}$$

where

$$M_2=\frac{1}{\alpha}B\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)},\frac{4\beta^2}{1+3\beta}\right],\qquad \mathcal{A}_2(t)=\int_0^t f^{\frac{1+4\beta}{\beta}}(s)A^{\frac{1+4\beta}{\beta}}(s)\,ds$$

and

$$V_2(t)=\exp\left(-K^{\frac{(q-p)(1+4\beta)}{p\beta}}M_2^{\frac{1+3\beta}{\beta}}\int_0^t s^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{\beta}}f^{\frac{1+4\beta}{\beta}}(s)b^{\frac{1+4\beta}{\beta}}(s)\,ds\right).$$

Proof. Define a function $v(t)$ by

$$v(t)=b(t)\int_0^t \left(t^\alpha-s^\alpha\right)^{\beta-1}s^{\gamma-1}f(s)u^q(s)\,ds,\qquad t\in R_+, \tag{2.4}$$

then

$$u^p(t)\leqslant a(t)+v(t)$$

or

$$u(t)\leqslant \left(a(t)+v(t)\right)^{\frac{1}{p}}. \tag{2.5}$$

By Lemma 2.1 and (2.5), for any $K>0$, we have

$$u^q(t)\leqslant \left(a(t)+v(t)\right)^{\frac{q}{p}}\leqslant \frac{q}{p}K^{\frac{q-p}{p}}\left(a(t)+v(t)\right)+\frac{p-q}{p}K^{\frac{q}{p}}.$$

Substituting the last relations into (2.4) we get

$$\begin{aligned} v(t)&\leqslant b(t)\int_0^t \left(t^\alpha-s^\alpha\right)^{\beta-1}s^{\gamma-1}f(s)\left[\frac{q}{p}K^{\frac{q-p}{p}}\left(a(s)+v(s)\right)+\frac{p-q}{p}K^{\frac{q}{p}}\right]ds\\ &=b(t)\int_0^t \left(t^\alpha-s^\alpha\right)^{\beta-1}s^{\gamma-1}f(s)A(s)\,ds+\frac{q}{p}K^{\frac{q-p}{p}}b(t)\int_0^t \left(t^\alpha-s^\alpha\right)^{\beta-1}s^{\gamma-1}f(s)v(s)\,ds, \end{aligned} \tag{2.6}$$

where $A(t)=\frac{q}{p}K^{\frac{q-p}{p}}a(t)+\frac{p-q}{p}K^{\frac{q}{p}}$.

If $[\alpha,\beta,\gamma]\in I$, let $p_1=1/\beta$, $q_1=1/(1-\beta)$; if $[\alpha,\beta,\gamma]\in II$, let $p_2=(1+4\beta)/(1+3\beta)$, $q_2=(1+4\beta)/\beta$, then $\frac{1}{p_i}+\frac{1}{q_i}=1$ for $i=1,2$, and then using Hölder's inequality with indexes p_i, q_i to (2.6) we get

$$\begin{aligned} v(t)&\leqslant b(t)\left[\int_0^t \left(t^\alpha-s^\alpha\right)^{p_i(\beta-1)}s^{p_i(\gamma-1)}\,ds\right]^{1/p_i}\left[\int_0^t f^{q_i}(s)A^{q_i}(s)\,ds\right]^{1/q_i}\\ &\quad+K^{\frac{q-p}{p}}b(t)\left[\int_0^t \left(t^\alpha-s^\alpha\right)^{p_i(\beta-1)}s^{p_i(\gamma-1)}\,ds\right]^{1/p_i}\left[\int_0^t f^{q_i}(s)v^{q_i}(s)\,ds\right]^{1/q_i}. \end{aligned}$$

By Lemmas 2.3 and 2.4, the last inequality can be rewritten as

$$v(t)\leqslant \left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}}\mathcal{A}_i^{\frac{1}{q_i}}(t)b(t)+K^{\frac{q-p}{p}}\left(M_i t^{\theta_i}\right)^{\frac{1}{p_i}}b(t)\left[\int_0^t f^{q_i}(s)v^{q_i}(s)\,ds\right]^{1/q_i} \tag{2.7}$$

for $t\in R_+$, where

$$M_i=\frac{1}{\alpha}B\left[\frac{p_i(\gamma-1)+1}{\alpha},\,p_i(\beta-1)+1\right],\qquad \mathcal{A}_i(t)=\int_0^t f^{q_i}(s)A^{q_i}(s)\,ds$$

and θ_i is given as in Lemma 2.4 for $i=1,2$.

