Journal of Algebra **233**, 594–613 (2000)
doi:10.1006/jabr.2000.8441, available online at http://www.idealibrary.com on IDEAL®

Hecke Algebra Actions on the Coinvariant Algebra

Ron M. Adin[1]

Department of Mathematics and Computer Science, Bar-Ilan University

Alexander Postnikov

Department of Applied Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
E-mail: apost@math.mit.edu

and

Yuval Roichman[1]

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat Gan 52900, Israel
E-mail: yuvalr@math.biu.ac.il

Communicated by G. D. James

Received August 12, 1999

Two actions of the Hecke algebra of type A on the corresponding polynomial ring are studied. Both are deformations of the natural action of the symmetric group on polynomials, and keep symmetric functions invariant. We give an explicit description of these actions, and deduce a combinatorial formula for the resulting graded characters on the coinvariant algebra. © 2000 Academic Press

1. INTRODUCTION

1.1. The symmetric group S_n acts on the polynomial ring $P_n = F[x_1, \ldots, x_n]$ (where F is a field of characteristic zero) by permuting variables. Let I_n be the ideal of P_n generated by the symmetric i.e.,

[1] Research supported in part by the Israel Science Foundation and by internal research grants from Bar-Ilan University.

0021-8693/00 $35.00

(S_n-invariant) polynomials without a constant term. The *coinvariant algebra* of type A is the quotient P_n/I_n. Schubert polynomials, constructed in the seminal papers [BGG, De], form a distinguished basis for the coinvariant algebra. These polynomials correspond to Schubert cells in the corresponding flag variety.

1.2. In this paper we present two deformations of this action. For these deformations we can take $F = \mathbf{C}(q)$, the field of rational functions in an indeterminate q. Most of the results actually hold when F is replaced by the ring $\mathbf{Z}[q]$ of polynomials in q with integer coefficients.

Let $T_1, \ldots, T_{n-1}$ be the standard generators of the Hecke algebra $\mathscr{H}_n(q)$ of type A; for definitions see Section 2.1 below.

The first action $\rho_1 \colon \mathscr{H}_n(q) \to \mathrm{Hom}_F(P_n, P_n)$ is defined using q-commutators

$$\rho_1(T_i) := \partial_i X_i - qX_i \partial_i \qquad (1 \le i < n), \tag{1.1}$$

where

$$\partial_i := \frac{1}{x_i - x_{i+1}}(1 - s_i)$$

is the divided difference operator (see Section 2.2), and X_i denotes multiplication by x_i. This action belongs to a family introduced in [LS] (see Section 7.1 below). For a geometric interpretation see [DKLLST]. In [DKLLST, Sect. 1] such families of operators are attributed to Hirzebruch [Hr].

The second action is naturally defined on monomials by the formula

$$\rho_2(T_i)\left(x_i^{\alpha} x_{i+1}^{\beta} m\right) := \begin{cases} q x_i^{\beta} x_{i+1}^{\alpha} m, & \text{if } \alpha > \beta, \\ (1-q) x_i^{\alpha} x_{i+1}^{\beta} m + x_i^{\beta} x_{i+1}^{\alpha} m, & \text{if } \alpha < \beta, \\ x_i^{\alpha} x_{i+1}^{\beta} m, & \text{if } \alpha = \beta. \end{cases} \tag{1.2}$$

Here m is a monomial involving neither x_i nor x_{i+1}.

For a closely related action (defined in the context of quantum groups) see [Ji].

Claim. The ideal I_n is invariant under both actions. The resulting graded characters on the coinvariant algebra have a common combinatorial formula.

This shows, in particular, that ρ_1 and ρ_2 lead to equivalent representations of $\mathscr{H}_n(q)$ on the coinvariant algebra. For $q = 1$ they both reduce to the natural S_n action.

1.3. Since the ideal I_n is invariant under both ρ_1 and ρ_2, the coinvariant algebra P_n/I_n carries appropriate actions $\tilde{\rho}_1$ and $\tilde{\rho}_2$. Let χ_1^k and χ_2^k be the characters of these representations on the kth homogeneous component of P_n/I_n. We shall give an explicit formula for these characters, using the following combinatorial function.

For any permutation $w \in S_n$, define

$$m_q(w) := \begin{cases} (-q)^m, & \text{if there exists a unique } 0 \le m < n \text{ so that} \\ & w(1) > \cdots > w(m+1) < w(m+2) < \cdots < w(n), \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

Let $\mu := (\mu_1, \ldots, \mu_t)$ be a partition of n, and let $S_\mu := S_{\mu_1} \times \cdots \times S_{\mu_t}$ be the corresponding Young subgroup of S_n. For any permutation $w \in S_n$ write $w = r \cdot (w_1 \times \cdots \times w_t)$, where $w_i \in S_{\mu_i}$ $(1 \le i \le t)$ and r is a representative of minimal length for the left coset wS_μ in S_n. Define

$$\text{weight}_q^\mu(w) := \prod_{i=1}^{t} m_q(w_i). \tag{1.4}$$

THEOREM. *For all $k \ge 0$ and $\mu \vdash n$,*

$$\chi_1^k(T_\mu) = \chi_2^k(T_\mu) = \sum_{\{w \in S_n : l(w) = k\}} \text{weight}_q^\mu(w),$$

where $T_\mu := T_1 T_2 \cdots T_{\mu_1 - 1} T_{\mu_1 + 1} \cdots \cdots T_{\mu_1 + \cdots + \mu_t - 1}$ is the subproduct of $T_1 T_2 \cdots T_{n-1}$ omitting $T_{\mu_1 + \cdots + \mu_i}$ for all $1 \le i < t$.

The proof relies on an explicit description of the action with respect to the Schubert basis of the coinvariant algebra. See Theorems 4.1 and 6.5 below.

Remark. This character formula is a natural q-analogue of a weight formula for S_n presented in [Ro2]. A formally similar result appears also in Kazhdan–Lusztig theory. Kazhdan–Lusztig characters may be represented as sums of exactly the same weights, but over different summation sets [Ro1, Corollary 4; Ra2].

