Available online at www.sciencedirect.com

SCIENCE DIRECT®

LINEAR ALGEBRA AND ITS APPLICATIONS

ELSEVIER

Linear Algebra and its Applications 369 (2003) 153–167

www.elsevier.com/locate/laa

Error analysis of signal zeros: a projected companion matrix approach

F.S.V. Bazán [1]

Departamento de Matemática, Universidade Federal de Santa Catarina, Florianópolis, SC 88040-900, Brazil

Received 27 April 2002; accepted 3 December 2002

Submitted by G.W. Stewart

Abstract

An error analysis of so-called signal zeros of polynomials associated with exponentially damped/undamped signals is performed and zero error bounds are derived. The bounds are in terms of the angle between the exact and approximate signal subspace, the signal parameter themselves, the polynomial degree, and the error on the polynomial coeficients. The key idea behind the analysis is to regard signal zeros as eigenvalues of projected companion matrices and then to generate error bounds by exploiting perturbation theorem for eigenvalues. The conclusion drawn from the bounds is that the signal zeros become relatively insensitive to small perturbations on the polynomial coefficients as long as the polynomial degree is large enough and the zeros are extracted as eigenvalues of projected companion matrices. Also, the bounds suggests that signal zero estimates derived from projected companion matrices are more accurate than those obtained from the companion matrices themselves. Illustrative numerical results are provided.

Keywords: Eigenvalue error analysis; Companion matrix; Time series; Linear prediction

E-mail address: fermin@mtm.ufsc.br (F.S.V. Bazán).

[1] This research was supported by CNPq, Brasil, grant 300487/94-0(NV).

0024-3795/03/$ - see front matter
doi:10.1016/S0024-3795(02)00723-1

1. Introduction

Let $P_N(z)$ denote a monic polynomial:

$$P_N(z) = f_0 + f_1 z + \cdots + f_{N-1}z^{N-1} + z^N \tag{1.1}$$

whose coefficients f_j satisfy the recurrence relation

$$f_0 h_k + f_1 h_{k+1} + \cdots + f_{N-1} h_{k+N-1} = h_{k+N}, \quad k = 0, 1, \ldots, \tag{1.2}$$

where h_k is defined by

$$h_k = \sum_{j=1}^{n} r_j \mathrm{e}^{s_j \Delta t\, k} = \sum_{j=1}^{n} r_j z_j^k. \tag{1.3}$$

In this definition, $r_j, s_j \in \mathbb{C}$, $s_j \neq s_k$ for $j \neq k$, $s_j = \alpha_j + \mathrm{i}\omega_j$, $\mathrm{i} = \sqrt{-1}$, $\alpha_j \leqslant 0$, and Δt, called the sampling rate, is assumed to satisfy $\max_j \omega_j \Delta t < \pi,\ 1 \leqslant j \leqslant n$. It is well known that if $N \geqslant n$, then $P_N(z)$ has $z_j = \mathrm{e}^{s_j \Delta t\, k}$ $(j = 1, 2, \ldots, n)$ as roots [2,20]. These z_j are often referred to as *signal zeros* and their estimation from noisy data $\tilde{h}_k = h_k + \epsilon_k,\ k = 0, 1, \ldots, L$, where ϵ_k represents noise, is an important problem in science and engineering. Applications of the problem include signal processing, radar, geophysics, and direction of arrival, among others [9–14,19].

Since in practical applications the polynomial coefficients are estimated from noisy data $\tilde{h}_k$ rather than being given exactly, the effect of uncertainties ϵ_k on the signal zeros z_j is an issue of considerable importance. The problem has received the attention of numerous researchers and several error analyses based on statistical properties of the noise have been carried out; for works in signal processing and system identification problems, see [19] and references therein. A recent work that does not rely on any statistical assumption can be found in [4]. The analysis of these authors relies on the fact that, since polynomial zeros can be regarded as companion matrix eigenvalues, both in theory and practice [6,18], the derivation of error bounds for the signal zeros from eigenvalue perturbation theory is always possible. Following this line of analysis, Bazán and Toint conclude that if the signal zeros are regarded as eigenvalues of the companion matrix F_N:

$$F_N = [e_2 \quad e_3 \quad \cdots \quad e_N \quad f] = \begin{bmatrix} 0 & 0 & \cdots & 0 & -f_0 \\ 1 & 0 & \cdots & 0 & -f_1 \\ 0 & 1 & \cdots & 0 & -f_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -f_{N-1} \end{bmatrix}, \tag{1.4}$$

the sensitivity of these eigenvalues to perturbations on the coefficients f_j is governed by $\kappa(W_N)$, the 2-norm condition number of W_N, where

$$W_N = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{N-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & z_n & z_n^2 & \cdots & z_n^{N-1} \end{bmatrix}. \tag{1.5}$$

Additionally, they provide an error bound which essentially depends on N and $\kappa(W_N)$ and show that the bound may be small as long as N is large enough.

In this work we derive new error bounds for the signal zeros, concentrating on bounds that relate the signal zeros to a small eigenvalue problem as opposed to the approach of Bazán and Toint that requires the solution of a large companion matrix eigenvalue problem. The key idea is to regard the signal zeros as eigenvalues of a small $n \times n$ matrix obtained by projecting the companion matrix onto the column space of W_N^* (the star symbol denotes complex conjugate transpose) and then to derive error bounds from the projected problem. A particularly nice thing behind this is that better error estimates can be obtained. Besides this, like the bound of Bazán and Toint, the ones derived here do not depend on the nature of the noise, i.e., the error analysis is free of statistical hypotheses.

For future reference, the columns subspace of W_N^* will be called *the signal subspace* and denoted by $\mathscr{S}_N$. Also, we note for later use that

$$W_N F_N = Z W_N, \tag{1.6}$$

where $Z = \text{diag}(z_1, \ldots, z_n)$. In the following the z_j's will be referred to as *signal eigenvalues*.