Using Lemma 2.5 to (2.7), we get

$$v(t) \leqslant \big(M_i t^{\theta_i}\big)^{\frac{1}{p_i}} \mathcal{A}_i^{\frac{1}{q_i}}(t)b(t) + K^{\frac{q-p}{p}}\big(M_i t^{\theta_i}\big)^{\frac{1}{p_i}} b(t)\Big[1-\big(1-V_i(t)\big)^{\frac{1}{q_i}}\Big]^{-1} \times \left(\int_0^t f^{q_i}(s)\big(M_i s^{\theta_i}\big)^{\frac{q_i}{p_i}} b^{q_i}(s)\mathcal{A}_i(s)V_i(s)\,ds\right)^{\frac{1}{q_i}}, \tag{2.8}$$

where

$$V_i(t) = \exp\left(-K^{\frac{q_i(q-p)}{p}} \int_0^t f^{q_i}(s)\big(M_i s^{\theta_i}\big)^{\frac{q_i}{p_i}} b^{q_i}(s)\,ds\right).$$

Finally, substituting (2.8) into (2.5), considering two situations for $i = 1, 2$ and using parameters α, β and γ to denote p_i, q_i and θ_i in (2.8), we can get the desired estimations (2.2) and (2.3), respectively. □

Remark 2.1. (i) In (2.2) and (2.3), we not only have given some new bounds to a class of nonlinear weakly singular integral inequalities, but also note that the functions $a(t)$ and $b(t)$ appearing in (2.2) and (2.3) are not required to satisfy the nondecreasing condition as some known results [7,8,10].

(ii) Using the generalized Bernoulli inequality [13] to (2.2) and (2.3), we can obtain some simpler formulas to the estimates of the solutions of (2.1) as follows.

Theorem 2.6′. *Let $u(t)$, $a(t)$, $b(t)$, $f(t)$, p and q be defined as in Theorem* 2.6, *$u(t)$ satisfy* (2.1). *Then for any $K > 0$ we have*

(i) *if $[\alpha, \beta, \gamma] \in I$,*

$$u(t) \leqslant \left\{a(t) + M_1^{\beta} t^{(\alpha+1)(\beta-1)+\gamma} b(t)\left[\mathcal{A}_1^{1-\beta}(t) + K^{\frac{q-p}{p}} \frac{M_1^{\beta}}{1-\beta} V_1^{-1}(t) \times \left(\int_0^t s^{\frac{(\alpha+1)(\beta-1)+\gamma}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s)\mathcal{A}_1(s)V_1(s)\,ds\right)^{1-\beta}\right]\right\}^{\frac{1}{p}}, \tag{2.2$'$}$$

where M_1, $\mathcal{A}_1(t)$ and $V_1(t)$ are defined as in Theorem 2.6 *for $t \in R_+$;*

(ii) *if $[\alpha, \beta, \gamma] \in II$,*

$$u(t) \leqslant \left\{a(t) + M_2^{\frac{1+3\beta}{1+4\beta}} t^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{\beta}} b(t)\left[\mathcal{A}_2^{\frac{\beta}{1+4\beta}}(t) + K^{\frac{q-p}{p}} M_2^{\frac{1+3\beta}{1+4\beta}} \times \left(\frac{1+4\beta}{\beta}\right) V_2^{-1}(t)\left(\int_0^t s^{\frac{[\alpha(\beta-1)+\gamma](1+4\beta)-\beta}{\beta}} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s)\mathcal{A}_2(s)V_2(s)\,ds\right)^{\frac{\beta}{1+4\beta}}\right]\right\}^{\frac{1}{p}}, \tag{2.3$'$}$$

where M_2, $\mathcal{A}_2(t)$ and $V_2(t)$ are defined as in Theorem 2.6 *for $t \in R_+$.*