1.4. The rest of this paper is organized as follows. Preliminaries and necessary background are given in Section 2. In Section 3 we introduce q-commutators and study their representation matrices. The character formula for q-commutators is proved in Section 4. Natural randomized operators are introduced in Section 5. In Section 6 we show that the representations induced by the two different actions are equivalent. Section 7 concludes the paper with remarks regarding related families of operators, connections with Kazhdan–Lusztig theory, and open problems.

2. PRELIMINARIES

2.1. *The Hecke Algebra of Type A*

The symmetric group S_n is generated by $n-1$ involutions $s_1, s_2, \ldots, s_{n-1}$ satisfying the Moore–Coxeter relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad (1 \le i < n-1) \tag{2.1}$$

and

$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1. \tag{2.2}$$

These involutions are known as the Coxeter generators of S_n.

All reduced expressions of a permutation $w \in S_n$ with respect to these generators have the same length, denoted by $l(w)$.

The Hecke algebra $\mathscr{H}_n(q)$ of type A is the algebra over $F := \mathbf{C}(q)$ generated by $n-1$ generators $T_1, \ldots, T_{n-1}$, satisfying the Moore–Coxeter relations (2.1) and (2.2) as well as the following “deformed involution” relation:

$$T_i^2 = (1-q)T_i + q \qquad (1 \le i < n). \tag{2.3}$$

It should be noted that the last relation is slightly non-standard; this is done in order to get more elegant q-analogues. In order to shift to the standard version, one should replace each T_i by $-T_i$.

Let w be a permutation in S_n and let $s_{i_1} \cdots s_{i_{l(w)}}$ be a reduced expression for w. It follows from the above relations that $T_w := T_{i_1} \cdots T_{i_{l(w)}}$ is independent of the choice of reduced expression; the set $\{T_w \mid w \in S_n\}$ forms a linear basis for $\mathscr{H}_n(q)$.

Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of n. Define $T_\mu \in \mathscr{H}_n(q)$ to be the product

$$T_\mu := T_1 T_2 \cdots T_{\mu_1 - 1} T_{\mu_1 + 1} T_{\mu_1 + 2} \cdots T_{\mu_1 + \mu_2 - 1} T_{\mu_1 + \mu_2 + 1} \cdots \cdots T_{\mu_1 + \cdots \mu_t - 1}.$$

This is the subproduct of the product $T_1 T_2 \cdots T_{n-1}$ of all generators (in the usual order), obtained by omitting $T_{\mu_1 + \cdots + \mu_i}$ for all $1 \le i < t$. These elements play an important role in the character theory of $\mathscr{H}_n(q)$. For $q = 1$, the elements T_μ are representatives of all conjugacy classes in S_n. It follows that, for $q = 1$, a character is determined by its values at these elements. This is also the case for arbitrary q, as the following theorem shows.

THEOREM 2.1 [Ra1, Theorem 5.1]. *For each $w \in S_n$ there exists a linear combination*

$$C_w = \sum_{\mu} a_{w,\mu} T_{\mu} \in \mathscr{H}_n(q),$$

with $a_{w,\mu} \in \mathbf{Z}[q]$, such that

$$\chi(T_w) = \chi(C_w)$$

for all characters χ of the Hecke algebra $\mathscr{H}_n(q)$.

Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of n. Each permutation $w \in S_n$ has an associated weight, $\text{weight}_q^{\mu}(w)$, as defined in (1.3), (1.4). The irreducible characters of $\mathscr{H}_n(q)$ are indexed by the partitions of n. These characters may be represented as weighted sums over Knuth equivalence classes.

THEOREM 2.2 [Ro1, Corollary 4]. *Let $\mathscr{C}$ be a Knuth equivalence class of shape λ. Then*

$$\chi^{\lambda}(T_{\mu}) = \sum_{w \in \mathscr{C}} \text{weight}_q^{\mu}(w),$$

where χ^{λ} is the irreducible character of $\mathscr{H}_n(q)$ corresponding to the shape λ.

2.2. *Schubert Polynomials and the Coinvariant Algebra*

2.2.1. *Basic Actions on the Polynomial Ring*

Let $x_1, x_2, \ldots, x_n$ be independent variables, and let P_n be the polynomial ring $F[x_1, x_2, \ldots, x_n]$. The symmetric group S_n acts on P_n by permuting the variables x_i. Let $\Lambda_n = \Lambda[x_1, x_2, \ldots, x_n]$ be the subring of symmetric functions (i.e., polynomials which are invariant under the action of S_n). Denote by $\Lambda_n(i)$ the ring of all polynomials which are invariant under the action of s_i for a fixed i, $1 \le i < n$. Clearly, $f \in \Lambda_n(i)$ if and only if f is symmetric in the variables x_i and x_{i+1}. We call the polynomials in $\Lambda_n(i)$ *i-symmetric* polynomials.

For $1 \le i < n$ define a divided difference operator ∂_i: $P_n \to P_n$ by

$$\partial_i := (x_i - x_{i+1})^{-1}(1 - s_i).$$

If $f \in P_n$ is a homogeneous polynomial of degree d, which is not i-symmetric, then $\partial_i(f)$ is homogeneous of degree $d - 1$. For i-symmetric polynomials $\partial_i(f) = 0$.

The operators ∂_i satisfy the nil-Coxeter relations [Ma, (2.1)]

$$\partial_i^2 = 0 \qquad (1 \leq i < n), \tag{2.4}$$

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \qquad (1 \leq i < n-1), \tag{2.5}$$

$$\partial_i \partial_j = \partial_j \partial_i \qquad \text{if } |i-j| > 1. \tag{2.6}$$

Let X_i be the operator on P_n corresponding to multiplication by x_i. Clearly, X_i increases degree by 1.

The algebra generated by the operators ∂_i, $1 \leq i < n$, and X_i, $1 \leq i \leq n$ was studied in [De, BGG]. The generators satisfy the following commutation relations:

$$\partial_i X_j = X_j \partial_i \qquad \text{if } |i-j| > 1, \tag{2.7}$$

$$\partial_i X_i = 1 + X_{i+1} \partial_i \qquad (1 \leq i < n), \tag{2.8}$$

$$X_i \partial_i = 1 + \partial_i X_{i+1} \qquad (1 \leq i < n), \tag{2.9}$$

2.2.2. *Schubert Polynomials*

For any sequence $a = (a_1, \dots, a_k)$ of positive integers less than n, define $\partial_a := \partial_{a_1} \cdots \partial_{a_k}$. It follows from the relations (2.5), (2.6) that if a, b are two reduced expressions for the same permutation $w \in S_n$ then $\partial_a = \partial_b$. We can therefore use the notation ∂_w for $w \in S_n$, and in particular $\partial_{s_i} := \partial_i$ for $1 \leq i < n$.