The paper is organized as follows. In Section 2 we review and derive a few basic results that are needed for our eigenvalue analysis. Section 3 contains a general bound for the eigenvalue error in terms of the angle between $\mathscr{S}_N$ and $\widetilde{\mathscr{S}}_N$, the polynomial degree and the signal parameter themselves. $\widetilde{\mathscr{S}}_N$ denotes an approximation for $\mathscr{S}_N$ and it is assumed to be extracted from available data. An important aspect of the bound is that it does not depend on any method used to compute the approximate signal subspace. When the approximate signal subspace is computed via the singular value decomposition (SVD), error bounds that provide insight into the problem are obtained. This is done in Section 4. Section 5 is devoted to numerically illustrate the theory, the superiority of our bounds over the one by Bazán and Toint is illustrated there. Finally, some conclusions are provided in Section 6.

2. Some preliminaries

The goal of the section is to introduce further notation, a few basic definitions and preliminary results. The conjugate transpose of A is A^* and $A^\dagger$ its Moore–Penrose pseudo-inverse. $\|\cdot\|_2$ denotes the 2-norm and $\|\cdot\|_{\text{F}}$ the Frobenius norm. The 2-norm condition number of A is $\kappa(A) = \|A\|_2\|A^\dagger\|_2$. The spectrum of $A \in \mathbb{C}^{n\times n}$ is denoted by $\lambda(A)$. The singular values of A are denoted by $\sigma_i(A)$ and ordered as $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_n(A) \geqslant 0$.

Definition 1. Let $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ be two subspaces in $\mathbb{R}^n$ of the same dimension, and let P and $\widetilde{P}$ be orthogonal projectors onto $\mathscr{S}$ and $\widetilde{\mathscr{S}}$, respectively. The distance between $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ is defined as (see, e.g., [7, p. 76])

$$\mathrm{d}(\mathscr{S}, \widetilde{\mathscr{S}}) = \|\mathscr{P} - \widetilde{\mathscr{P}}\|_2. \tag{2.1}$$

A known result regarding separation of equidimensional subspaces is

$$\mathrm{d}(\mathscr{S}, \widetilde{\mathscr{S}}) = \sin(\Theta), \tag{2.2}$$

where Θ denotes the largest canonical angle between $\mathscr{S}$ and $\widetilde{\mathscr{S}}$. For details on canonical angles, see [5].

Let $A \in \mathbb{C}^{n\times n}$ have simple eigenvalues and let λ_i an eigenvalue of A with right eigenvector v_i and left eigenvector u_i. Define

$$s_i = \frac{u_i^* v_i}{\|u_i\|_2 \|v_i\|_2}. \tag{2.3}$$

It is well known that s_i is always nonzero and that the real number $\kappa_i = 1/|s_i|$ serves as a measure for the sensitivity of eigenvalues λ_i to perturbations on A. Such a number is referred to as *condition number* of eigenvalue λ_i (see, e.g., [22, p. 314]). A precise result regarding eigenvalue perturbation along this line is as follows. Let $\widetilde{\lambda}_i$ be an eigenvalue of $\widetilde{A} = A + E \in \mathbb{C}^{n\times n}$ that is closest to eigenvalue λ_i. Then, for small enough E, the following first-order estimate holds:

$$|\widetilde{\lambda}_j - \lambda_j| \leqslant \left| \frac{u_j^* E v_j}{u_j^* v_j} \right| \leqslant \kappa_j \|E\|_2, \quad j = 1, \ldots, n. \tag{2.4}$$

Definition 2. For each $A \in \mathbb{C}^{n\times n}$, the Departure from normality of A, in the Frobenius norm, is

$$D^2(A) = \|A\|_{\mathrm{F}}^2 - \sum_{i=1}^{n} |\lambda_i|^2.$$

Number $D(A)$ measures how close is A of being a normal matrix. For normal matrices $D(A) = 0$ and the A-matrix eigenvalue problem is perfectly conditioned. Consequently, the smaller $D(A)$, the better the conditioning of the matrix eigenvalue problem. An informative result explaining this, due to Smith [15], is as follows.

Proposition 1. *Let $A \in \mathbb{C}^{n\times n}$ have simple eigenvalues λ_j. Define*

$$\delta_i = \min_{\substack{j \\ j\neq i}} |\lambda_i - \lambda_j|, \quad 1 \leqslant i, j \leqslant n.$$

Then it holds

$$1 \leqslant \kappa_i \leqslant \left[1 + \frac{D^2(A)}{(n-1)\delta_i^2}\right]^{(n-1)/2}, \quad i = 1, 2, \ldots, n. \tag{2.5}$$

We end the section by describing a technical result concerning the behavior of the polynomial coefficients f_j as a function of the degree of $P_N(z)$.

Proposition 2. *Let $H_{M,N}(\ell)$, $\ell \geqslant 0$, $M \geqslant N \geqslant n$, denote an $M \times N$ Hankel matrix whose (i, j) entry is $h_{\ell+i+j-2}$, with h_k defined in* (1.3). *Let $f^\dagger$ denote the minimum 2-norm solution of the linear system*

$$H_{M,N}(\ell)x = H_{M,N}(\ell+1)e_N, \tag{2.6}$$

*where e_N is the N*th *canonical vector in $\mathbb{R}^N$. Then $\|f^\dagger\|_2$ is a decreasing function of N, and*

$$\lim_{N\to\infty} \|f^\dagger\|_2 = 0. \tag{2.7}$$

Proof. It is well known that matrix $H_{M,N}(\ell)$ has a Vandermonde decomposition of the form

$$H_{M,N}(\ell) = W_M^{\mathrm{T}} Z^\ell RW_N, \tag{2.8}$$

where W_N and W_M are as in (1.5), $R = \operatorname{diag}(r_1, \ldots, r_n)$, and Z as in (1.6). From this it is straightforward that

$$f^\dagger = H_{M,N}(\ell)^\dagger H_{M,N}(\ell+1)e_N \quad \Leftrightarrow \quad f^\dagger = W_N^\dagger Z^N e, \tag{2.9}$$

with $e = [1 \ \cdots \ 1]^{\mathrm{T}} \in \mathbb{R}^n$. The statements of the proposition follow from this relation upon using Theorem 1 from [1]. □