Proof. By the generalized Bernoulli inequality [13], we have

$$\big(1-V_i(t)\big)^{\frac{1}{q_i}} < 1 - \frac{1}{q_i} V_i(t)$$

or

$$\big[1-\big(1-V_i(t)\big)^{\frac{1}{q_i}}\big]^{-1} < q_i V_i^{-1}(t)$$

for $i = 1, 2$, where $V_i(t)$ is defined as in Theorem 2.6. Substituting the last inequalities into (2.2) and (2.3) we can obtain (2.2)′ and (2.3)′, respectively. □

Corollary 2.7. *Let functions $u(t), a(t), b(t)$ and $f(t)$ be defined as in Theorem* 2.6. *Suppose that*

$$u(t) \leqslant a(t) + b(t) \int_0^t (t-s)^{\beta-1} f(s)u(s)\,ds, \quad t \in R_+. \tag{2.9}$$

Then we have

(i) *if* $\beta \in (\frac{1}{2}, 1)$,

$$u(t) \leqslant a(t) + M_{11}^{\beta} t^{2\beta-1} b(t)\left[\mathcal{A}_{11}^{1-\beta}(t) + \frac{M_{11}^{\beta}}{1-\beta} V_{11}^{-1}(t) \int_0^t s^{\frac{2\beta-1}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s) \mathcal{A}_{11}(s) V_{11}(s)\,ds\right], \tag{2.10}$$

where

$$M_{11} = B\left[1, \frac{2\beta-1}{\beta}\right], \qquad \mathcal{A}_{11}(t) = \int_0^t f^{\frac{1}{1-\beta}}(s) a^{\frac{1}{1-\beta}}(s)\,ds$$

and

$$V_{11}(t) = \exp\left(-M_{11}^{\frac{\beta}{1-\beta}} \int_0^t s^{\frac{2\beta-1}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s)\,ds\right)$$

for $t \in R_+$;

(ii) *if* $\beta \in (0, \frac{1}{2}]$,

$$u(t) \leqslant a(t) + M_{12}^{\frac{1+3\beta}{1+4\beta}} t^{4\beta} b(t)\left[\mathcal{A}_{12}^{\frac{\beta}{1+4\beta}}(t) + \frac{1+4\beta}{\beta} M_{12}^{\frac{1+3\beta}{1+4\beta}} V_{12}^{-1}(t) \times \int_0^t s^{4\beta} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s) \mathcal{A}_{12}(s) V_{12}(s)\,ds\right], \tag{2.11}$$

where

$$M_{12} = B\left[1, \frac{4\beta^2}{1+3\beta}\right], \qquad \mathcal{A}_{12}(t) = \int_0^t f^{\frac{1+4\beta}{\beta}}(s) a^{\frac{1+4\beta}{\beta}}(s)\,ds$$

and

$$V_{12}(t) = \exp\left(-M_{12}^{\frac{1+3\beta}{\beta}} \int_0^t s^{4\beta} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s)\,ds\right)$$

for $t \in R_+$.

Proof. (2.10) and (2.11) follow by letting $p = q = \alpha = \gamma = 1$ in Theorem 2.6′ and by simple computation, we omit the details. □

Remark 2.2. Inequality (2.9) has been studied in [7], but here we not only have given some new estimates which are not in complicated power series, but also eliminated the nondecreasing condition to function $b(t)$.

Let $p = 2$, $q = \alpha = \gamma = 1$, we can get the following interesting Henry–Ou-Iang type singular integral inequality. About Ou-Iang type inequalities and their applications we refer to [4] and references cited therein.

