MATHEMATICS

DIFFERENT REALIZATIONS OF A NON SUFFICIENT JET

BY

JACEK BOCHNAK AND TZEE-CHAR KUO [1])

(Communicated by Prof. N. H. Kuiper at the meeting of June 26, 1971)

§ 1. The result

Conjecture (Thom, [11], Problem 3, p. 229). If an r-jet $w \in J^r(n, p)$ is not C^0-sufficient, then w has an infinite family of realizations $\{f_\alpha\}$ such that for $\alpha \neq \beta$, (the germs of) f_α and f_β are not topologically equivalent.

The purpose of this paper is to establish this conjecture when $p=1$. The case $n=2$, $p=1$ has been proved in [4].

The terminology involved is explained as follows. The jet space $J^r(n, 1)$ consists of all real polynomials $w(x_1, \ldots, x_n)$ of degree $\leqslant r$ with $w(0)=0$. If $w \in J^r(n, 1)$ coincides with the Taylor's expansion up to degree r of a given C^r- (resp. C^{r+1}-) function f, then f is called a realization of w in $\mathscr{E}_{[r]}$ (resp. $\mathscr{E}_{[r+1]}$). Here $\mathscr{E}_{[s]}$ denotes the set of all germs of C^s-functions. An r-jet $w \in J^r(n, 1)$ is v-sufficient in $\mathscr{E}_{[r]}$ (resp. $\mathscr{E}_{[r+1]}$) ([2], [3], [4]), if for any two realizations f and g of w, the germs of the varieties $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic. If there exists a local homeomorphism h: $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f \circ h = g$, then w is called C^0-sufficient.

It has been proved in [2] that C^0-sufficiency is equivalent to v-sufficiency in $J^r(n, 1)$. Hence in this paper we shall abreviate both notions simply as sufficiency.

Theorem. If $w \in J^r(n, 1)$ is not sufficient in $\mathscr{E}_{[r]}$ or in $\mathscr{E}_{[r+1]}$, then there exists an infinite sequence $\{f_i\}_{i \in \mathbb{N}}$ of realizations of w with mutually non homeomorphic (germs of) varieties $f_i^{-1}(0)$.

§ 2. The proof

We shall only consider the case $\mathscr{E}_{[r+1]}$; that for $\mathscr{E}_{[r]}$ is similar. The case $n=1$ is trivial, we shall assume $n \geqslant 2$.

Illustrative Example. The 5-jet $w(x, y) = x^4 - 2x^2y^3 \in J^5(2, 1)$ admits a realization $f(x, y) = (x^2 - y^3)^2$. The variety $f(x, y) = 0$ is singular (i.e. grad $f = 0$) along the arcs $x = \pm y^{3/2}$. Now adding perturbations such as $\pm y^{2N}$ ($N \geqslant 3$) and $(\cos 1/y) \exp(-1/y^2)$ to f will cause catastrophic changes to the variety near $x = \pm y^{3/2}$. Hence w is not sufficient and in this way

*) The author is partially supported by the British Royal Society European Program.

one can find infinitely many different realizations. (Similar examples:

$$w=x^2-2xy^2 \in J^3, \text{ with } f=(x-y^2)^2;$$

$$w=x^3-3xy^5 \in J^6 \text{ with } f=(x-y^{5/2})^2(x+2y^{5/2}).)$$

These examples suggest that the proof of the theorem should be divided into three steps. The first is to detect an arc for w (such as $x=y^{3/2}$) along which a realization is singular. The second step is to construct such a realization. The final step is to construct infinitely many perturbations giving rise to infinitely many different realizations.

Step 1. For a given (germ of) polynomial function g of n variables, let

$$E_g=\{u \in \mathbf{R}^n \colon |\text{grad } g(u)| = \min_{|x|=|u|} |\text{grad } g(x)|\}.$$

(On the sphere $|x|=$ constant, $|\text{grad } g|$ takes its minimum on the points of E_g). By the Seidenberg-Tarski theorem ([5], p. 17), E_g is a semi-algebraic set. Applying the Curve Selection Lemma ([7], p. 103; [10], § 3), one can find an analytic arc

$$L_g\colon x_i=\lambda_i(t), \quad 0\leqslant t<\eta$$

(each $\lambda_i(t)$ is a convergent power series) such that $\lambda(t)=(\lambda_1(t), \ldots, \lambda_n(t)) \in E_g$ $\lambda(0)=0$, $\lambda(t)\neq 0$ for $t>0$. Observe that $\lambda([0, \eta))$ is a semi-analytic set. We shall call such an arc in E_g a Łojasiewicz arc for g.