3. Signal eigenvalue error bounds

As was mentioned in the Introduction, the bounds for the eigenvalue error that we state in the work will appear from analyzing a projected companion matrix eigenvalue problem. To explain this in a precise way we need to describe how the companion matrix eigenvalue problem and the projected one are related. Let the columns of $\mathcal{Q}_N \in \mathbb{C}^{N\times n}$ form an orthonormal basis for the signal subspace $\mathscr{S}_N = \mathscr{R}(W_N^*)$. It then follows that there exists a nonsingular matrix $G \in \mathbb{C}^{n\times n}$ such as

$$W_N^* = \mathcal{Q}_N G.$$

Substitution of this decomposition in (1.6) yields an equivalence of the type

$$W_N F_N = ZW_N \quad \Leftrightarrow \quad \mathcal{Q}_N^* F_N \mathcal{Q}_N = G^{-*}ZG^*, \tag{3.1}$$

which describes in a precise way the link between the projected eigenvalue problem described in the right equality and the related F_N-companion matrix eigenvalue problem. Equivalence (3.1) plays an important role in our analysis. First, for using the right equality we reduce the original (generally large) eigenvalue problem to one of order n, which yields substantial computational savings if $n \ll N$, and secondly, for we shall see that the eigenvalue error estimation problem from the projected companion matrix becomes simple. We now introduce some notation to describe the eigenvalue error associated with the projected problem. If the data are free of noise and $\mathscr{Q}_N$ and F_N are available, the $n \times n$ projected matrix in (3.1) will be denoted by $\mathscr{F}_N$, i.e.,

$$\mathscr{F}_N = \mathscr{Q}_N^* F_N \mathscr{Q}_N. \tag{3.2}$$

The counterpart of $\mathscr{F}_N$ for the case where the data are contaminated by noise is

$$\widetilde{\mathscr{F}}_N = \widetilde{\mathscr{Q}}_N^* \widetilde{F}_N \widetilde{\mathscr{Q}}_N, \tag{3.3}$$

where the columns of $\widetilde{\mathscr{Q}}_N \in \mathbb{C}^{N\times n}$ form an orthonormal basis for an n-dimensional subspace $\widetilde{\mathscr{S}}_N$, close to $\mathscr{S}_N$ (in some sense), and $\widetilde{F}_N$ denotes a perturbation of F_N that preserves the companion structure. Thus, if $\tilde{z}_j$ denotes the eigenvalue of $\widetilde{\mathscr{F}}_N$ that is closest to z_j, our goal is to estimate the eigenvalue error $|\tilde{z}_j - z_j|$, $j = 1 : n$.

With the above preparation, many bounds for the eigenvalue error may be generated by using appropriate perturbation theorems for eigenvalues. For example, using the Bauer–Fike Theorem [7, Theorem 7.2.2, p. 312] one may obtain estimates of the form

$$|\tilde{z}_j - z_j| \leqslant \kappa(X)\|\widetilde{\mathscr{F}}_N - \mathscr{F}_N\|_2, \quad j = 1 : n, \tag{3.4}$$

where $\kappa(X)$ is the 2-norm condition number of a right eigenvector matrix of $\mathscr{F}_N$. The difficulty found with this bound is that the error $\|\widetilde{\mathscr{F}}_N - \mathscr{F}_N\|_2$ is rather involved in our case. So, the bounds that we derive will arise from a related eigenproblem which can be described as follows. Let $\mathscr{P}$ and $\widetilde{\mathscr{P}}$ denote the orthogonal projectors on $\mathscr{S}_N$ and $\widetilde{\mathscr{S}}_N$, respectively, and introduce $\mathscr{F}_{N,\mathscr{P}}$ and $\widetilde{\mathscr{F}}_{N,\mathscr{P}}$ defined by

$$\mathscr{F}_{N,\mathscr{P}} = \mathscr{Q}_N \mathscr{Q}_N^* F_N \doteq \mathscr{P} F_N, \quad \widetilde{\mathscr{F}}_{N,\mathscr{P}} = \widetilde{\mathscr{Q}}_N \widetilde{\mathscr{Q}}_N^* \widetilde{F}_N \doteq \widetilde{\mathscr{P}} \widetilde{F}_N. \tag{3.5}$$

Our eigenvalue error bounds will thus arise from applying the eigenvalue perturbation result described in (2.4) to the matrix $\mathscr{F}_{N,\mathscr{P}}$. The reason of this is that, discarding zero eigenvalues

$$\lambda(\mathscr{F}_{N,\mathscr{P}}) = \lambda(\mathscr{F}_N) \quad \text{and} \quad \lambda(\widetilde{\mathscr{F}}_{N,\mathscr{P}}) = \lambda(\widetilde{\mathscr{F}}_N).$$

Note that the choice of $\mathscr{F}_{N,\mathscr{P}}$ is to simplify the analysis; in practice, signal zeros have to be extracted from the small $n \times n$ matrix $\mathscr{F}_N$ defined in (3.2).

The following propositions provide information on a measure for the sensitivity of eigenvalue z_j and an estimate for the error matrix $\|\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}}\|_2$.

