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Existence Theorem for Periodic Solutions of Higher Order Nonlinear Differential Equations

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We study the existence of periodic solutions to differential equations of the form $L(x) + g(t, x, x', \dots, x^{(m)}) = f(t)$ with $L(x) = x^{(m)} + a_{m-1}x^{(m-1)} + \cdots + a_1x'$.

1. INTRODUCTION

The purpose of this paper is to establish the existence of periodic solutions to the nonlinear differential equation

$$x^{(m)} + a_{m-1}x^{(m-1)} + \cdots + a_1x' + g(t, x, x', \dots, x^{(m)}) = f(t), \quad (1.1)$$

where $a_1, a_2, \dots, a_{m-1}$ are constants, $g : \mathbb{R} \times \mathbb{R}^{m+1} \to \mathbb{R}$ is continuous and T-periodic $(T > 0)$ in its first variable, and $f(t)$ is a continuous T-periodic function.

This and similar types of problems have recently received considerable attention. (See [2, 4, 6, 7, 9–12, 14–18], etc.) In most known existence results, the nonlinearity g depends at most on the lower order derivatives $x', x'', \dots,$ and $x^{(m-1)}$ and, hence, defines a compact nonlinear operator between some appropriate Banach spaces. Therefore, the abstract results used there are not applicable to (1.1). We extend the result of [17] and allow the nonlinearity g to depend on the highest derivative $x^{(m)}$. In our case, the nonlinearity g defines a k-set contractive operator between some

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Banach spaces. Our method is based on the continuation theory for k-set contractions [7].

As in [17], the interest of our conditions lies in the possibility of proving an existence theorem for the problem (1.1) without needing an assumption on the growth of g for $x \geq 0$ or else for $x \leq 0$. In [14] and [16], the authors studied the similar nonlinear periodic boundary value problems and allowed the nonlinearity g to depend on the highest derivative of $x(t)$. However, our conditions on g in this paper are different from theirs.

To show the existence of solutions to the considered problems we will use the continuation theory for k-set contractions [7, 10]. Our method in this direction relies on an abstract theorem developed in [16] and *a priori* bounds on solutions. We will state this abstract theorem in Section 2.

2. ABSTRACT EXISTENCE THEOREMS

In this section we will briefly state the part of the abstract continuation theory for k-set contractions that will be used in our study of Eq. (1.1).

Let Z be a Banach space. For a bounded subset $A \subset Z$, let $\Gamma_Z(A)$ denote the (Kuratovski) measure of non-compactness defined by

$$\Gamma_Z(A) = \inf\{\delta > 0 : \exists \text{ a finite number of subsets } A_i \subset A, \\ A = \cup_i A_i, \operatorname{diam}(A_i) \leq \delta\}. \quad (2.1)$$

Here, $\operatorname{diam}(A_i)$ denotes the maximum distance between the points in the set A_i. Let X and Y be Banach spaces and Ω a bounded open subset of X. A continuous and bounded map $N : \overline{\Omega} \to Y$ is called k-set-contractive if for any bounded $A \subset \overline{\Omega}$ we have

$$\Gamma_Y(N(A)) \leq k\Gamma_X(A). \quad (2.2)$$

Also, for a continuous and bounded map $T : X \to Y$ we define

$$l(T) = \sup\{r \geq 0 : \forall \text{ bounded subset } A \subset X, r\Gamma_X(A) \leq \Gamma_Y(T(A))\}. \quad (2.3)$$

Now, let $L : X \to Y$ be a Fredholm operator of index zero, and $N : \overline{\Omega} \to Y$ be k-set-contractive with $k < l(L)$. Using the approach of Mawhin's, it was shown by Hetzer [10] that if $Lx \neq Nx$ for all $x \in \partial\Omega$, then one can associate with the pair (L, N) a topological degree $D[(L, N), Q]$ which has most of the important properties of the so-called Leray-Schauder degree. In particular, it has a homotopy invariance property that allows one to prove the following

THEOREM 2.1 [16]. *Let $L : X \to Y$ be a Fredholm operator of index zero, and $y \in Y$ be a fixed point. Suppose that $N : \overline{\Omega} \to Y$ is k-set-contractive with*

$k < l(L)$ where $\Omega \subset X$ is bounded, open, and symmetric about $0 \in \Omega$. Suppose further that:

(A) $Lx \neq \lambda Nx + \lambda y$, *for* $x \in \partial\Omega$, $\lambda \in (0,1)$, *and*

(B) $[QN(x) + Qy, x] \cdot [QN(-x) + Qy, x] < 0$, *for* $x \in \operatorname{Ker}(L) \cap \partial\Omega$,

where $[\ ,\ ]$ *is some bilinear form on* $Y \times X$ *and* Q *is the projection of* Y *onto* $\operatorname{coker}(L)$. *Then there exists* $x \in \overline{\Omega}$ *such that* $Lx - Nx = y$.