The most important property of L_g is the following. Let $\varrho<\infty$, $\mu\leqslant\infty$ be the numbers such that $|\lambda(t)| \sim t^\varrho$, $|\text{grad } g(\lambda(t))| \sim t^\mu$, ($A(t) \sim B(t)$ means that A/B lies between two positive constants, for $t>0$ and t small). Then $|\text{grad } g(x)| \geqslant \varepsilon |x|^{\mu/\varrho}$ for x near 0, where $\varepsilon>0$ is a constant.

Now for the given non sufficient jet w, choose a Łojasiewicz arc L_w and let μ, ϱ be defined as above. Then we must have

$$\frac{\mu}{\varrho} \geqslant r, \tag{2.1}$$

since otherwise w would be sufficient ([4], Theorem 0 with $\delta=r-\mu/\varrho$).

(For non-sufficiency in $\mathscr{E}_{[r]}$ we must have $\mu/\varrho>r-1$ by Kuiper's theorem [3]).

Step 2. By rotating $\mathbf{R}^n$, if necessary, we can assume that L_w is tangent to the positive x_1-axis. Then

$$0(\lambda_i(t))>0(\lambda_1(t)), \quad i\geqslant 2, \tag{2.2}$$

where $0(\lambda(t))$ is the (lowest) order of the series $\lambda(t)$.

Moreover, we may assume

$$\lambda_1(t)=t^q, \quad q=0(\lambda_1(t)).$$

This can be achieved by changing the parameter t analytically (if necessary). (If $\lambda_1(t)=a_qt^q+\ldots$, $a_q>0$, then change t to $s=t(a_q+a_{q+1}t+\ldots)^{1/q}$.)

To sum up, we have put L_w in the form

$$\lambda_1(t)=t^q,\ \lambda_i(t)=\sum_{j\geqslant q+1} a_{ij}t^j,\ 2<i<n. \tag{2.3}$$

With the above preparation, we now state the main result in this step.

Proposition 1. For a non sufficient jet $w\in J^r(n, 1)$ with a Łojasiewicz arc L_w of the form (2.3), there exists a realization φ having the following properties:

(1) $\varphi=0$ and grad $\varphi=0$ along the arc L_w.
(2) $\varphi=\psi_1+\psi_2$, with ψ_1 analytic and ψ_2 of the form $\psi_2(x_1, x_2, \dots, x_n)=$ $=f(\sigma_q(x_1), x_2, \dots, x_n)$, where $\sigma_q(x_1)=x_1|x_1|^{1/q-1}$, $\sigma_q(0)=0$, $q\in \mathrm{N}$ and f is an analytic function.
(3) $\varphi^{-1}(0)$ is semi-analytic (in a neighborhood of 0).
(4) For any $x_1>0$ (x_1 small), the function

$$(x_2, \dots, x_n)\to \varphi(x_1, x_2, \dots, x_n)$$

admits the point $(x_2, \dots, x_n)=(\lambda_2(x_1^{1/q}), \dots, \lambda_n(x_1^{1/q}))$ as a non-degenerate critical point.

Proof. Consider the local C^1-coordinate transformation

$$(x_1, x_2, \dots, x_n)\to(x_1, x_2-\lambda_2(\sigma_q(x_1)), \dots, x_n-\lambda_n(\sigma_q(x_1)))=(y_1, \dots, y_n).$$

This transformation is C^1 since $0(\lambda_i(t))>q$. The arc L_w is transformed into the positive y_1-axis. Let us write w in the coordinate system $(y_1, \dots, y_n)$:

$$w(y_1, \dots, y_n)=g(\sigma_q(y_1))+\sum_{j=2}^{n} g_j(\sigma_q(y_1))y_j+\sum_{2\leqslant i_2+\dots+i_n\leqslant r} g_{i_2\dots i_n}(\sigma_q(y_1))y_2^{i_2}\dots y_n^{i_n},$$

where $g, g_j, g_{i_2\dots i_n}$ are analytic functions in a neighborhood of 0 in $\mathbf{R}$.
Observe that

$$\frac{\partial w}{\partial y_1}(y_1, 0)=\frac{d(g\circ\sigma_q)}{dy_1}(y_1), \frac{\partial w}{\partial y_j}(y_1, 0)=g_j\circ\sigma_q(y_1),\ j=2, \dots, n\,.$$