Proposition 3. *Let $\kappa_{j,\mathscr{P}}$ denote the condition number of eigenvalue z_j regarded as eigenvalue of matrix $\mathscr{F}_{N,\mathscr{P}}$. Then*

$$\kappa_{j,\mathscr{P}} = \|e_j^* W_N\|_2 \|W_N^\dagger e_j\|_2, \quad j = 1, \ldots, n, \tag{3.6}$$

where W_N *is defined in* (1.5), *and this condition number coincides with the condition number of eigenvalue* z_j *regarded as eigenvalue of matrix* $\mathscr{F}_N$. *Additionally, it holds that*

$$1 \leqslant \kappa_{j,\mathscr{P}} \leqslant \left[1 + \frac{n-1+\|f^\dagger\|_2^2\|p_1\|_2^2 + \prod_{i=1}^n |z_i|^2 - |f_0|^2 - \sum_{i=1}^n |z_i|^2}{(n-1)\delta_j^2}\right]^{(n-1)/2}, \quad j = 1:n, \tag{3.7}$$

where

$$\delta_i = \min_{\substack{j \\ j \neq i}} |z_i - z_j|, \quad 1 \leqslant i, j \leqslant n,$$

p_1 *is the first column of the projector* $\mathscr{P}$, $f^\dagger$ *is the same as in Proposition* 2 *and* f_0 *its first component.*

Proof. We first note that $\mathscr{Q}_N\mathscr{Q}_N^* = W_N^\dagger W_N$. Using this equality in (3.5) we have

$$\mathscr{F}_{N,\mathscr{P}} = W_N^\dagger W_N F_N = W_N^\dagger Z W_N, \tag{3.8}$$

where the last equality comes from (1.6). Hence it is straightforward to see that $W_N^\dagger e_j$ and $W^* e_j$ are, respectively, right and left eigenvectors of $\mathscr{F}_{N,\mathscr{P}}$ corresponding to eigenvalue z_j, and that they satisfy the property $e_j^* W_N W_N^\dagger e_j = 1$. Substitution of these eigenvectors into the definition of $\kappa_{j,\mathscr{P}}$ (see (2.3)) leads to (3.6).

We now analyze the condition number κ_j of eigenvalue z_j regarded as eigenvalue of $\mathscr{F}_N$. In fact, using (3.1) and (3.2) it follows that $\mathscr{F}_N$ has a spectral decomposition of the form

$$\mathscr{F}_N = (\mathscr{Q}_N^* W_N^\dagger) Z (W_N \mathscr{Q}_N) = (\mathscr{Q}_N W_N^\dagger) Z (\mathscr{Q}_N^* W^\dagger)^{-1}, \tag{3.9}$$

and so $v_i = \mathscr{Q}_N^* W_N^\dagger e_i$ and $u_i = \mathscr{Q}_N^* W_N^* e_i$ are right and left eigenvectors of $\mathscr{F}_N$ related to eigenvalue z_j, respectively. Since these eigenvectors satisfy the normalization condition $u_i^* v_i = 1$, the condition number κ_j of eigenvalue z_j regarded as eigenvalue of $\mathscr{F}_N$ is

$$\kappa_j = \|u_i\|_2 \|v_i\|_2 = \|\mathscr{Q}_N^* W_N^* e_i\|_2 \|\mathscr{Q}_N^* W_N^\dagger e_i\|_2.$$

Now, since both $W_N^* e_j$ and $W_N^\dagger e_j$ belong to the signal subspace $\mathscr{S}_N$, it is clear that $\|\mathscr{Q}_N^* W_N^* e_j\|_2 = \|W_N^* e_j\|_2$ and $\|\mathscr{Q}_N^* W_N^\dagger e_j\|_2 = \|W_N^\dagger e_j\|_2$ so $\kappa_j = \kappa_{j,\mathscr{P}}$.

To prove (3.7) we need to compute $D(\mathscr{F}_N)$, the departure from normality of matrix $\mathscr{F}_N$ as given in Definition 2. For this, the singular values of matrix $\mathscr{F}_N$ can be used (see, e.g., [1, Theorem 4]). The proof of the proposition ends by substituting this $D(\mathscr{F}_N)$ into inequality (2.5) in Proposition 1. □

We thus see that for z_j's not extremely close to each other and not much smaller in modulus than 1, the only condition needed to ensure small values of κ_j is to keep N large enough for in this case $\|f^\dagger\|_2^2 \approx 0$.

Proposition 4. *For every $N \geqslant n$,*

$$\|\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}}\|_2 \leqslant \sqrt{\sin(\Theta)^2 + \epsilon_2 - \epsilon_1}, \tag{3.10}$$

where Θ is the largest canonical angle between $\mathscr{S}_N$ and $\widetilde{\mathscr{S}}_N$, and ϵ_1, ϵ_2 are positive real numbers. Additionally, if $\tilde{f} \in \widetilde{\mathscr{S}}_N$ and $\|f^\dagger - \tilde{f}\|_2^2 \leqslant \|f^\dagger\|_2^2$, where $f^\dagger$ and $\tilde{f}$ are the last column vector of matrices F_N and $\widetilde{F}_N$, respectively, then for N large enough,

$$\|\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}}\|_2 \approx \sin(\Theta). \tag{3.11}$$

Proof. Let p_i and $\tilde{p}_i$ denote the columns of $\mathscr{P}$ and $\widetilde{\mathscr{P}}$, respectively. Set $\varepsilon_i = p_i - \tilde{p}_i,\ i = 1:N$, and $\zeta = f^\dagger - \widetilde{\mathscr{P}}\tilde{f}$. Using the definitions of $\mathscr{F}_{N,\mathscr{P}}$ and $\widetilde{\mathscr{F}}_{N,\mathscr{P}}$ given in (3.5) we have that

$$\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}} = \mathscr{P}F_N - \widetilde{\mathscr{P}}\widetilde{F}_N = [\varepsilon_2 \cdots \varepsilon_N\ \zeta],$$

and hence that

$$\begin{aligned}&(\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}})(\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}})^* \\ &\quad = \varepsilon_1\varepsilon_1^* + \varepsilon_2\varepsilon_2^* + \cdots + \varepsilon_N\varepsilon_N^* + \zeta\zeta^* - \varepsilon_1\varepsilon_1^* \\ &\quad = (\mathscr{P} - \widetilde{\mathscr{P}})(\mathscr{P} - \widetilde{\mathscr{P}})^* + \zeta\zeta^* - \varepsilon_1\varepsilon_1^*.\end{aligned}$$