3. MAIN RESULTS

Let C_T^0 denote the linear space of real valued continuous T-periodic functions on $\mathbb{R}$. The linear space C_T^0 is a Banach space with the usual norm for $x \in C_T^0$ given by $|x|_0 = \max_{t \in \mathbb{R}} |x(t)|$. Let C_T^m $(m \geq 1)$ denote the linear space of T-periodic functions with m continuous derivatives. C_T^m is a Banach space with norm $|x|_m = \max\{|x^{(i)}|_0 : 0 \leq i \leq m\}$.

Let $X = C_T^m$ and $Y = C_T^0$ and let $L: X \to Y$ be given by

$$L(x) = x^{(m)} + a_{m-1}x^{(m-1)} + \cdots + a_1 x'.$$

It is obvious that L is a bounded linear map. Next define a (nonlinear) map $N: X \to Y$ by

$$N(x)(t) = -g\big(t, x(t), x'(t), x''(t), \ldots, x^{(m)}(t)\big).$$

Now, the problem (1.1) has a solution $x(t)$ if and only if $Lx - Nx = f$ for some $x \in X$.

We put the following conditions on g and f. They are similar to the ones contained in [17].

(H3.1) $g: \mathbb{R} \times \mathbb{R}^{m+1} \to \mathbb{R}$ is continuous and T-periodic $(T > 0)$ in its first variable.

(H3.2) There exist measurable functions $\mu_+, \mu_- : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ such that

$$\mu_+(t) \leq \liminf_{x \to +\infty} g(t, x, x_1, x_2, \ldots, x_m), \qquad t \in \mathbb{R},$$

$$\mu_-(t) \geq \limsup_{x \to -\infty} g(t, x, x_1, x_2, \ldots, x_m), \qquad t \in \mathbb{R}$$

uniformly for $(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$.

(H3.3) There exist constants $c_1, c_2 \in \mathbb{R}$, such that

$$g(t, x, x_1, x_2, \ldots, x_m) \geq c_1 \qquad \text{for } x \geq 0, (t, x_1, x_2, \ldots, x_m) \in \mathbb{R} \times \mathbb{R}^m$$

and

$$g(t, x, x_1, x_2, \dots, x_m) \le c_2 \qquad \text{for } x \le 0, (t, x_1, x_2, \dots, x_m) \in \mathbb{R} \times \mathbb{R}^m.$$

THEOREM 3.1. *Let* (H3.1)–(H3.3) *be satisfied and assume*

(a) *The only T-periodic solutions to the equation* $Lx = 0$ *are the constants.*

(b) *There exists a* $k \in [0, 1)$, *such that*

$$\left|g(t, x, x_1, \dots, x_{m-1}, p) - g(t, x, x_1, \dots, x_{m-1}, q)\right| \le k|p - q|$$

for any $(t, x, x_1, \dots, x_{m-1}, p), (t, x, x_1, \dots, x_{m-1}, q) \in \mathbb{R} \times \mathbb{R}^{m+1}$.

(c) *There exist positive constants* $p_0, p_1, p_2, \dots, p_m$, p *such that*

$$\begin{aligned} \left|g(t, x, x_1, x_2)\right| \le g(t, x, x_1, x_2) + p_0|x| + p_1|x_1| \\ + p_2|x_2| + \cdots + p_m|x_m| + p, \end{aligned}$$

$\forall (t, x, x_1, x_2, \dots, x_m) \in \mathbb{R} \times \mathbb{R}^{m+1}$, *or*

$$\begin{aligned} \left|g(t, x, x_1, x_2)\right| \le -g(t, x, x_1, x_2) + p_0|x| + p_1|x_1| \\ + p_2|x_2| + \cdots + p_m|x_m| + p \end{aligned}$$

$\forall (t, x, x_1, x_2, \dots, x_m) \in \mathbb{R} \times \mathbb{R}^{m+1}$.

(d) *There exists*

$$\int_0^T \mu_-(t)\, dt < \int_0^T f(t)\, dt < \int_0^T \mu_+(t)\, dt.$$

Then there exists an $\delta > 0$ *such that when* $\max\{p_0, p_1, p_2, \dots, p_m\} < \delta$, *the problem* (1.1) *has a solution.*

Notice that if g is non-negative or nonpositive then our key condition (c) in the above theorem is automatically satisfied.