Let ϱ, μ be the numbers such that

$$|\lambda(t)|\sim t^\varrho$$

and

$$|\text{grad}\, w(\lambda(t))|\sim t^\mu.$$

Then by (2.1), $\mu/\varrho\geqslant r$. By (2.3), $|\lambda(t)|\sim|\lambda_1(t)|=t^q$ and so $\varrho=q$, $t^\mu=|y_1|^{\mu/\varrho}$.
Hence

$$\left|\left(\frac{\partial w}{\partial y_1}(y_1, 0), \frac{\partial w}{\partial y_2}(y_1, 0), \dots, \frac{\partial w}{\partial y_n}(y_1, 0)\right)\right|\sim|y_1|^{r_1}$$

where $r_1=\mu/\varrho>r$.

Thus the order of g (resp. g_j, $j=2, \ldots, n$) is greater than or equal to $q(r+1)$ (resp. qr). This implies that

$$P(x_1, \ldots, x_n)=g(\sigma_q(x_1))+\sum_{j=2}^{n} g_j(\sigma_q(x_1))(x_j-\lambda_j(\sigma_q(x_1))$$

is a C^{r+1} function, r-flat at 0. Now

$$\psi_2=w-P$$

is a realization satisfying the property (1) of the Proposition.

We shall construct an analytic function ψ_1, sufficiently flat at 0, singular along L_w, so that $\varphi=\psi_1+\psi_2$ also satisfies (4). We repeat here some arguments due to Łojasiewicz [2].

As we mentioned above, the image $\lambda([0, \eta))$ of the arc L_w is semi-analytic. This implies that there exists a system of pseudo-polynomials of two variables $H_i(x_1, \xi)$, $i=2, \ldots, n$, of the form

$$H_i(x_1, \xi)=\xi^{m_i}+\sum_{j=0}^{m_i-1} a_{ij}(x_1)\xi^j$$

where a_{ij} are analytic functions, such that for each $x_1>0$ (sufficiently small)

$$H_i(x_1, \lambda_i(|x_1|^{1/q}))=0 \tag{2.4}$$

$$\frac{\partial H_i}{\partial \xi}(x_1, \lambda_i(|x_1|^{1/q}))\neq 0. \tag{2.5}$$

This follows as a special case from the general theory of normal decomposition compatible with a given semi-analytic set (Łojasiewicz [6] p. 451, or [7]). (In our case, however, we can simply eliminate t in (2.3) to find an H_i satisfying (2.4). Then an H_i satisfying (2.4) with minimal degree also satisfies (2.5).)

Put

$$\psi_1(x_1, \ldots, x_n)=\tfrac{1}{2}x_1{}^N \sum_{i=2}^{n} (H_i(x_1, x_i))^2.$$

The value of N will be decided later.

By (2.4) and (2.5)

$$\det\left[\frac{\partial^2\psi_1}{\partial x_i\partial x_j}\right]_{i,j\geq 2}=x_1^{(n-1)N}\prod_{i=2}^{n}\left[\left(\frac{\partial H_i}{\partial \xi}\right)^2_{\xi=x_i}\right]$$

along L_w. It is different from 0 for $x_1, >0$.

Now along L_w, we can write

$$\det\left[\frac{\partial^2(\psi_1+\psi_2)}{\partial x_i\partial x_j}\right]_{i,j\geq 2}=\sum_{j=0}^{q(n-1)} x_1{}^{jN}\,\Gamma_j(x)$$

where

$$\Gamma_0(x)=\det\left[\frac{\partial^2\psi_1}{\partial x_i\partial x_j}\right],\quad \Gamma_{q(n-1)}(x)=\prod_i\left(\frac{\partial H_i}{\partial \xi}\right)^2$$

(along L_w), and each Γ_j is independent of N. Let $D_N(t)$ denote the function obtained by substituting (2.3) into $\sum x_1{}^{jN}\Gamma_j$, then

$$D(s^2) = \sum s^{2jN}\gamma_j(s)$$

where $\gamma_j(s) = \Gamma_j(\lambda(s^2))$.