Let λ be the largest eigenvalues of $(\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}})(\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}})^*$ and let ϕ denote a related unit eigenvector. The Rayleigh–Ritz characterization of λ together with Definition 1 imply then that

$$\|\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}}\|_2^2 \leqslant \sin(\Theta)^2 + |\phi^*\zeta|^2 - |\phi^*\varepsilon_1|^2.$$

The proof of (3.10) follows from this inequality upon defining $\epsilon_1 = |\phi^*\varepsilon_1|^2$ and $\epsilon_2 = |\phi^*\zeta|^2$.

To prove (3.11) observe that, since $\tilde{f} \in \widetilde{\mathscr{S}}_N$ and $\|f^\dagger - \tilde{f}\|_2^2 \leqslant \|f^\dagger\|_2^2$, both by assumption, we have that $\epsilon_2 \leqslant \|f^\dagger\|_2^2$. Now as $\|f^\dagger\|_2^2 \approx 0$ for N large enough, by Proposition 2, this last inequality in (3.10) ensures (3.11). □

Note from Proposition 4 that for the error matrix to be small not only N has to be large enough but also $\widetilde{\mathscr{S}}_N$ has to approximate the exact signal subspace $\mathscr{S}_N$ well. Having a small error matrix is important because in this event one is allowed to carry out a first-order eigenvalue perturbation analysis near zero to assess the accuracy of the eigenvalues $\tilde{z}_j$. Indeed, we shall see that taking N large enough is usually sufficient to ensure that $\widetilde{\mathscr{S}}_N \approx \mathscr{S}_N$, as long as the approximate matrix $\widetilde{\mathscr{Q}}_N$ is conveniently computed and the amount of noise on the data is not very large.

The following theorem states a first-order estimate for the absolute eigenvalue error.

Proposition 5. *Let* z_j *and* $\tilde{z}_j$ *be eigenvalues of* $\mathscr{F}_N$ *and* $\widetilde{\mathscr{F}}_N$, *respectively, as described before. Then the following first-order estimate holds*:

$$|\tilde{z}_j - z_j| \leqslant \left[1 + \frac{n-1+\|f^\dagger\|_2^2\|p_1\|_2^2 + \prod_{j=1}^n |z_j|^2 - \sum_{i=1}^n |z_j|^2}{(n-1)\delta_j^2}\right]^{(n-1)/2} \times\sqrt{\sin(\Theta)^2 + \|f^\dagger - \tilde{f}\|_2^2}, \quad j = 1, \ldots, n. \tag{3.12}$$

Proof. It is sufficient to apply the result described in (2.4) to matrix $\mathscr{F}_{N,\mathscr{P}}$ and then to use Propositions 3 and 4. □

Thus, taking for grant that N is large enough, that $\mathscr{S}_N \approx \widetilde{\mathscr{S}}_N$, and that the z_j's are near to the unit circle but not extremely closed to each other, we conclude that a reasonable estimate for the eigenvalue error is $\sin(\Theta)$, the subspace angle between $\mathscr{S}_N$ and $\widetilde{\mathscr{S}}_N$. That is,

$$|\tilde{z}_j - z_j| \approx \sin(\Theta), \quad j = 1, \ldots, n. \tag{3.13}$$

The simplicity and significance of this result suggests that the eigenvalue error bound (and hence the eigenvalue error itself) obtained by using the projected companion matrix is smaller than the one derived from the related companion matrix derived by Bazán and Toint in [4]. Numerical experiments that illustrate the validity of (3.13) are presented in Section 5.

4. SVD-based estimates

Reintroduce the Hankel matrix containing pure signal, and for ease of notation, for fixed ℓ, e.g., $\ell = 0$, let it and the right-hand side in (2.6) be denoted by H and b, respectively. The goal here is to estimate bound (3.12) for the case when the projected matrix is computed via the singular value decomposition (SVD) of $\widetilde{H} = H + E$ where E stands for noise. Let

$$H = U\Sigma V^* = [U_1 \quad U_2]\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^*\\ V_2^*\end{bmatrix} \tag{4.1}$$

be a partitioned SVD of H where both U_1 and V_1 have n columns, with Σ_1 containing the n nonzero singular values of H, and let

$$\widetilde{H} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^* = [\widetilde{U}_1 \quad \widetilde{U}_2]\begin{bmatrix}\widetilde{\Sigma}_1 & 0\\ 0 & \widetilde{\Sigma}_2\end{bmatrix}\begin{bmatrix}\widetilde{V}_1^*\\ \widetilde{V}_2^*\end{bmatrix} \tag{4.2}$$

be a conformal decomposition of $\widetilde{H}$. For convenience, we write $\widetilde{H} = \widetilde{U}_1\widetilde{\Sigma}_1\widetilde{V}_1^* + \widetilde{U}_2\widetilde{\Sigma}_2\widetilde{V}_2^* \doteq \widetilde{H}_1 + \widetilde{H}_2$. From these decompositions, the signal subspace is

$\mathscr{R}(H^*) = \mathscr{R}(V_1)$, and the approximate signal subspace is $\mathscr{R}(\widetilde{H}_1^*) = \mathscr{R}(\widetilde{V}_1)$. Also, the exact and approximate minimum norm solution of the system (2.6) are, respectively, $f^\dagger = H^\dagger b = V_1\Sigma_1^{-1}U_1^*b$, and $\tilde{f} = \widetilde{V}_1\widetilde{\Sigma}_1^{-1}\widetilde{U}_1^*\tilde{b}$. To estimate bound (3.12) we therefore need to estimate $\sin(\Theta)$, the angle between $\mathscr{R}(\widetilde{H}_1^*)$ and $\mathscr{R}(H^*)$, and the error $\|f^\dagger - \tilde{f}\|_2$, both in terms of signal characteristics and the amount of noise $\|E\|_2$.