The relation $\partial_i^2 = 0$ implies that for any $w \in S_n$ and any $1 \leq i < n$

$$\partial_i \partial_w = \begin{cases} \partial_{s_i w}, & \text{if } l(s_i w) > l(w), \\ 0, & \text{if } l(s_i w) < l(w). \end{cases} \tag{2.10}$$

For each $w \in S_n$ we define the *Schubert polynomial* $\mathfrak{S}_w$ by

$$\mathfrak{S}_w := \partial_{w^{-1} w_0} \left(x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \right),$$

where w_0 is the longest element in S_n.

By definition, $\mathfrak{S}_w$ is a homogeneous polynomial of degree $l(w)$.

It follows from (2.10) that

$$\partial_i(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_{w s_i}, & \text{if } l(w s_i) < l(w), \\ 0, & \text{if } l(w s_i) > l(w). \end{cases} \tag{2.11}$$

Denote $\mathfrak{S}_{s_i}$ by $\mathfrak{S}_i$. For any $1 \le i < n$,

$$\mathfrak{S}_i = x_1 + \cdots + x_i. \tag{2.12}$$

See [Ma, (4.4)]. The following is an important variant of Monk's formula.

MONK'S FORMULA [Ma, (4.11)]. *Let $1 \le i < n$ and $w \in S_n$. Then*

$$\mathfrak{S}_i \mathfrak{S}_w = \sum_t \mathfrak{S}_{wt},$$

where the sum extends over all transitions $t = t_{jk}$ interchanging j and k, with $1 \le j \le i < k \le n$ and $l(wt) = l(w) + 1$.

The description of the action of the operator X_i on Schubert polynomials follows from Monk's formula and (2.12),

$$\begin{aligned} X_i(\mathfrak{S}_w) = (\mathfrak{S}_i - \mathfrak{S}_{i-1})\mathfrak{S}_w &= \sum_{j \le i < k} \mathfrak{S}_{wt_{jk}} - \sum_{j < i \le k} \mathfrak{S}_{wt_{jk}} \\ &= \sum_{j = i < k} \mathfrak{S}_{wt_{jk}} - \sum_{j < i = k} \mathfrak{S}_{wt_{jk}}, \end{aligned} \tag{2.13}$$

where all summations are over the transpositions $t = t_{jk}$ satisfying $l(wt) = l(w) + 1$, with j and k in the indicated ranges.

2.2.3. *The Coinvariant Algebra*

Recall that $\Lambda_n = \Lambda[x_1, \ldots, x_n]$ is the subring of P_n consisting of symmetric functions, and let I_n be the ideal of P_n generated by symmetric functions without a constant term. The quotient P_n/I_n is called the *coinvariant algebra* of S_n. S_n acts naturally on this algebra. The resulting representation is isomorphic to the regular representation of the symmetric group. See, e.g., [Hu, Sect. 3.6; Hi, Sect. II.3].

Let R^k $(0 \le k \le \binom{n}{2})$ be the kth homogeneous component of the coinvariant algebra: $P_n/I_n = \oplus_{k=0}^{\binom{n}{2}} R^k$. Each R^k is an $F[S_n]$-module; let χ^k be the corresponding character. The set $\{\mathfrak{S}_w \mid w \in S_n\}$ of Schubert polynomials forms a basis for P_n/I_n, and the set $\{\mathfrak{S}_w \mid l(w) = k\}$ forms a basis for R^k.

The action of the simple reflections on Schubert polynomials is described by the following proposition, which is a reformulation of [BGG, Theorem 3.14(iii)].

PROPOSITION 2.3. *For any simple reflection s_i and any $w \in S_n$,*

$$s_i(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_w, & \text{if } l(ws_i) > l(w), \\ -\mathfrak{S}_w + \sum_{k<i} \mathfrak{S}_{w(k,i+1,i)} - \sum_{k<i} \mathfrak{S}_{w(k,i,i+1)} \\ +\sum_{k>i+1} \mathfrak{S}_{w(k,i,i+1)} - \sum_{k>i+1} \mathfrak{S}_{w(k,i+1,i)}, \\ & \text{if } l(ws_i) < l(w), \end{cases}$$

where $(k, i, i+1), (k, i+1, i)$ are cycles of length 3, and the sums extend over those values of k (in the prescribed ranges) for which $w(k, i, i+1)$ (respectively, $w(k, i+1, i)$) has the same length as w.

Note that the signs in this proposition may depend on notational conventions.

Let μ be a partition of n, and let χ^k be the S_n-character on R^k as above. The following character formula is analogous to Theorem 2.2.

THEOREM 2.4 [Ro2, Theorem 2]. *With the notations of Theorem 2.2,*

$$\chi^k(w_\mu) = \sum_{l(w)=k} \text{weight}_1^\mu(w),$$

where $\text{weight}_1^\mu(w)$ *is the weight* (1.4) *with* $q = 1$, *and* w_μ *is any permutation of cycle-type* μ.

The goal of this paper is to define a Hecke algebra action on the polynomial ring P_n which produces a q-analogue of Theorem 2.4.

3. q-COMMUTATORS

For $1 \le i < n$ define the q-commutator $[\partial_i, X_i]_q$ as follows:

$$[\partial_i, X_i]_q := \partial_i X_i - qX_i\partial_i.$$

It should be noted that for $q = 1$, $[\partial_i, X_i]_1 = s_i$. Let $A_i := [\partial_i, X_i]_q$.

Claim 3.1. The operators A_i, $1 \le i < n$, satisfy the Hecke algebra relations (2.1)–(2.3).