By the definition of ψ_2, $D_N(s^2)$ is an analytic function of s. We now show that $D_N(s^2) \not\equiv 0$ for all large values of N. In fact, the set of values of N for which $D_N \equiv 0$ consists of at most $q(n-1)$ elements. For if $D_N \equiv 0$ for $q(n-1)+1$ different values $N = N_1, \ldots, N_{q(n-1)+1}$, then since $\gamma_{q(n-1)}(s) \neq 0$ ((2.5)), we would have

$$\begin{vmatrix} 1 & \sigma^{N_1} & \ldots & \sigma^{pN_1} \\ & \cdots & & \\ 1 & \sigma^{N_{p+1}} & \ldots & \sigma^{pN_{p+1}} \end{vmatrix} \equiv 0$$

where $\sigma = s^2$, $p = q(n-1)$. But the value of the above determinant is $(-1)^k \prod_{1\leqslant i<j\leqslant p+1}(\sigma^{N_i} - \sigma^{N_j})$ (for some k), which is not identically zero.

Now choose N so that $D_N(s^2) \not\equiv 0$, then due to analyticity, $D_N(s^2) \neq 0$ for all positive s (s small). This implies condition (4), for the function $\varphi = \psi_1 + \psi_2$. Moreover, if $N > r$ also, then φ remains a realization of w.

It is clear that condition (3) is implied by (2). The proof of Proposition 1 is complete.

Step 3. A proof of the following lemma will be given in § 3.

Lemma 1. There exists a sequence $\{A_i\}_{i\in\mathbf{N}}$ of closed subsets of $[0, \infty)$ with mutually non-homeomorphic germs at 0. Moreover, the germ of each A_i is not locally connected.

Notation. For a subspace X of $\mathbf{R}^n$, let $\tilde{X}$ denote the set of all $x \in X$, such that no neighborhood of x in X is homeomorphic with $\mathbf{R}^{n-1}$.

The ideas in this step are best explained in the following.

Example. Let $w(x_1, \ldots, x_n) = \sum_{i=2}^n \varepsilon_i x_i^2$ in $J^r(n, 1)$, $r \geqslant 3$, $\varepsilon_i^2 = 1$. Take $\varphi = w$, $L_w =$ the positive x_1-axis. Let $\beta_i : \mathbf{R} \to [0, \infty)$ be a C^∞-function, flat at 0, with $\beta_i^{-1}(0) = A_i$. Then the realizations $f_i = \varphi + \beta_i$ give rise to mutually non-homeomorphic germs of varieties. Observe that $\widetilde{f_i^{-1}(0)}$ is homeomorphic with A_i.

The general case is slightly more complicated.

In the following, φ is the function in Proposition 1. Firstly, we shall select an open semianalytic subset U_w of $\mathbf{R}^n$ containing $L_w\backslash\{0\}$, $U_w \subset (0, \infty) \times \mathbf{R}^{n-1}$, in which

$$\left|\left(\frac{\partial\varphi}{\partial x_2}, \ldots, \frac{\partial\varphi}{\partial x_n}\right)\right| \geqslant |x_1|^m \sum_{i=2}^n |x_i - \lambda_i(|x_1|^{1/q})|, \tag{2.6}$$

where $m \geqslant 1$ is a constant.

In fact, U_w will be a horn-shaped set (a so-called horn-neighborhood) of the form

$$\{(x_1, \ldots, x_n) \in \mathbf{R}^n : \sum_{i=2}^{n} |x_i - \lambda_i(|x_1|^{1/q})|^2 < |x_1|^{2m}\}$$

where the value of m is to be determined.