Corollary 2.8. *Let functions $u(t)$, $a(t)$, $b(t)$ and $f(t)$ be defined as in Theorem* 2.6. *Suppose that*

$$u^2(t) \leqslant a(t) + b(t)\int_0^t (t-s)^{\beta-1} f(s)u(s)\,ds, \quad t \in R_+. \tag{2.12}$$

Then for any $K > 0$ we have

(i) *if $\beta \in (\frac{1}{2}, 1)$,*

$$u(t) \leqslant \left\{a(t) + M_{11}^{\beta} t^{2\beta-1} b(t)\left[\widetilde{\mathcal{A}}_{11}^{1-\beta}(t) + K^{-\frac{1}{2}}\frac{M_{11}^{\beta}}{1-\beta}\widetilde{V}_{11}^{-1}(t) \times \int_0^t s^{\frac{2\beta-1}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s)\widetilde{\mathcal{A}}_{11}(s)\widetilde{V}_{11}(s)\,ds\right]\right\}^{\frac{1}{2}}, \tag{2.13}$$

where

$$\widetilde{\mathcal{A}}_{11}(t) = \left(\frac{1}{2}K^{\frac{1}{2}}\right)^{\frac{1}{1-\beta}}\int_0^t f^{\frac{1}{1-\beta}}(s)\left(\frac{a(s)}{K}+1\right)^{\frac{1}{1-\beta}}ds,$$

$$\widetilde{V}_{11}(t) = \exp\left[-\left(\frac{M_{11}^{\beta}}{K^{\frac{1}{2}}}\right)^{\frac{1}{1-\beta}}\int_0^t s^{\frac{2\beta-1}{1-\beta}} f^{\frac{1}{1-\beta}}(s) b^{\frac{1}{1-\beta}}(s)\,ds\right]$$

and M_{11} is defined in Corollary 2.7 *for $t \in R_+$;*

(ii) *if $\beta \in (0, \frac{1}{2}]$,*

$$u(t) \leqslant \left\{a(t) + M_{12}^{\frac{1+3\beta}{1+4\beta}} t^{\frac{4\beta^2}{1+4\beta}} b(t)\left[\widetilde{\mathcal{A}}_{12}^{\frac{\beta}{1+4\beta}}(t) + K^{-\frac{1}{2}} M_{12}^{\frac{1+3\beta}{1+4\beta}}\left(\frac{1+4\beta}{\beta}\right)\widetilde{V}_{12}^{-1}(t) \times \left(\int_0^t s^{\frac{4\beta^2}{1+4\beta}} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s)\widetilde{\mathcal{A}}_{12}(s)\widetilde{V}_{12}(s)\,ds\right)^{\frac{\beta}{1+4\beta}}\right]\right\}^{\frac{1}{2}}, \tag{2.14}$$

where

$$\widetilde{\mathcal{A}}_{12}(t) = \left(\frac{1}{2}K^{\frac{1}{2}}\right)^{\frac{1+4\beta}{\beta}}\int_0^t f^{\frac{1+4\beta}{\beta}}(s)\left(\frac{a(s)}{K}+1\right)^{\frac{1+4\beta}{\beta}}(s)\,ds,$$

$$\widetilde{V}_{12}(t) = \exp\left[-\left(\frac{M_{12}^{1+3\beta}}{K^{\frac{1+4\beta}{2}}}\right)^{\frac{1}{\beta}}\int_0^t s^{\frac{4\beta^2}{1+\beta}} f^{\frac{1+4\beta}{\beta}}(s) b^{\frac{1+4\beta}{\beta}}(s)\,ds\right]$$

and M_{12} is defined as in Corollary 2.7 *for $t \in R_+$.*

Proof. Inequalities (2.13) and (2.14) follow by letting $p = 2$, $q = \alpha = \gamma = 1$ in Theorem 2.6′ and by simple computation, we omit the details. □

3. Applications

In this section, we will indicate the usefulness of our main results in the study of the boundedness of certain fractional differential equations with Riemann–Liouville (R–L) fractional operator and Erdélyi–Kober (E–K) operator.