Proof. Combine the nil-Coxeter relations (2.4)–(2.6) for the operators ∂_i with the commutation relations (2.7)–(2.9) for the operators ∂_i and X_j. ∎

It follows that the mapping $T_i \mapsto A_i$ $(1 \le i < n)$ may be extended to a representation ρ_1 of $\mathscr{H}_n(q)$ on P_n:

$$\rho_1(T_i) := A_i = [\partial_i, X_i]_q.$$

Remark. The polynomial action of the Coxeter generators of S_n is *multiplicative*; i.e., for any generator s_i and any two polynomials $f, g \in P_n$,

$$s_i(fg) = s_i(f)s_i(g). \tag{3.1}$$

Thus each s_i acts on P_n as an *algebra* automorphism. It follows that if f is i-symmetric (see Section 2.2.1) then $s_i(fg) = fs_i(g)$. In contrast to that, the operators A_i are not multiplicative. Actually, (2.3) implies that the eigenvalues of any linear action of a Hecke algebra generator T_i are 1 and $-q$, and taking f to be a $(-q)$-eigenvector of A_i, one would get (if A_i were multiplicative) that f^2 is a q^2-eigenvector, which is impossible for generic q.

Claim 3.2. For any $1 \le i < n$, any i-symmetric polynomial $f \in \Lambda_n(i)$, and any polynomial $g \in P_n$,

$$A_i(f) = f \tag{3.2}$$

and

$$A_i(fg) = fA_i(g). \tag{3.3}$$

Proof. (3.2) is the special case $g = 1$ of (3.3). The latter follows from the fact that for arbitrary polynomials $f, g \in P_n$

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g).$$

Therefore, if $f \in \Lambda_n(i)$ then $\partial_i(fg) = f\partial_i(g)$, so that

$$A_i(fg) = \partial_i(x_i fg) - qx_i\partial_i(fg) = f[\partial_i(x_i g) - qx_i\partial_i(g)] = fA_i(g).$$

∎

It follows that the ideal I_n of P_n is invariant under all the operators A_i, giving rise to a representation $\tilde{\rho}_1$ of $\mathscr{H}_n(q)$ on the quotient P_n/I_n, namely, on the coinvariant algebra. Let χ_1^k be the character of this representation on the kth homogeneous component R^k of P_n/I_n $(0 \le k \le \binom{n}{2})$.

Recall from Section 2.2.3 that the set of Schubert polynomials $\{\mathfrak{S}_w \mid l(w) = k\}$ forms a basis for R^k.

The representation $\tilde{\rho}_1$ yields a q-analogue of Proposition 2.3.

THEOREM 3.3. *For any* $1 \le i < n$ *and* $w \in S_n$,

$$\tilde{\rho}_1(T_i)(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_w, & \text{if } l(ws_i) > l(w), \\ -q\mathfrak{S}_w + q\sum_{k<i}\mathfrak{S}_{w(k,i+1,i)} - \sum_{k<i}\mathfrak{S}_{w(k,i,i+1)} \\ +\sum_{k>i+1}\mathfrak{S}_{w(k,i,i+1)} - q\sum_{k>i+1}\mathfrak{S}_{w(k,i+1,i)}, \\ & \text{if } l(ws_i) < l(w), \end{cases}$$

where $(k,i,i+1),(k,i+1,i)$ *are cycles of length* 3, *and the sums extend over those values of* k (*in the prescribed ranges*) *for which* $w(k,i,i+1)$ (*respectively*, $w(k,i+1,i)$) *has the same length as* w.

Proof. By the commutation relation (2.8),

$$A_i = 1 + (X_{i+1} - qX_i)\partial_i.$$

Applying (2.11) and (2.13) completes the proof. ∎

4. CHARACTERS OF q-COMMUTATORS

In this section we prove the following q-analogue of Theorem 2.4.

THEOREM 4.1. *For any partition* $\mu \vdash n$ *and* $k \ge 0$,

$$\chi_1^k(T_\mu) = \sum_{l(w)=k} \text{weight}_q^\mu(w),$$

where $\text{weight}_q^\mu(w)$ *is defined as in* (1.4), *and the subproduct* T_μ *is defined as in Section* 2.1.

First recall that, by Theorem 3.3, for any $1 \le i < n$ and $w \in S_n$

$$\tilde{A}_i(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_w, & \text{if } l(ws_i) > l(w), \\ -q\mathfrak{S}_w + \sum_{l(zs_i)>l(z)=l(w)} a_{w,z}(q)\mathfrak{S}_z, & \text{if } l(ws_i) < l(w), \end{cases} \tag{4.1}$$

where $\tilde{A}_i := \tilde{\rho}_1(T_i)$, $a_{w,z}(q) \in \mathbf{Z}[q]$, and the summation is over all $z \in S_n$ with $l(zs_i) > l(z) = l(w)$.

Denote by $\langle \cdot , \cdot \rangle$ the inner product on P_n/I_n defined by $\langle \mathfrak{S}_v, \mathfrak{S}_w \rangle := \delta_{vw}$, where δ_{vw} is the Kronecker delta. In order to prove Theorem 4.1 we need the following lemma.

LEMMA 4.2. *Let* $w \in S_n$ *be a permutation satisfying* $l(ws_i) < l(w)$. *Then, for any* $\pi \in S_n$,

$$\left\langle \tilde{A}_i \tilde{A}_\pi(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle = -q\left\langle \tilde{A}_\pi(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle.$$

Proof of Lemma 4.2. It follows from (4.1) that if $l(ws_i) < l(w)$ and $v \in S_n$ then

$$\left\langle \tilde{A}_i(\mathfrak{S}_v), \mathfrak{S}_w \right\rangle = \begin{cases} -q, & \text{if } v = w, \\ 0, & \text{if } v \neq w. \end{cases} \tag{4.2}$$

Substituting (4.2) into

$$\left\langle \tilde{A}_i \tilde{A}_\pi(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle = \left\langle \tilde{A}_i\left(\sum_v \langle \tilde{A}_\pi(\mathfrak{S}_w), \mathfrak{S}_v \rangle \mathfrak{S}_v \right), \mathfrak{S}_w \right\rangle$$
$$= \sum_v \left\langle \tilde{A}_\pi(\mathfrak{S}_w), \mathfrak{S}_v \right\rangle \left\langle \tilde{A}_i(\mathfrak{S}_v), \mathfrak{S}_w \right\rangle$$

we obtain the desired conclusion. ∎

Proof of Theorem 4.1. In order to prove Theorem 4.1, it suffices to prove that for any partition $\mu = (\mu_1, \dots, \mu_t)$ of n

$$\left\langle \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle = \text{weight}_q^\mu(w),$$

where $\tilde{A}_\mu = \tilde{\rho}_1(T_\mu)$ is the subproduct of $\tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_{n-1}$ obtained by omitting $\tilde{A}_{\mu_1 + \cdots + \mu_i}$ for all $1 \leq i < t$.