Observe that, by the conditions (2) and (4) in Proposition 1, the inequality

$$\sum_{i=2}^{n} \left(\frac{\partial \varphi}{\partial x_i}\right)^2 \geqslant 2\eta^2 \sum_{i=2}^{n} |x_i - \lambda_i(|x_1|^{1/q})|^2 \tag{2.7}$$

defines in the $(x_1, \ldots, x_n, \eta)$-space a semi-analytic set E, containing the arc $(L_w \backslash \{0\}) \times \{0\}$ in its interior. Therefore by Łojasiewicz regular separation theorem for semi-analytic sets ([7]) (cf. [8], p. 14), there exists m such that the set

$$H_m = \{(x, \eta) : \eta^2 + \sum |x_i - \lambda_i(|x_1|^{1/q})|^2 < 2|x_1|^{2m}\}$$

which contains $(L_w \backslash \{0\}) \times \{0\}$, is contained in E.

Now for $x \in U_w$, $(x_1, |x_1|^m) \in H_m$ and so by (2.7), we have (2.6).

Let $\gamma(x_1, \ldots, x_n) \geqslant 0$ be a C^∞-function, flat at 0, $\gamma = 0$ outside U_w, $\gamma > 0$ on $L_w \backslash \{0\}$ and

$$\left| \left(\frac{\partial \gamma}{\partial x_2}, \ldots, \frac{\partial \gamma}{\partial x_n} \right) \right| \leqslant \tfrac{1}{2} |x_1|^m \sum_{i=2}^{n} |x_i - \lambda_i(|x_1|^{1/q})|.$$

(The construction of such a γ is easy).

By (2.6)

$$\operatorname{grad}(\varphi + \gamma)(x) \neq 0 \tag{2.8}$$

in U_w, except possibly along L_w. But $\gamma > 0$ along $L_w \backslash \{0\}$, the variety $(\varphi + \gamma)^{-1}(0)$ is disjoint from $L_w \backslash \{0\}$. Hence $(\varphi + \gamma)^{-1}(0)$ is a manifold of codimension 1 in U_w.

Now let $\beta_i : \mathbf{R} \to [0, 1]$ be C^∞-function, flat at $0 \in \mathbf{R}$, and $\beta_i^{-1}(0) = A_i$, where A_i are the sets in Lemma 1. We also assume $|d\beta/dx_1| \leqslant 1$. We claim that

$$f_i(x) = \varphi(x) + \beta_i(x_1)\gamma(x)$$

are the desired realizations.

Consider $V_i = f_i^{-1}(0)$. We shall now show that for $i \neq j$ the germs of V_i and V_j are non-homeomorphic. Clearly it is sufficient to prove that $\tilde{V}_i$ and $\tilde{V}_j$ are non-homeomorphic. Suppose that a homeomorphism h exists between these two, we shall then derive a contradiction.

By (2.8) and by the choice of β_i,

$$L_w^{(i)} = \tilde{V}_i \cap \overline{U}_w = \{x \in \mathbf{R}^n : x \in L_w, x_1 \in A_i\}.$$

So by our construction, we have homeomorphisms

$$\tilde{V}_i \cap \overline{U}_w \approx A_i, \text{ for every } i \in \mathbf{N}.$$

The germ of the image $h(L_w^{(i)})$ intersects $\bar{U}_w$ only at 0, since otherwise we would have $A_i \approx A_j$. Hence

$$h(L_w^{(i)} \backslash \{0\}) \subset \tilde{V}_j \backslash \bar{U}_w.$$

Clearly $\tilde{V} \backslash \bar{U}_w = \tilde{V}_i \backslash \bar{U}_w$, where $V = \varphi^{-1}(0)$, for any $i \in \mathrm{N}$. The set $\tilde{V}$ is semi-analytic (see Lemma 2 below), hence so is $\tilde{V}_i \backslash \bar{U}_w$ and hence both are locally connected ([7], Prop. 3, p. 76). But $h(L_w^{(i)} \backslash \{0\})$ is open in $\tilde{V}_j \backslash \bar{U}_w$ (since $L_w^{(i)} \backslash \{0\}$ is open in $\tilde{V}_i$) and is not locally connected (Lemma 1); this gives rise to another contradiction. Therefore h does not exist.

Lemma 2. If X is a semi-analytic set, then so is $\tilde{X}$.
A proof is given in the next section.