Riemann–Liouville derivative and integral, and Erdélyi–Kober (E–K) operator are defined as below, respectively:

Definition 3.1. (See [14].) The fractional derivative of order $0 < \alpha < 1$ of a function $f(x) \in C(R_+, R)$ is given by

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} f(t)\, dt$$

provided that the right side is pointwise defined on R_+.

Definition 3.2. (See [14].) The fractional primitive of order $\alpha > 0$ of a function $f: R_+ \to R$ is given by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

provided the right side is pointwise defined on R_+.

Definition 3.3. (See [15,16].) The Erdélyi–Kober fractional integral of a continuous $f: R_+ \to R$ is defined by

$$I_\beta^{\gamma,\delta} f(x) = \frac{x^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^x \left(x^\beta - t^\beta\right)^{\delta-1} t^{\beta\gamma} f(t) d\left(t^\beta\right)$$

with real δ, γ and $\beta > 0$, provided the right side is pointwise defined on R_+.

(I) Consider the following initial value problem of Podlubny [14] in terms of the Riemann–Liouville fractional derivatives:

$$D^\alpha y(t) = f\bigl(t, y(t)\bigr), \tag{3.1}$$

$$D^{\alpha-1} y(t)|_{t=0} = \eta, \tag{3.2}$$

where $0 < \alpha < 1$, $0 \leqslant t < T \leqslant +\infty$, $f: [0,T) \times R \to R$; and D^α denotes R–L derivative operator.

From the problem (3.1)–(3.2) we can get a fractional integral equation

$$y(t) = \frac{\eta}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\bigl(\tau, y(\tau)\bigr)\, d\tau, \tag{3.3}$$

which is equivalent to the initial value problem (3.1)–(3.2) (cf. [14, pp. 127–128]).

Theorem 3.4. *Let $0 < \alpha \leqslant 1$ and f be continuous and satisfy the condition*

$$\bigl|f(t,y)\bigr| \leqslant g(t)|y|^q, \tag{3.4}$$

where $0 < q \leqslant 1$ is a constant, $g(t)$ is nonnegative continuous function for $0 \leqslant t < T \leqslant +\infty$. Then for any solutions $y(t)$ of the initial value problem (3.1)–(3.2)

(i) *if $\alpha \in (\frac{1}{2}, 1)$,*

$$\bigl|y(t)\bigr| \leqslant \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1} + \frac{\widetilde{M}_{11}^\alpha t^{2\alpha-1}}{\Gamma(\alpha)} \Biggl[\mathcal{A}_{1q}^{1-\alpha}(t) + \frac{K^{q-1}\widetilde{M}_{11}^\alpha}{(1-\alpha)\Gamma(\alpha)} \widetilde{V}_{1q}^{-1}(t) \times \int_0^t s^{\frac{2\alpha-1}{1-\alpha}} g^{\frac{1}{1-\alpha}}(s) \mathcal{A}_{1q}(s) \widetilde{V}_{1q}(s)\, ds\Biggr], \quad 0 < t < T \leqslant +\infty, \tag{3.5}$$

where

$$A_q(t) = \frac{q|\eta|}{K^{1-q}\Gamma(\alpha)} t^{\alpha-1} + (1-q)K^q,$$

$$\widetilde{M}_{11}=B\left[1,\frac{2\alpha-1}{\alpha}\right],\qquad \mathcal{A}_{1q}(t)=\int_0^t g^{\frac{1}{1-\alpha}}(s)A_q^{\frac{1}{1-\alpha}}(s)\,ds$$

and

$$\widetilde{V}_{1q}(t)=\exp\left[-\left(\frac{K^{1-q}\widetilde{M}_{11}^{\alpha}}{\Gamma(\alpha)}\right)^{\frac{1}{1-\alpha}}\int_0^t s^{\frac{2\alpha-1}{1-\alpha}}g^{\frac{1}{1-\alpha}}(s)\,ds\right];$$