Assume now that there is an index i such that $\tilde{A}_i$ and $\tilde{A}_{i+1}$ are factors of $\tilde{A}_\mu$, $l(ws_i) > l(w)$, and $l(ws_{i+1}) < l(w)$. Then, by Lemma 4.2,

$$\left\langle \tilde{A}_{i+1} \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle = -q \left\langle \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle. \tag{4.3}$$

On the other hand, by the Hecke algebra relations, $\tilde{A}_{i+1} \tilde{A}_\mu = \tilde{A}_\mu \tilde{A}_i$. Hence

$$\left\langle \tilde{A}_{i+1} \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle = \left\langle \tilde{A}_\mu \tilde{A}_i(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle = \left\langle \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle. \tag{4.4}$$

The last equality follows from (4.1).

Comparing (4.3) and (4.4) we obtain

$$-q \left\langle \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle = \left\langle \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle.$$

We conclude that, if there is an index i so that $\tilde{A}_i$ and $\tilde{A}_{i+1}$ are factors of $\tilde{A}_\mu$, $l(ws_i) > l(w)$, and $l(ws_{i+1}) < l(w)$, then (since q is indeterminate)

$$\left\langle \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle = 0.$$

Note that in this case i, $i+1$, and $i+2$ belong to the same "block" in the partition μ, and $w(i) < w(i+1) > w(i+2)$. Thus indeed

$$\text{weight}_q^\mu(w) = 0.$$

It remains to check the case in which there is no index i so that both $\tilde{A}_i$ and $\tilde{A}_{i+1}$ appear as factors in the product $\tilde{A}_\mu$, with $l(ws_i) > l(w)$ and $l(ws_{i+1}) < l(w)$.

In this case, the relation $\tilde{A}_i \tilde{A}_j = \tilde{A}_j \tilde{A}_i$ for $|i - j| > 1$ gives

$$\tilde{A}_\mu = \tilde{A}_{i_1} \cdots \tilde{A}_{i_m} \tilde{A}_{i_{m+1}} \cdots \tilde{A}_{i_{\mu_1 + \cdots + \mu_t - t}},$$

where $l(ws_{i_j}) < l(w)$ for $j \le m$, and $l(ws_{i_j}) > l(w)$ for $j > m$. Applying (4.1) and Lemma 4.2 iteratively implies

$$\left\langle \tilde{A}_\mu(\mathfrak{S}_w), \mathfrak{S}_w \right\rangle = (-q)^m = \text{weight}_q^\mu(w),$$

where $m = \#\{i \mid l(ws_i) < l(w) \text{ and } \tilde{A}_i \text{ is a factor of } \tilde{A}_\mu\}$. ∎

5. RANDOMIZED OPERATORS

In this section we define a natural "randomized" action of the Coxeter generators on the polynomial ring P_n, and show that this action satisfies the Hecke algebra relations. This action will be defined initially on monomials, and then extended by linearity to all polynomials in P_n.

Let $e_{\alpha, \beta, m} := x_i^\alpha x_{i+1}^\beta m$, where $m \in P_n$ is a monomial involving neither x_i nor x_{i+1}, and α, β are nonnegative integers. Note that the linear subspace $V_{\alpha, \beta, m} := \text{span}\{e_{\alpha, \beta, m}, e_{\beta, \alpha, m}\}$ is invariant under the action of s_i. In this space s_i acts as a transposition of the two basis elements (if $\alpha \neq \beta$).

A natural randomization of $e_{\alpha, \beta, m}$ is $(1 - q)e_{\alpha, \beta, m} + qe_{\beta, \alpha, m}$, where the parameter q may be interpreted as transition probability $0 \le q \le 1$. Motivated by well-known asymmetric physical processes (simulated annealing etc.), we define

$$R_i^*(e_{\alpha, \beta, m}) := \begin{cases} e_{\beta, \alpha, m}, & \text{if } \alpha \ge \beta, \\ (1 - q)e_{\alpha, \beta, m} + qe_{\beta, \alpha, m}, & \text{if } \alpha < \beta, \end{cases} \tag{5.1}$$

and extend this randomized action to the whole polynomial ring P_n by linearity. See also [Ji].

Claim 5.1. The operators R_i^*, $1 \le i < n$, satisfy the Hecke algebra relations (2.1)–(2.3).

Proof. This is easily verified by an explicit calculation of the action on the monomials $e_{\alpha, \beta, m}$. ∎

The operators R_i^* lead, therefore, to a representation of $\mathscr{H}_n(q)$ on P_n. Unfortunately, the symmetric functions are not invariant under this action. Consider, therefore, the operators whose representation matrices with respect to the basis of monomials are the transposes of those representing R_i^*; i.e., define

$$R_i(e_{\alpha,\beta,m}) := \begin{cases} qe_{\beta,\alpha,m}, & \text{if } \alpha > \beta, \\ (1-q)e_{\alpha,\beta,m} + e_{\beta,\alpha,m}, & \text{if } \alpha < \beta, \\ e_{\alpha,\beta,m}, & \text{if } \alpha = \beta. \end{cases} \tag{5.2}$$

Of course, the operators R_i, $1 \le i < n$, also satisfy the Hecke relations (2.1)–(2.3). It follows that the mapping $T_i \mapsto R_i$ $(1 \le i < n)$ may be extended to a representation ρ_2 of $\mathscr{H}_n(q)$ on P_n:

$$\rho_2(T_i) := R_i.$$

The following claim is analogous to Claim 3.2.