Before proving Theorem 3.1, we need the following lemmas.

LEMMA 3.2. *L is a Fredholm map of index* 0 *and satisfies*

$$l(L) \ge 1.$$

Proof. It is easy to verify that L is a Fredholm map of index 0 due to the condition (a) of Theorem 3.1. In fact, for $y \in Y$ we define

$$Q(y) = \frac{1}{T} \int_0^T y(t)\, dt.$$

Then $\mathrm{Im}(L) \subseteq \ker(Q)$. Applying the L^2 theory of Fourier series to the equation $Lx = y$, we also can see that $\mathrm{Im}(L) \supseteq \mathrm{Ker}(Q)$. Therefore, $\mathrm{Im}(L)$ is closed and $\dim \ker(L) = \operatorname{codim} \mathrm{Im}(L) = 1$. Let $A \subset X$ be a bounded subset and let $\eta = \Gamma_Y(L(A)) > 0$. Given $\epsilon > 0$, according to the definition, there is a finite number of subsets A_i of A such that $\mathrm{diam}_0(L(A_i) \leq + \epsilon$. Since X is compactly embedded into C_T^{m-1} and since A_i are bounded in X, it follows that there is a finite number of subsets A_{ij} of A_i such that $\mathrm{diam}_{m-1}(A_{ij}) < \epsilon$, and hence, $\mathrm{diam}_m(A_{ij}) \leq \eta + \epsilon + ma\epsilon$, where $a = \max_{1 \leq i \leq m-1}\{|a_i|\}$ and $\mathrm{diam}_k(\cdot)$ are defined with respect to the norms $|\cdot|_m$, $0 \leq k \leq m$. This proves

$$\Gamma_X(A) \leq \eta = \Gamma_Y(L(A)),$$

that is, $l(L) \geq 1$. ∎

LEMMA 3.3. *$N: X \to Y$ is a k-set-contractive map with $k < 1$ as given in condition* (b) *of Theorem* 3.1.

Proof. Let $A \subset X$ be a bounded subset and let $\eta = \Gamma_X(A)$. Then for any $\epsilon > 0$, there is a finite family of subsets $\{A_i\}$ with $A = \bigcup_i A_i$ and $\mathrm{diam}_m(A_i) \leq \eta + \epsilon$. Now it follows from the fact that g is uniformly continuous on any compact subset of $\mathbb{R} \times \mathbb{R}^{m+1}$, and from the fact that A and A_i are precompact in C_T^{m-1} with norm $|\cdot|_{m-1}$, that there is a finite family of subsets $\{A_{ij}\}$ of A_i such that $A_i = \bigcup_j A_{ij}$ with

$$\begin{aligned}&\big|g\big(t, x(t), x'(t), \dots, x^{(m-1)}(t), u^{(m)}(t)\big)\\ &\quad -g\big(t, u(t), u'(t), \dots, u^{(m-1)}(t), u^{(m)}(t)\big)\big| < \epsilon\end{aligned}$$

for any $x, u \in A_{ij}$. Therefore, for $x, u \in A_{ij}$ we have

$$\begin{aligned}\|Nx - Nu\|_0 &= \sup_{0 \leq t \leq 1} \big|g(t, x, x', \dots, x^{(m-1)}, x^{(m)})\\ &\qquad -g(t, u, u', \dots, u^{(m-1)}, u^{(m)})\big|\\ &\leq \sup_{0 \leq t \leq 1} \big|g(t, x, x', \dots, x^{(m-1)}, x^{(m)})\\ &\qquad -g(t, x, x', \dots, x^{(m-1)}, u^{(m)})\big|\\ &\quad + \sup_{0 \leq t \leq 1} \big|g(t, x, x', \dots, x^{(m-1)}, u^{(m)})\\ &\qquad -g(t, u, u', \dots, u^{(m-1)}, u^{(m)})\big|\\ &\leq k\|x^{(m)} - u^{(m)}\|_0 + \epsilon \leq k\eta + (k+1)\epsilon.\end{aligned}$$

That is,

$$\Gamma_Y(N(A)) \leq k\Gamma_X(A).$$

∎

The next lemma is from [17].

LEMMA 3.4. *Under the assumption* (a) *of Theorem* 3.1, *there is a constant* $\mu > 0$ *such that*

$$\sum_{i=1}^{m-1} |x^{(i)}|_0 + \int_0^T \left| x^{(m)}(t) \right| dt \le \mu \int_0^T \left| Lx(t) \right| dt$$

for all $x \in C_m^T$.