(ii) *if* $\alpha\in(0,\frac{1}{2}]$,

$$\left|y(t)\right|\leqslant\frac{|\eta|}{\Gamma(\alpha)}t^{\alpha-1}+\frac{\widetilde{M}_{12}^{\frac{1+3\alpha}{1+4\alpha}}t^{4\alpha}}{\Gamma(\alpha)}\left[\mathcal{A}_{2q}^{\frac{\alpha}{1+4\alpha}}(t)+\frac{K^{q-1}\widetilde{M}_{12}^{\frac{1+3\alpha}{1+4\alpha}}(1+4\alpha)}{\alpha\Gamma(\alpha)}\widetilde{V}_{2q}^{-1}(t)\right.$$
$$\left.\times\left(\int_0^t s^{4\alpha}g^{\frac{1+4\alpha}{\alpha}}(s)\mathcal{A}_{2q}(s)\widetilde{V}_{2q}(s)\,ds\right)^{\frac{\alpha}{1+4\alpha}}\right],\quad 0<t<T\leqslant+\infty, \tag{3.6}$$

where

$$\widetilde{M}_{12}=B\left[1,\frac{4\alpha^2}{1+3\alpha}\right],\qquad \mathcal{A}_{2q}(t)=\int_0^t g^{\frac{1+4\alpha}{\alpha}}(s)A_q^{\frac{1+4\alpha}{\alpha}}(s)\,ds$$

and

$$\widetilde{V}_{2q}(t)=\exp\left[-\left(\frac{K^{q-1}}{\Gamma(\alpha)}\right)^{\frac{1+3\alpha}{\alpha}}\widetilde{M}_{12}^{\frac{1+3\alpha}{\alpha}}\int_0^t s^{4\alpha}g^{\frac{1+4\alpha}{\alpha}}(s)\,ds\right].$$

Proof. From (3.3) and (3.4) we have

$$\left|y(t)\right|\leqslant\frac{|\eta|}{\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\left|f\left(\tau,y(\tau)\right)\right|d\tau$$
$$\leqslant\frac{|\eta|}{\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}g(\tau)\left|y(\tau)\right|^q d\tau.$$

An application of Theorem 2.6′ (with $a(t)=\frac{|\eta|}{\Gamma(\alpha)}t^{\alpha-1}$, $b(t)=\frac{1}{\Gamma(\alpha)}$, $f(t)=g(t)$, $p=1$, $\alpha=\gamma=1$ and $\beta=\alpha$) to the last inequality yields the desired estimations (3.5) and (3.6). □

(II) Consider the following Volterra type integral equations of second kind, involving an E–K fractional integral with parameters δ, γ and β,

$$y^p(t)-\lambda t^{-\beta\gamma}\int_0^t\frac{(t^\beta-\tau^\beta)^{\delta-1}}{\Gamma(\delta)}\tau^{\beta(1+\gamma)-1}y^q(\tau)\,d\tau=f(t), \tag{3.7}$$

which arises very often in various problems, especial describing physical processes with aftereffects. When (3.7) is a linear equation, i.e., $p=q=1$, the other parameters satisfy some conditions and $y(t)$ belong to a space of weighted continuous functions, Al-Saqabi and Kiryakova [16] have found the solutions of (3.7) in the explicit form with convolutional type integral involving Mittag–Leffler function. Here we give the explicit bound of the solutions of nonlinear equation (3.7) under some suitable conditions.

Theorem 3.5. *Let* $y(t)$, $f(t)\in C[0,+\infty)$, $p\geqslant q>0$ *be constants and* $y(t)$ *satisfy* (3.7). *Then for any constant* $K>0$ *we have*