Claim 5.2. For any $1 \le i < n$, any i-symmetric polynomial $f \in \Lambda_n(i)$, and any polynomial $g \in P_n$,

$$R_i(f) = f \tag{5.3}$$

and

$$R_i(fg) = fR_i(g). \tag{5.4}$$

Proof. This may be shown by direct calculation. ∎

It follows from the first part of the claim that symmetric functions are pointwise invariant under $\rho_2(\mathscr{H}_n(q))$. By the second part, the ideal I_n is also invariant under $\rho_2(\mathscr{H}_n(q))$. Thus, ρ_2 gives rise to a representation $\tilde{\rho}_2$ of $\mathscr{H}_n(q)$ on the coinvariant algebra P_n/I_n.

The action of R_i on monomials is transparent. Section 6 is devoted to a better understanding of the action on the coinvariant algebra.

6. PROPERTIES OF THE RANDOMIZED ACTION

The following sequence of assertions concerns the connections between the operators A_i and R_i.

Claim 6.1. The operators A_i and R_i have the same invariant vectors,

$$\ker(A_i - 1) = \ker(R_i - 1) = \Lambda_n(i),$$

where $\Lambda_n(i)$ is the set (actually, subalgebra) of all polynomials invariant under s_i.

Proof. By the definition of A_i and the commutation relations (2.8),

$$\ker(A_i - 1) = \ker\left[(X_{i+1} - qX_i)\partial_i\right] = \ker \partial_i = \Lambda_n(i).$$

As for $R_i - 1$, let $V_{\alpha,\beta,m} := \text{span}\{e_{\alpha,\beta,m}, e_{\beta,\alpha,m}\}$ as in the beginning of Section 5. Note that

$$P_n = \bigoplus_{\{(\alpha,\beta,m) \mid \alpha \geq \beta\}} V_{\alpha,\beta,m}$$

is a decomposition of P_n into a direct sum of R_i-invariant subspaces.

By (5.2), in $V_{\alpha,\beta,m}$,

$$(R_i - 1)(e_{\alpha,\beta,m}) = \begin{cases} -e_{\alpha,\beta,m} + qe_{\beta,\alpha,m}, & \text{if } \alpha > \beta, \\ -qe_{\alpha,\beta,m} + e_{\beta,\alpha,m}, & \text{if } \alpha < \beta, \\ 0, & \text{if } \alpha = \beta. \end{cases}$$

Thus

$$\ker(R_i - 1) \cap V_{\alpha,\beta,m} = \begin{cases} \text{span}\{e_{\alpha,\beta,m} + e_{\beta,\alpha,m}\}, & \text{if } \alpha \neq \beta, \\ \text{span}\{e_{\alpha,\beta,m}\}, & \text{if } \alpha = \beta, \end{cases}$$

implying

$$\ker(R_i - 1) = \bigoplus_{\{(\alpha,\beta,m) \mid \alpha \geq \beta\}} \text{span}\{e_{\alpha,\beta,m} + e_{\beta,\alpha,m}\} = \Lambda_n(i).$$

∎

Claim 6.2. (a) For any positive integers $i < n$, $j \leq n$ and m,

$$(A_i - R_i)\left(x_j^m\right) = (1 - q)\partial_i\left(x_j^{m+1}\right). \tag{6.1}$$

(b) For any polynomial $f \in \mathbf{Z}[x_1, \ldots, x_n]$, the polynomial $(A_i - R_i)f$ is i-symmetric and divisible by $1 - q$:

$$(A_i - R_i)f \in \Lambda_n(i) \cap (1 - q) \cdot \mathbf{Z}[x_1, \ldots, x_n].$$

Proof. (a) If $j \notin \{i, i+1\}$ then both sides of (6.1) equal zero. If $j = i$ and $m \geq 1$ then

$$(A_i - R_i)(x_i^m) = (1 + (x_{i+1} - qx_i)\partial_i - R_i)(x_i^m)$$

$$= x_i^m + (x_{i+1} - qx_i) \sum_{t=1}^{m} x_i^{m-t} x_{i+1}^{t-1} - qx_{i+1}^m$$

$$= (1-q) \sum_{t=0}^{m} x_i^{m-t} x_{i+1}^{t} = (1-q)\,\partial_i\big(x_i^{m+1}\big).$$

If $j = i + 1$ and $m \geq 1$ then, by Claim 6.1.

$$(A_i - R_i)(x_{i+1}^m) = (A_i - R_i)(x_i^m + x_{i+1}^m - x_i^m) = (A_i - R_i)(-x_i^m)$$
$$= -(1-q)\,\partial_i\big(x_i^{m+1}\big) = (1-q)\,\partial_i\big(x_{i+1}^{m+1}\big).$$

(b) It suffices to prove this claim for monomials $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$. Any such monomial has the form gx_j^m, where $g \in \Lambda_n(i)$, $j \in \{i, i+1\}$, and m is a nonnegative integer. If $m = 0$ then $(A_i - R_i)(g) = 0$. Otherwise, it follows from (3.3), (5.4), and (6.1) that

$$(A_i - R_i)\big(gx_j^m\big) = g(A_i - R_i)\big(x_j^m\big) = (1-q)\,g\,\partial_i\big(x_j^{m+1}\big), \quad (6.2)$$

as claimed. ∎

LEMMA 6.3. *$\Lambda_n(i)$ is spanned, as a Λ_n-module, by the Schubert polynomials $\mathfrak{S}_w$ with $l(ws_i) > l(w)$. The same holds when the ground field F is replaced by* $\mathbf{Z}$.

Proof. First of all, if $l(ws_i) > l(w)$ then, by Proposition 2.3, $s_i(\mathfrak{S}_w) = \mathfrak{S}_w$ and therefore $\mathfrak{S}_w \in \Lambda_n(i)$.