Proof. See [17]. ∎

LEMMA 3.5. *There is a number* r_0, *such that for each solution* $x(t)$ *to* $Lx + \lambda Nx = \lambda f$, $0 < \lambda < 1$, *there is a* $z \in [0, T]$, *with* $|x(z)| \le r_0$. *Here* z *may depend on* $x(t)$ *and* λ.

Proof. The proof may be found in [17] but for the sake of completeness we give the proof here. Suppose that for each positive integer n there is a $\lambda_n \in (0, 1)$ and a solution x_n of $Lx + \lambda_n Nx = \lambda_n \tilde{f}$ with $x_n(t) \ge n$ for $t \in [0, T]$. Then we would have

$$\int_0^T Nx_n(t)\, dt = \int_0^T f(t)\, dt.$$

In other words,

$$\int_0^T g\big(t, x_n(t), x_n'(t), \dots, x_n^{(m)}(t)\big)\, dt = \int_0^T f(t)\, dt.$$

On the other hand, we also have $\lim_{n\to\infty}\inf g(t, x_n(t), x_n'(t), \dots, x_n^{(m)}(t)) \ge \mu_+(t)$. Now, using this and Fatou's lemma, we get

$$\int_0^T f(t)\, dt \ge \int_0^T \mu_+(t)\, dt$$

contradicting condition (d). Thus there is a number r_1 such that if x is a solution of $Lx + \lambda Nx = \lambda \tilde{f}$, $\lambda \in (0, 1)$ then there is a number $s_1 \in [0, 1]$ with $x(s_1) \le r_1$.

Similarly, by using μ_- and Fatou's lemma we can show that there must be a number $r_2 > 0$ such that for any solution x there is a corresponding value $s_2 \in [0, 1]$ with $x(s_2) \ge -r_2$. By continuity we conclude that for any solution x there is some $z_x \in [0, 1]$ with $|x(z_x)| \le r_0$, $r_0 = \max\{r_1, r_2\}$. ∎

LEMMA 3.6. *There exist numbers* $M, \delta > 0$ *such that if conditions of Theorem* 3.1 *hold and* $\max\{p_0, p_1, p_2, \dots, p_m\} < \delta$ *then every solution* $x(t)$ *of the problem*

$$Lx - \lambda Nx = \lambda f, \qquad \lambda \in (0, 1)$$

satisfies $|x|_m \le M$.

Proof of Lemma 3.6. Let $Lx - \lambda Nx = \lambda f$ for $x(t) \in X$, i.e.,

$$L(x)(t) + \lambda g\big(t, x(t), x'(t), \dots, x^{(m)}(t)\big) = \lambda f. \tag{3.1}$$

Integrating this identity we have

$$\int_0^T g\big(t, x(t), x'(t), \dots, x^{(m)}(t)\big)\, dt = \int_0^T f(t)\, dt. \tag{3.2}$$

Using the condition (c) of Theorem 3.1, (3.1), and (3.2) we have

$$\begin{aligned}
\int_0^T |L(x)(t)|\, dt &\le \int_0^T \big|g\big(t, x(t), x'(t), \dots, x^{(m)}(t)\big)\big|\, dt + \int_0^T |f(t)|\, dt \\
&\le 2\int_0^T |f(t)|\, dt + p_0 T|x|_0 + p_1 T|x|_0 + \cdots \\
&\quad + p_{m-1} T|x^{(m-1)}|_0 + p_m \int_0^T \big|x^{(m)}(t)\big|\, dt + pT \\
&\le T \sum_{i=1}^{m-1} p_i |x^i|_0 + p_0 T|x|_0 + \; + p_m \int_0^T |x^m(t)|\, dt + \tilde{p},
\end{aligned}$$

where $\tilde{p} = pT + 2\int_0^T |f|$ is a constant. Combining Lemma 3.4 and the above inequality, we get

$$\sum_{i=1}^{m-1} (1 - T\mu p_i)|x^i|_0 + (1 - \mu p_m) \int_0^T |x^m(t)|\, dt \le p_0 T\mu |x|_0 + \mu \tilde{p}. \tag{3.3}$$

On the other hand, according to Lemma 3.5, there exist $z_x \in [0, T]$ such that $|x(z_x)| \le r_0$. Therefore,

$$\begin{aligned}
|x|_0 &= \sup_{0 \le t \le T} |x(t)| \\
&= \sup_{0 \le t \le T} \left| x(z_x) + \int_{z_x}^t x'(t)\, dt \right| \\
&\le r_0 + T|x'|_0.
\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\begin{aligned}
&\sum_{i=2}^{m-1} (1 - T\mu p_i)|x^i|_0 + \big(1 - T\mu p_1 - T^2 \mu p_0\big)|x'|_0 \\
&\quad + (1 - \mu p_m) \int_0^T |x^m(t)|\, dt \le \mu \tilde{p} + r_0 T\mu p_0.
\end{aligned}$$