(i) *if* $[\beta, \delta, \beta(1+\gamma)] \in I$,

$$\left|y(t)\right| \leqslant \left\{\left|f(t)\right| + \frac{|\lambda|\bar{M}_1^{\delta}}{\Gamma(\delta)} t^{\delta(\beta+1)-1}\left[\bar{\mathcal{A}}_1^{1-\delta}(t) + K^{\frac{q-p}{p}} \frac{|\lambda|\bar{M}_1^{\delta}}{(1-\delta)\Gamma(\delta)} \bar{V}_1^{-1}(t) \times \left(\int_0^t s^{\frac{\delta(\beta+1)-1}{1-\delta}} \bar{\mathcal{A}}_1(s)\bar{V}_1(s)\,ds\right)^{1-\delta}\right]\right\}^{\frac{1}{p}}, \quad t > 0, \tag{3.8}$$

where

$$\bar{M}_1 = \frac{1}{\beta} B\left[\frac{\delta+\beta(1+\gamma)-1}{\beta\delta}, \frac{\beta\delta-1}{\delta}\right],$$

$$\bar{A}(t) = \frac{q}{p} K^{\frac{q-p}{p}} \left|f(t)\right| + \frac{p-q}{p} K^{\frac{q}{p}}, \qquad \bar{\mathcal{A}}_1(t) = \int_0^t \bar{A}^{\frac{1}{1-\delta}}(s)\,ds$$

and

$$\bar{V}_1(t) = \exp\left[-\frac{(1-\delta)K^{\frac{p-q}{p(1-\delta)}}}{\beta\delta} \left(\frac{\bar{M}_1^{\delta}|\lambda|}{\Gamma(\delta)}\right)^{\frac{1}{1-\delta}} t^{\frac{\delta\beta}{1-\delta}}\right];$$

(ii) *if* $[\beta, \delta, \beta(1+\gamma)] \in II$,

$$\left|y(t)\right| \leqslant \left\{\left|f(t)\right| + \frac{|\lambda|\bar{M}_2^{\frac{1+3\delta}{1+4\delta}}}{\Gamma(\delta)} t^{\frac{\beta(\delta+\gamma+4\delta^2+3\delta\gamma)-\delta}{\delta}}\left[\bar{\mathcal{A}}_2^{\frac{\delta}{1+4\delta}}(t) + \frac{K^{\frac{q-p}{p}} \bar{M}_2^{\frac{1+3\delta}{1+4\delta}}(1+4\delta)|\lambda|}{\delta\Gamma(\delta)} \times \bar{V}_2^{-1}(t)\left(\int_0^t s^{\beta(4\delta+1)-1} \bar{\mathcal{A}}_2(s)\bar{V}_2(s)\,ds\right)^{\frac{\delta}{1+4\delta}}\right]\right\}^{\frac{1}{p}}, \quad t > 0, \tag{3.9}$$

where

$$\bar{M}_2 = \frac{1}{\beta} B\left[\frac{\beta(1+\gamma)(1+4\delta)-\delta}{\beta(1+3\delta)}, \frac{4\delta^2}{1+3\delta^2}\right], \qquad \bar{\mathcal{A}}_2(t) = \int_0^t \bar{A}^{\frac{1+4\delta}{\delta}}(s)\,ds$$

and

$$\bar{V}_2(t) = \exp\left[-\frac{K^{\frac{(q-p)(1+4\delta)}{p\delta}} \bar{M}_2^{\frac{1+3\delta}{\delta}}}{\beta(1+4\delta)} \left(\frac{|\lambda|}{\Gamma(\delta)}\right)^{\frac{1+4\delta}{\delta}} t^{\beta(1+4\delta)}\right].$$

Proof. From (3.7) we have

$$|y|^p(t) \leqslant \left|f(t)\right| + \frac{|\lambda|}{\Gamma(\delta)} t^{-\beta\gamma} \int_0^t \left(t^{\beta} - \tau^{\beta}\right)^{\delta-1} \tau^{\beta(1+\gamma)-1} |y|^q(\tau)\,d\tau.$$

An application of Theorem 2.6′ (with $a(t) = |f(t)|$, $b(t) = \frac{|\lambda|}{\Gamma(\delta)} t^{-\beta\gamma}$, $\alpha = \beta$, $\beta = \delta$ and $\gamma = \beta(1+\gamma)$) to the last inequality yields the desired estimations (3.8) and (3.9). □

Remark 3.1. Obviously, the boundedness of the solutions of (3.1)–(3.2) and (3.7) cannot be derived by the known results in [5–8,10].