By the same proposition, for any $w \in S_n$

$$(1 + s_i)(\mathfrak{S}_w) \in \operatorname{span}\{\mathfrak{S}_z \mid l(zs_i) > l(z) = l(w)\} \qquad (\text{in } P_n/I_n),$$

and therefore, for any $f \in P_n/I_n$,

$$(1 + s_i)(f) \in \operatorname{span}\{\mathfrak{S}_z \mid l(zs_i) > l(z)\} \qquad (\text{in } P_n/I_n).$$

If $f \in \Lambda_n(i)/I_n$ then $(1 + s_i)(f) = 2f$, so that $\Lambda_n(i)/I_n$ is spanned, as a vector space, by the above Schubert polynomials. Since I_n is the ideal of P_n generated by the homogeneous elements in Λ_n of positive degree, a standard argument yields the claimed result for $\Lambda_n(i)$ as a Λ_n-module.

The result for $\mathbf{Z}$ instead of F follows from the fact that Schubert polynomials also form a "basis" for polynomials with integer coefficients. ∎

The following proposition provides a description of the action of $\tilde{\rho}_2(\mathscr{H}_n(q))$ on Schubert polynomials.

PROPOSITION 6.4. *For each* $1 \le i < n$ *and* $w \in S_n$,

$$\tilde{\rho}_2(T_i)(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_w, & \text{if } l(ws_i) > l(w), \\ -q\mathfrak{S}_w + \sum_{l(zs_i) > l(z)} [(1-q)b_{w,z} + c_{w,z}]\mathfrak{S}_z, & \\ & \text{if } l(ws_i) < l(w), \end{cases}$$

where $b_{w,z} \in \mathbf{Z}$, $c_{w,z} \in \{-1, 0, 1\}$, *and the sum extends over all permutations* $z \in S_n$ *with* $l(zs_i) > l(z) = l(w)$.

Proof. Since $\mathfrak{S}_w \in \Lambda_n(i)$ for $w \in S_n$ with $l(ws_i) > l(w)$, Claim 6.1 implies that $\rho_2(T_i)(\mathfrak{S}_w) = \mathfrak{S}_w$ for these w.

Homogeneous components of P_n are invariant under the action of each R_i, $1 \le i < n$. It follows that the homogeneous components of the coinvariant algebra are invariant under $\tilde{\rho}_2(\mathscr{H}_n(q))$, so that each $\tilde{\rho}_2(T_i)(\mathfrak{S}_w)$ is spanned by Schubert polynomials of degree $l(w)$. Combining this fact with Claim 6.2(b) and Lemma 6.3 shows that for any $1 \le i < n$ and $w \in S_n$

$$(\tilde{\rho}_2(T_i) - \tilde{\rho}_1(T_i))(\mathfrak{S}_w) = (1-q) \sum_{l(zs_i) > l(z) = l(w)} d_{w,z}\mathfrak{S}_z, \quad (6.3)$$

where $d_{w,z} \in \mathbf{Z}$, and the sum extends over all permutations $z \in S_n$ with $l(zs_i) > l(z) = l(w)$.

Combining (6.3) with (4.1) gives, for any $w \in S_n$ with $l(ws_i) < l(w)$,

$$\begin{aligned} \tilde{\rho}_2(T_i)(\mathfrak{S}_w) &= \tilde{\rho}_1(T_i)(\mathfrak{S}_w) + (\tilde{\rho}_2(T_i) - \tilde{\rho}_1(T_i))(\mathfrak{S}_w) \\ &= -q\mathfrak{S}_w + \sum_{l(zs_i) > l(z) = l(w)} a_{w,z}\mathfrak{S}_z \\ &\quad + (1-q) \sum_{l(zs_i) > l(z) = l(w)} d_{w,z}\mathfrak{S}_z \\ &= -q\mathfrak{S}_w + \sum_{l(zs_i) > l(z) = l(w)} [(1-q)d_{w,z} + a_{w,z}]\mathfrak{S}_z, \end{aligned}$$

where $a_{w,z} \in \{0, \pm 1, \pm q\}$, $d_{w,z} \in \mathbf{Z}$, and the sum extends over all $z \in S_n$ with $l(zs_i) > l(z) = l(w)$,

Substituting

$$b_{w,z} = \begin{cases} d_{w,z}, & \text{if } a_{w,z} \neq \pm q \\ d_{w,z} - a_{w,z}/q, & \text{if } a_{w,z} = \pm q \end{cases}$$

and

$$c_{w,z} = \begin{cases} a_{w,z}, & \text{if } a_{w,z} \neq \pm q \\ a_{w,z}/q, & \text{if } a_{w,z} = \pm q \end{cases}$$

completes the proof. ∎

Imitating the proof of Theorem 4.1 we obtain

THEOREM 6.5.

$$\chi_2^k(T_\mu) = \sum_{l(w)=k} \text{weight}_q^\mu(w),$$

where $\text{weight}_q^\mu(w)$ *are the same weights as in Theorem* 4.1.

Combining Theorems 6.5 and 4.1 together with Ram's result (Theorem 2.1) shows that

THEOREM 6.6. *The representation of* $\mathscr{H}_n(q)$ *induced by the* q*-commutators* A_i *on the homogeneous components of the coinvariant algebra and the representation induced by the transposed randomized operators* R_i *on these components are equivalent.*

Problem 6.7. Calculate the coefficients $b_{w,z}$ in Proposition 6.4.

We conjecture that $b_{w,z} \in \{-1, 0, 1\}$.

7. FINAL REMARKS

7.1. *Related Families of Operators*

Consider the following family of q-commutators:

$$B_i := -[\partial_i, X_{i+1}]_q \qquad (1 \leq i < n).$$

This family is closely related to the q-commutators A_i.

Fact 7.1. The operators B_i satisfy the Hecke algebra relations (2.1)–(2.3).

PROPOSITION 7.2. *Let* D_i *be operators on the polynomial ring* P_n *of the form* $c_i + P_i(X_i, X_{i+1})\partial_i$, $1 \leq i < n$, *where* P_i *are polynomials of two variables, and* c_i *are constants. If*

(1) D_i *satisfy the Hecke algebra relations* (2.1)–(2.3) (*with* $q \neq 0, -1$),

(2) D_i *are degree preserving* (*as operators on* P_n),

(3) Λ_n, *the subring of symmetric functions, is pointwise invariant under all* D_i, $1 \le i < n$,

then, for $n > 2$, *either* $D_i = A_i$ $(\forall i)$ *or* $D_i = B_i$ $(\forall i)$.