§ 3. Proofs of Lemmas 1 and 2

Proof of Lemma 1. Let $F \subset \mathbf{R}$ be given. We define by recurrence the derived sets $F(1) = \{x \in F : \exists \{x_n\}_{i \in \mathrm{N}},\ x_n \in F,\ x_n \neq x,\ x_n \to x\}$ and $F(n) = F(n-1)(1)$. It is clear that for compact subsets $E, F \subset \mathbf{R}$, if for some k, $E(k) = \emptyset$ and $F(k) \neq \emptyset$, then there does not exist a continuous injection $F \to E$.

Now let $A_n = \{\sum_{j=1}^{n} a_j : a_j \in A\}$ where $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.
Since

$$\lim_{k \to \infty} \left(\sum_{j=1}^{i-1} a_j + \frac{1}{k} \right) = \sum_{j=1}^{i-1} a_j, \text{ we have } A_i(1) \supset A_{i-1}.$$

We now show that $A_i(1) = A_{i-1}$. For a given $y \in A_i(1)$, choose a sequence $x_n \in A_i$, $x_n \neq y$, $x_n \to y$, and write

$$x_n = a_{1,n} + \ldots + a_{i,n}$$

where each $a_{k,n}$ is of the form $1/m$. Then $\lim_{n \to \infty} a_{k,n} = 0$ for at least one $k (1 \leqslant k \leqslant i)$, since otherwise each $a_{j,n}$ could take only a finite set of different values, x_n can not tend to y. Now, by replacing x_n by a suitable subsequence, if necessary, we can assume that $\lim_{n \to \infty} a_{j,n}$ exists for each j. Hence $\lim x_n = \sum_{j \neq k} \lim a_{j,n} \in A_{i-1}$.

Now take $A_0 = \{0\}$, $A_{-1} = \emptyset$, then $A_n(m) = A_{n-m} \begin{cases} = \emptyset \text{ if } n < m \\ \neq \emptyset \text{ if } n \geqslant m \end{cases}$.
Hence if $p < n$, there does not exist a continuous injection $A_n \to A_p$. In particular, A_n is not homeomorphic to A_p if $n \neq p$.

Proof of Lemma 2. Since X is semi-analytic, X admits a regular stratification $X = \cup M_i$ in the sense of Whitney [7], where each stratum M_i is a connected semi-analytic manifold. By Corollary (10.2) in [9], any two points of a same stratum have homeomorphic neighborhoods in X. Hence for each i, either $M_i \subset \tilde{X}$ or $M_i \cap \tilde{X} = \emptyset$. That is, $\tilde{X}$ is a (locally-finite) union of semi-analytic strata, hence is semi-analytic.

ACKNOWLEDGEMENT

We wish to thank Professors N. H. Kuiper and S. Łojasiewicz for helpful conversations.

Institut des Hautes Études Scientifiques
91 Bures-sur-Yvette, France

REFERENCES

1. BOCHNAK, J., Jets sufficients et germs de détermination finie, C. R. Acad. Sci. Paris, 271, 1162–1164 (1970).
2. ——— and S. ŁOJASIEWICZ, A converse of the Kuiper-Kuo theorem, Proc. of Liverpool Singularities Symp. I., Lectures Notes in Math. 192, Springer, 254–261 (1971).
3. KUIPER, N., C^1-equivalence of functions near isolated critical points, Symp. Infinite Dimensional Topology, Baton Rouge 1967.
4. KUO, T. C., A complete determination of C^0-sufficiency in J^r (2, 1), Inv. Math. 8, 226–235 (1969).
5. LEVINE, H., Singularities of differentiable mappings. Proc. of Liverpool Singularities Symposium I. Lectures Notes in Math. 192, 1–90 Springer (1971).
6. ŁOJASIEWICZ, S., Triangulation of semi-analytic sets, Ann. Scuola Normale Sup. Pisa, 18.4 443–474 (1964).
7. ———, Ensembles Semi-Analytiques. Lectures Notes IHES (Bures-sur-Yvette), 1965.
8. MALGRANGE, B., Ideals of Differentiable Functions. Oxford 1966.
9. MATHER, J. N., Notes on Topological Stability. Lectures Notes, Harvard Univ. 1970.
10. MILNOR, J., Singular points of Complex Hypersurfaces. Ann. of Math. Studies 61, Princeton Univ. Press 1968.
11. THOM, R., Manifolds, Amsterdam 1970, Edited by N. H. Kuiper, Springer Lectures Notes, 197, 1971.