As a preparation, recall from Wedin [21] that if B is a perturbation of $A \in \mathbb{C}^{n\times n}$, i.e., $B = A + \mathscr{E}$, with $\operatorname{rank}(B) = \operatorname{rank}(A) = r$, then $\sin(\Phi)$, the subspace angle between $\mathscr{R}(B)$ and $\mathscr{R}(A)$, satisfies

$$\sin(\Phi) \leqslant \frac{\|\mathscr{E}\|_2}{\sigma_r(A)}. \tag{4.3}$$

This bound is not applicable to our case since $\|\widetilde{H}_1 - H\|_2$ is not available. However, a similar bound as that of (4.3) can be derived for $\sin(\Theta)$ involving the norm of the error $\|\widetilde{H} - H\|_2$. Of course, following Stewart [16] (see also [17, Lemma 3.5]), it is not difficult to prove that

$$\sin(\Theta) \leqslant \frac{\|E\|_2}{\gamma} \quad \text{with } \gamma = \min\{\sigma_n(H), \sigma_n(\widetilde{H})\}. \tag{4.4}$$

Assuming that $\|E\|_2 < \sigma_n(H)$, we obtain the bounds

$$\sin(\Theta) \leqslant \begin{cases} \dfrac{\|E\|_2}{\sigma_n(H) - \|E\|_2} & \text{if } \gamma = \sigma_n(\widetilde{H}), \\ \dfrac{\|E\|_2}{\sigma_n(H)} & \text{if } \gamma = \sigma_n(H), \end{cases} \tag{4.5}$$

from which it is clear that $\|E\|_2$ must be much smaller than $\sigma_n(H)$ for $\widetilde{\mathscr{S}}_N$ to approximate $\mathscr{S}_N$ well. Note that this suggests we should discuss the behavior of $\sigma_n(H)$ as a function of the signal parameters r_j and z_j and the dimension of the Hankel matrix. For this, note from (2.8) that

$$\frac{1}{\sigma_n(H)} = \|H^\dagger\|_2 \leqslant \|W_N^{\mathrm{T}\dagger}\|_2\|R^\dagger\|_2\|W_M^\dagger\|_2. \tag{4.6}$$

But since by (3.6)

$$\|W_N^\dagger e_i\|_2 = \frac{\kappa_{i,\mathscr{P}}}{\|e_i^* W_N\|_2},$$

we have that

$$\|W_N^\dagger\|_2^2 \leqslant \|W_N^\dagger\|_{\mathrm{F}}^2 = \sum_{i=1}^{n} \frac{\kappa_{i,\mathscr{P}}^2}{\|e_i^* W_N\|_2^2} \leqslant n \cdot \max_i \frac{\kappa_{i,\mathscr{P}}^2}{\|e_i^* W_N\|_2^2}. \tag{4.7}$$

Since $\|W_N^\dagger\|_2$ decreases with N (see, e.g., [1]), inequality (4.7) in (4.6) gives that

$$\frac{1}{\sigma_n(H)} \leqslant n \cdot \max_i \frac{1}{|r_i|} \cdot \max_i \frac{\kappa_{i,\mathscr{P}}^2}{\|e_i^* W_N\|_2^2}. \tag{4.8}$$

Using this inequality in bound (4.5) corresponding to the condition $\gamma = \sigma_n(H)$, we deduce that

$$\sin(\Theta) \leqslant \frac{n}{\rho}\left[1+\frac{n-1+\|f^{\dagger}\|_2^2\|p_1\|_2^2+\prod_{j=1}^{n}|z_j|^2-\sum_{j=1}^{n}|z_j|^2}{(n-1)\delta^2}\right]^{(n-1)} \times\frac{(1-\beta^2)}{(1-\beta^{2N})}\|E\|_2, \tag{4.9}$$

where $\rho = \min_i\{|r_i|\}$, $\delta = \min_i\{\delta_i\}$, and $\beta = \min_i\{|z_i|\}$.

Bound (4.9) shows that if the approximate signal subspace is computed from a large Hankel matrix via the SVD, the signal subspace of signals comprising well-resolved components and modes z_j near to the unit circle is relatively insensitive to noise.

As for the error $\|f^{\dagger}-\tilde{f}\|$, it essentially depends on the conditioning of H (see, e.g., [8, Theorem 3.2.3]) and it can be proved that if H is well conditioned and $\|E\|_2 \ll \sigma_n(H)$, then the error $\|f^{\dagger}-\tilde{f}\|$ must be small. Concerning $\kappa(H)$, since it depends on $\kappa(W_N)$ (see (2.8)), good conditioning of W_N will result in good conditioning of H. This ensures that whenever N is large enough and $|z_j| \approx 1$ (which ensure W_N is well conditioned, see [1]), H will be well conditioned unless the zeros z_j are extremely close to each other, see [3].