Proposition 7.2 is related to a general theorem of Lascoux and Schützenberger.

LS THEOREM [LS, Theorem 1]. *Let* x_1, x_2, x_3 *be variables, and let* s_i $i = 1, 2$ *be the simple transpositions as above. Let* D_i, $i = 1, 2$ *be linear operators on the ring of* rational functions $\mathbf{C}(x_1, x_2, x_3)$ (*considered as a vector space over* $\mathbf{C}$) *defined by*

$$D_i = P_i + Q_i s_i,$$

where $P_i, Q_i \in \mathbf{C}(x_i, x_{i+1})$ *are rational functions of the corresponding pair of variables.*

Assume that:

(1) $D_1 D_2 D_1 = D_2 D_1 D_2$;

(2) D_1 *is invertible and* $P_1 \neq 0$.

Then D_1 *and* D_2 *preserve the ring of* polynomials $\mathbf{C}[x_1, x_2, x_3]$ *if and only if there exist* $\alpha, \beta, \gamma, \delta, \eta \in \mathbf{C}$, *so that* $\Delta := \alpha\delta - \beta\gamma \neq 0$, $\eta \neq 0$, $\eta \neq \Delta$, *and*

$$P_i(x_i, x_{i+1}) = (x_i - x_{i+1})^{-1}(\alpha x_i + \beta)(\gamma x_{i+1} + \delta) \qquad \text{and}$$
$$Q_i = \eta - P_i.$$

Also, in that case, both D_1 *and* D_2 *satisfy* $D_i^2 = \Delta D_i + \eta(\eta - \Delta)$.

Obviously, the initial conditions in this theorem are quite different from those of Proposition 7.2; but the two families A_i and B_i are common solutions of both problems (for the LS theorem in the special case $\Delta = 1 - q$, $\eta = 1$). Note that for $\Delta = 1 - q$, $\eta = -q$ one gets two other families of the q-commutator type, for which Λ_n is *not* pointwise invariant.

It should be mentioned that the family R_i of Sections 5 and 6 is *not* obtainable from the LS theorem (or from Proposition 7.2).

7.2. *Connections with Kazhdan–Lusztig Theory*

Theorem 3.3 has a remarkable analogue in Kazhdan–Lusztig theory. In their seminal paper [KL] Kazhdan and Lusztig constructed a canonical basis to Hecke algebra representations. A basic theorem in this theory describes the action of the generators T_s on the canonical basis elements C_w.

THEOREM 7.3 [KL, (2.3a)–(2.3c)]. *Let W be a Coxeter group, s a Coxeter generator of W, $w \in W$, and C_w the corresponding Kazhdan–Lusztig basis element. Then*

$$T_s(C_w) = \begin{cases} -C_w, & \text{if } l(sw) < l(w), \\ qC_w + q^{1/2} \quad \sum_{l(sz) < l(z) = l(w)} a_{w,z} C_z, & \text{if } l(sw) > l(w), \end{cases}$$

where the coefficients $a_{w,z} \in \mathbf{Z}$ are independent of q.

This analogy leads to similar character formulas in the two theories; see Theorems 2.2 and 4.1. This surprising phenomenon seems to warrant further study.

7.3. *Probabilistic Aspects*

The parameter q in the definition of the Hecke algebra may be interpreted as a *transition probability*. This gives a natural interpretation to the appearance of the coefficients q and $1 - q$ in the basic Hecke relation (2.3). This observation was fundamental to the definition of the randomized action in Section 5. The operators defined there interpolate between two well-studied extreme cases: sorting ($q = 0$) and mixing ($q = 1$) by means of adjacent transpositions. They also form an interesting link between algebra and physics-motivated optimization.

7.4. *Other Weyl Groups*

Extension of all the above to other Weyl and Coxeter groups is highly desirable. Preliminary computations indicate that this may not be straightforward.

REFERENCES

[BGG] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, Schubert cells and cohomology of Schubert spaces G/P, *Usp. Mat. Nauk.* **28** (1973), 3–26.

[De] M. Demazure, Invariants symétriques entiers de groupes de Weyl et torsion, *Invent. Math.* **21** (1973), 287–301.

[DKLLST] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf, and J.-Y. Thibon, Euler–Poincaré characteristic and polynomial representations of Iwahori–Hecke algebras, *Publ. Res. Inst. Math. Sci.* **31** (1995), 179–201.

[Hi] H. L. Hiller, "Geometry of Coxeter Groups," Res. Notes in Math., Vol. 54, Pitman, Boston, 1982.

[Hr] F. Hirzebruch, "Topological Methods in Algebraic Geometry," Springer-Verlag, Berlin, 1966.

[Hu] J. E. Humphreys, "Reflection Groups and Coxeter Groups," Cambridge Studies in Advanced Math., Vol. 29, Cambridge Univ. Press, Cambridge, UK, 1992.

[Ji] M. Jimbo, A q-difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation, *Lett. Math. Phys.* **10** (1985), 63–69.

[KL] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[LS] A. Lascoux and M. P. Schützenberger, Symmetrization operators on polynomial rings, *Funct. Anal. Appl.* **21** (1987), 324–326.

[Ma] I. G. Macdonald, Schubert polynomials, *in* "Surveys in Combinatorics, 1991 (Guildford, 1991)," London Math. Soc. Lecture Note Ser., Vol. 166, Cambridge Univ. Press, Cambridge, UK, 1991.

[Ra1] A. Ram, A Frobenius formula for the characters of the Hecke algebras, *Invent. Math.* **106** (1991), 461–488.

[Ra2] A. Ram, An elementary proof of Roichman's rule for irreducible characters of Iwahori–Hecke algebras of type A, *in* "Mathematical Essays in Honor of Gian-Carlo Rota," Progr. Math., Vol. 161, pp. 335–342, Birkhäuser, Boston, 1998.

[Ro1] Y. Roichman, A recursive rule for Kazhdan–Lusztig characters, *Adv. Math.* **129** (1997), 25–45.

[Ro2] Y. Roichman, Schubert polynomials, Kazhdan–Lusztig basis and characters, (with an appendix: On characters of Weyl groups, co-authored with R. M. Adin and A. Postnikov), *Discrete Math.*, to appear.