Therefore, there exist constants $\delta > 0$ and $M_1 > 0$ such that, if

$$p_i \leq \delta, \qquad \text{for all } i = 0, 1, \dots, m,$$

then

$$|x^{(i)}|_0 \leq M_1, \qquad \text{for } i = 0, 1, \dots, m-1.$$

Moreover, given such a solution of $L(x)(t) + \lambda g(t, x, x', \dots, x^{(m)}) = \lambda f(t)$ we have

$$\begin{aligned}
\left|x^{(m)}(t)\right| &\leq \left|g(t, x, x', \dots, x^{(m)})\right| + |f(t)| + \sum_{i=1}^{m-1} |a_i|\,|x^{(i)}|_0 \\
&\leq \left|g(t, x, x', \dots, x^{(m)}) - g(t, x, x', \dots, 0)\right| \\
&\quad + |g(t, x, x', \dots, 0)| + |f(t)| + (m-1) \max_{1 \leq i \leq m-1} |a_i| M_1 \\
&\leq k\left|x^{(m)}(t)\right| + |g(t, x, x', \dots, 0)| + |f(t)| \\
&\quad + (m-1) \max_{1 \leq i \leq m-1} |a_i| M_1 \\
&\leq k\left|x^{(m)}(t)\right| + M_2.
\end{aligned}$$

Here, M_2 is some constant. From this we see that $|x^{(m)}|_0 \leq M_2/(1-k)$ so that if we let $M = \max\{M_1, M_2/(1-k)\}$ then

$$|x|_m \leq M$$

for some sufficiently small $\delta > 0$ and $p_0, p_1, \dots, p_m < \delta$. ∎

Proof of Theorem 3.1. Let $r > M$, where M is in the Lemma 3.6. We apply Theorem 2.1 for $\Omega = \{x \in X : |x|_0 < r\}$. It is easy to see now that all of the necessary conditions in Theorem 2.2 hold except for condition (B). We will now show that condition (B) also holds. Define a bounded bilinear form $[\cdot,\cdot]$ on $Y \times X$ by $[y, x] = \int_0^T y(t)x(t)\,dt$. Also, define $Q : Y \to \operatorname{coker}(L)$ by $y \mapsto \int_0^T y(t)\,dt$. Notice that for $x \in \ker(L) \cap \partial\Omega$ we must have $x = r$ or $x = -r$ so that for such an x

$$\begin{aligned}
&[QN(x) + Qy, x] \cdot [QN(-x) + Qy, x] \\
&\quad = r^2 \int_0^T (g(t, r, 0, \dots, 0) - f(t))\,dt \\
&\qquad \cdot \int_0^T (g(t, -r, 0, \dots, 0) - f(t))\,dt.
\end{aligned}$$

By condition (d), there is a number $\tilde{M} > 0$ such that if $r > \tilde{M}$ then

$$\int_0^T (g(t, r, 0, \dots, 0) - f(t))\,dt \cdot \int_0^T (g(t, -r, 0, \dots, 0) - f(t))\,dt < 0.$$

Thus if we pick $r > \max\{M, \tilde{M}\}$ then all of the conditions required in Theorem 2.1 hold. It now follows by Theorem 2.1 that there is a function $x(t) \in X$, such that

$$Lx - Nx = f.$$

This finishes the proof of Theorem 3.1. ∎

EXAMPLE 3.7. Let $g(t, x, x_1, \dots, x_m) = h_1(t, x_1, \dots, x_m)e^{-x^2} + h_2(t, x_1, \dots, x_{m-1})e^x$, here $h_1 \geq 0$ and $h_2 > 0$ are bounded continuous T-periodic functions in their first variable and h_1 satisfies

$$\sup\left|\frac{\partial}{\partial x_m} h_1(t, x, x_1, \dots, x_m)\right| < 1.$$

(For example, $h_1(t, x, x_1, \dots, x_m) = \frac{1}{4}\sin^2(x_m)e^{-x^2}$.) By Theorem 3.1, the differential equation

$$x^{(m)} + h_1(t, x', \dots, x^{(m)})e^{-x^2} + h_2(t, x', \dots, x^{(m-1)})e^x = f(t)$$

has solutions provided $\int_0^T f(t)\,dt > 0$.

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