To conclude, note that the estimates for $\sin(\Theta)$ and $\|f^{\dagger}-\tilde{f}\|$ derived from the SVD approach could be substituted in (3.12) to produce an interesting bound for the eigenvalue error. Although we do not explicitly carry through the substitution here, note that the estimate (4.9) for $\sin(\Theta)$ and the estimate of $\|f^{\dagger}-\tilde{f}\|$ (not included here) are sufficient to predict that the signal eigenvalues become relatively insensitive to noise when N is sufficiently large. Finally, observe that all bounds described throughout simplify considerably when the signal zeros fall on the unit circle. As an illustration, by setting $|z_j| = 1$, the bound on $\kappa_{j,\mathscr{P}}$ in (3.7) becomes

$$\kappa_{j,\mathscr{P}} \leqslant \left[1+\frac{\|f^{\dagger}\|_2^2\|p_1\|_2^2}{(n-1)\delta_j^2}\right]^{(n-1)/2}, \quad j=1,\ldots,n. \tag{4.10}$$

while that for $\sin(\Theta)$ is

$$\sin(\Theta) \leqslant \frac{n}{\rho}\left[1+\frac{\|f^{\dagger}\|_2^2\|p_1\|_2^2}{(n-1)\delta^2}\right]^{(n-1)} \frac{\|E\|_2}{N}. \tag{4.11}$$

These bounds show that if N is sufficiently large, then the eigenvalue problem is almost perfectly conditioned as $\kappa_{j,\mathscr{P}} \approx 1$, and the signal subspace is quite insensitive to noise as $\sin(\Theta) \approx 0$. A by-product of this is that the eigenvalues themselves are in this case quite insensitive to noise.

5. Numerical experiment

In this section we shall numerically illustrate the behavior of the eigenvalue error bounds described before. Specifically, we wish to illustrate the superiority of the projected companion matrix approach in resolving closely spaced signal zeros over the polynomial approach described in [4]. To achieve this goal, we compute bounds (5.1), (3.12) and (2.4). Bound (5.1) is due to Bazán and Toint and it is described as

$$|z_j - \tilde{z}_j| \leqslant |e_j^* W_N (f^\dagger - \tilde{f})| \|W_N^\dagger\|_2^2 \sqrt{n} \left(1 + \frac{\alpha^N}{1-\alpha}\right), \quad 1 \leqslant j \leqslant n, \quad (5.1)$$

where $\alpha = \max_j\{|z_j|\}$.

The exact eigenvalue errors for both approaches were also computed. To describe these errors we use the notation:

- $|z_j - \tilde{z}_j|_{\mathrm{PA}}$: eigenvalue error obtained from the polynomial approach,
- $|z_j - \tilde{z}_j|_{\mathrm{PCA}}$: eigenvalue error obtained from the projected companion matrix approach.

The signal used in our experiment is the same as that used by Bazán and Toint in [4]. The signal parameters are reproduced in Table 1. As the signal comprises closely spaced zeros (see the separations δ_j in Table 1), we choose the dimension of the Hankel matrix relatively large in order to enforce the associated Vandermonde matrix W_N to be well conditioned (see the theory in [1]). The Hankel matrix H used to construct the signal subspace and the polynomial coefficients f_j (or equivalently the companion matrix F_N) is then chosen of order 256×256 (i.e., $N = 256$).

All quantities needed were computed using the SVD. The data in the experiment consists of pure signal plus Gaussian noise generated by MATLAB, with the seed value set to zero, and at a level of 5%.

As can be seen from (5.1), (3.6) and (3.12), the quantities $\|W_N^\dagger\|_2^2$, $\|f^\dagger\|_2^2$, $\|f^\dagger - \tilde{f}\|_2^2$, and $\sin(\Theta)$, all play a crucial role in the eigenvalue error bounds. For this example we have

$$\|W_N^\dagger\|_2^2 \approx 1.1264 \times 10^{-2},$$
$$\|f^\dagger\|_2^2 \approx 2.2285 \times 10^{-3},$$
$$\|f^\dagger - \tilde{f}\|_2^2 \approx 3.8635 \times 10^{-5}.$$

The important quantity $\sin(\Theta)$ in (3.12) is in this case

$$\sin(\Theta) = 3.0414 \times 10^{-2}.$$

Table 1
Signal parameters corresponding to a vibrating system

j	$z_j, \bar{z}_j$	$\|z_j\|, \|\bar{z}_j\|$	$r_j, \bar{r}_j$	δ_j^2
1	$0.9699 \pm 0.2248\imath$	0.9956	$-0.1366 \pm 0.2490\imath$	0.0042
2	$0.9532 \pm 0.2931\imath$	0.9972	$0.7294 \pm 0.5743\imath$	0.0049
3	$0.9844 \pm 0.1619\imath$	0.9976	$-0.3162 \pm 0.0844\imath$	0.0031
4	$0.9921 \pm 0.1055\imath$	0.9977	$1.3284 \pm 0.6265\imath$	0.0023
5	$0.9972 \pm 0.0585\imath$	0.9989	$-0.0591 \pm 0.1958\imath$	0.0023

Table 2
Eigenvalue error and eigenvalue error bounds

j	1	2	3	4	5
$\lvert z_j - \tilde{z}_j \rvert_{\mathrm{PA}}$	8.6248×10^{-4}	9.8894×10^{-5}	1.7676×10^{-4}	3.2045×10^{-5}	1.8996×10^{-4}
$\lvert z_j - \tilde{z}_j \rvert_{\mathrm{PCA}}$	2.3791×10^{-4}	6.3195×10^{-5}	1.6163×10^{-4}	3.1855×10^{-5}	8.9152×10^{-5}
Bound (2.4)	3.1076×10^{-2}	3.0784×10^{-2}	3.1198×10^{-2}	3.0951×10^{-2}	3.0773×10^{-2}
Bound (3.12)	4.1439×10^{-2}	3.9648×10^{-2}	4.5428×10^{-2}	5.1395×10^{-2}	5.1395×10^{-2}
Bound (5.1)	2.2825×10^{-1}	6.7112×10^{-2}	1.8134×10^{-1}	3.3282×10^{-2}	4.7127×10^{-1}

All numerical results (shown to five significant figures) are reported in Table 2. Two facts should be emphasized from this table. First, that the superiority of our eigenvalue error bound (3.12) over that by Bazán and Toint, i.e., the bound derived from the companion matrix approach described in (5.1), is very apparent, and second, that the eigenvalues computed through the projected companion matrix approach are in fact more accurate than the eigenvalues computed through the polynomial approach. Table 2 also includes the eigenvalue error bounds described in (2.4). Notice that although this bound uses the exact condition numbers $\kappa_{j,\mathscr{P}}$ and the exact matrix error $\|\mathscr{F}_{N,\mathscr{P}} - \widetilde{\mathscr{F}}_{N,\mathscr{P}}\|_2$, it is not much better than our error bound (3.12). Notice also that the estimate for the eigenvalue error given by $\sin(\Theta)$ (see (3.13)) is appropriate in this case.