Letting $p = q = 1$ in Theorem 3.5, we can obtain an interesting result as follows.

Corollary 3.6. *Let $y(t)$, $f(t) \in C[0,+\infty)$ and $y(t)$ satisfy the equation*

$$y(t) - \lambda t^{-\beta\gamma} \int_0^t \frac{(t^\beta - \tau^\beta)^{\delta-1}}{\Gamma(\delta)} \tau^{\beta(1+\gamma)-1} y(\tau)\, d\tau = f(t). \tag{3.10}$$

Then we have

(i) *if* $[\beta, \delta, \beta(1+\gamma)] \in I$,

$$\begin{aligned} \left|y(t)\right| \leqslant \left|f(t)\right| + \frac{|\lambda| \bar{M}_1^\delta}{\Gamma(\delta)} t^{\delta(\beta+1)-1} \Bigg[& \bar{\mathcal{A}}_1^{*1-\delta}(t) + \frac{|\lambda| \bar{M}_1^\delta}{(1-\delta)\Gamma(\delta)} \bar{V}_1^{*-1}(t) \\ & \times \left(\int_0^t s^{\frac{\delta(\beta+1)-1}{1-\delta}} \bar{\mathcal{A}}_1^*(s) \bar{V}_1^*(s)\, ds \right)^{1-\delta} \Bigg], \quad t > 0, \end{aligned} \tag{3.11}$$

where

$$\bar{M}_1 = \frac{1}{\beta} B\left[\frac{\delta + \beta(1+\gamma) - 1}{\beta\delta}, \frac{\beta\delta - 1}{\delta} \right], \qquad \bar{\mathcal{A}}_1^*(t) = \int_0^t \left|f(s)\right|^{\frac{1}{1-\delta}} ds$$

and

$$\bar{V}_1^*(t) = \exp\left[-\frac{1-\delta}{\beta\delta} \left(\frac{\bar{M}_1^\delta |\lambda|}{\Gamma(\delta)} \right)^{\frac{1}{1-\delta}} t^{\frac{\delta\beta}{1-\delta}} \right];$$

(ii) *if* $[\beta, \delta, \beta(1+\gamma)] \in II$,

$$\begin{aligned} \left|y(t)\right| \leqslant \left|f(t)\right| + \frac{|\lambda| \bar{M}_2^{\frac{1+3\delta}{1+4\delta}}}{\Gamma(\delta)} t^{\frac{\beta(\delta+\gamma+4\delta^2+3\delta\gamma)-\delta}{\delta}} \Bigg[& \bar{\mathcal{A}}_2^{*\frac{\delta}{1+4\delta}}(t) + \frac{\bar{M}_2^{\frac{1+3\delta}{1+4\delta}} (1+4\delta)|\lambda|}{\delta\Gamma(\delta)} \\ & \times \bar{V}_2^{*-1}(t) \left(\int_0^t s^{\beta(4\delta+1)-1} \bar{\mathcal{A}}_2^*(s) \bar{V}_2^*(s)\, ds \right)^{\frac{\delta}{1+4\delta}} \Bigg], \quad t > 0, \end{aligned} \tag{3.12}$$

where

$$\bar{M}_2 = \frac{1}{\beta} B\left[\frac{\beta(1+\gamma)(1+4\delta) - \delta}{\beta(1+3\delta)}, \frac{4\delta^2}{1+3\delta^2} \right], \qquad \bar{\mathcal{A}}_2^*(t) = \int_0^t \left|f(s)\right|^{\frac{1+4\delta}{\delta}} ds$$

and

$$\bar{V}_2^*(t) = \exp\left[-\frac{\bar{M}_2^{\frac{1+3\delta}{\delta}}}{\beta(1+4\delta)} \left(\frac{|\lambda|}{\Gamma(\delta)} \right)^{\frac{1+4\delta}{\delta}} t^{\beta(1+4\delta)} \right].$$

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