As a final comment, it is worth mentioning that many numerical experiments were made using signals arising from modal analysis of dynamic structures and nuclear resonance spectroscopy in which we observed the behavior of the eigenvalue error bounds as well as the eigenvalue error themselves. The conclusion drawn from these experiments was that except for signal comprising extremely closely spaced signal zeros, moderate values of N are sufficient to ensure small values of the bounds and hence of the eigenvalue error themselves. These experiment confirmed that in general signal zeros become relatively insensitive to noise, under appropriate conditions.

6. Conclusions

In this work an error analysis of so-called signal zeros was carried out and eigenvalue error bounds were provided. The bounds rely on the observation that signal zeros can be regarded as eigenvalues of projected companion matrices as well as on Bauer–Fike-type perturbation theorems for eigenvalues. The bounds are in terms of the subspace angle, the error of the polynomial coefficients, the polynomial degree and the signal parameters themselves. Specifically, it is shown that if the approximate signal subspace and the polynomial coefficients are extracted from large Hankel matrices via the SVD, the bounds ensure the signal zeros become rather insensitive to noise on the data, the result being strengthened when the signal zeros fall close to the unit circle. In addition to this, the bounds suggests that signal zero estimates derived from projected companion matrices are more accurate than those obtained from the companion matrices themselves. This is confirmed by the numerical simulation. Finally, it is worth emphasizing that the results obtained are theoretical contributions; further work would be required to develop a reliable algorithm for extracting signal zeros using the projected companion matrix $\mathscr{F}_N$.

Acknowledgment

The author wishes to thank G.W. Stewart for helpful comments regarding the presentation of this paper.

References

[1] F.S.V. Bazán, Conditioning of rectangular Vandermonde matrices with nodes in the unit disk, SIAM J. Matrix Anal. Appl. 21 (2) (2000) 679–693.
[2] F.S.V. Bazán, L.H. Bezerra, On zero locations of predictor polynomials, Numer. Linear Algebra Appl. 4 (1997) 459–468.
[3] F.S.V. Bazán, Ph.L. Toint, Conditioning of Infinite Hankel Matrices of finite rank, Syst. Control Lett. 41 (2000) 347–359.
[4] F.S.V. Bazán, Ph.L. Toint, Error analysis of signal zeros from a related companion matrix eigenvalue problem, Appl. Math. Lett. 14 (2001) 859–866.
[5] Å. Björk, G.H. Golub, Numerical methods for computing angles between linear subspaces, Math. Comput. 27 (1973) 579–594.
[6] A. Edelman, H. Murakami, Polynomial roots from companion matrix eigenvalues, Math. Comput. 64 (210) (1995) 763–776.
[7] G.H. Golub, C.F. Van Loan, Matrix Computations, third ed., Johns Hopkins University Press, Baltimore, 1996.
[8] P.C. Hansen, Rank-Deficient and Discrete Ill-Posed Problems—Numerical Aspects of Linear Inversion, SIAM, Philadelphia, PA, 1998.
[9] R. Kumaresan, On the zeros of the linear prediction-error filters for deterministic signals, IEEE Trans. Acoust. Speech Signal Process., ASSP 31 (1) (1983) 217–221.
[10] S. Kung, K.S. Arun, D.V. Bhaskar Rao, State-space and singular value decomposition-based approximation methods for the harmonic retrieval problem, Opt. Soc. Amer. 73 (December) (1983) 1799–1811.
[11] F. Li, R.J. Vaccaro, Unified analysis for DOA estimation algorithms in array signal processing, Signal Process. 25 (1991) 147–169.
[12] J. Makhoul, Linear prediction: a tutorial review, Proc. IEEE 63 (4) (1975) 561–580.
[13] S. Mayrargue, J.P. Jouveau, A new application of SVD to harmonic retrieval, in: E.F. Deprettere (Ed.), SVD and Signal Processing, North-Holland, Amsterdam, 1988, pp. 467–472.
[14] V. Pisarenko, The retrieval of harmonics from a covariance function, Geophys. J. Roy. Astron. Soc. 33 (1973) 347–366.
[15] R.A. Smith, The condition numbers of the matrix eigenvalue problem, Numer. Math. 10 (1967) 232–240.
[16] G.W. Stewart, On the perturbation of pseudo-inverses, projections and linear least squares problems, SIAM Rev. 19 (4) (1977) 634–662.
[17] G.W. Stewart, J.-G. Sun, Matrix Perturbation Theory, Academic Press, San Diego, 1990.

[18] K.C. Toh, L.N. Trefethen, Pseudozeros of polynomials and pseudospectra of companion matrices, Numer. Math. 68 (1994) 403–425.
[19] A. Van Der Veen, E.F. Deprettere, A. Lee Swindlehurst, Subspace-based signal analysis using singular value decomposition, Proc. IEEE 81 (9) (1993) 1277–1309.
[20] M. Wei, G. Majda, A new theoretical approach for Prony's method, Linear Algebra Appl. 136 (1990) 119–132.
[21] P.Ä. Wedin, On angles between subspaces of a finite dimensional inner product space, in: B. Kägstrom, A. Ruhe (Eds.), Matrix Pencils, Lecture Notes in Mathematics, vol. 973, Springer-Verlag, Berlin, 1983, pp. 263–285.
[22] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Oxford University Press, Oxford, UK, 1965.