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MINIMAL MULTIPLICATIVE COVERS OF AN INTEGER

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Let $|S| = n$. The numbers $m(n,k) = |\{(S_1, \ldots, S_k) : \bigcup S_i = S$ and $\forall t \in [1,k]$, $\bigcup_{i \neq t} S_i \neq S\}|$ have been studied previously by Hearne and Wagner. The present paper treats three arrays, $\tilde{m}(n,k)$, $\bar{m}(n,k)$, and $\hat{m}(n,k)$, which extend $m(n,k)$ in the sense that $\tilde{m}(p_1 \cdots p_s, k) = \bar{m}(p_1 \cdots p_s, k) = \hat{m}(p_1 \cdots p_s, k) = m(s,k)$ for all sequences $(p_1, \ldots, p_s)$ of distinct primes.

1. Introduction

A sequence $(S_1, \ldots, S_k)$ of sets (called *blocks*) with $\bigcup S_i = S$ is called a *minimal ordered cover of* S if, $\forall t \in [1,k]$, $\bigcup_{i \neq t} S_i$ is a proper subset of S. It is shown in [2] that the number of minimal ordered covers of an n-set, with k blocks, is given by

$$m(n,k) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} (2^k - 1 - r)^n. \tag{1.1}$$

If we set

$$\tilde{m}(n,k) := |\{(d_1, \ldots, d_k) : \text{l.c.m.}\,(d_i) = n, \text{ and } \forall t \in [1,k],\ \text{l.c.m.}\,(d_i)_{i \neq t} < n\}|, \tag{1.2}$$

and

$$\bar{m}(n,k) = \left|\left\{(d_1, \ldots, d_k) : d_i \mid n,\ n \mid \prod d_i, \text{ and } \forall t \in [1,k],\ n \nmid \prod_{i \neq t} d_i\right\}\right|, \tag{1.3}$$

then it is clear that $\tilde{m}(n,k)$ and $\bar{m}(n,k)$ *extend* $m(n,k)$ in the sense that $\tilde{m}(p_1 \cdots p_s, k) = \bar{m}(p_1 \cdots p_s, k) = m(s,k)$ for all sequences $(p_1, \ldots, p_s)$ of distinct primes. We derive here explicit formulas for $\tilde{m}(n,k)$ and $\bar{m}(n,k)$, and consider in addition a third extension, $\hat{m}(n,k)$, of $m(n,k)$ given by

$$\hat{m}(n,k) := \sum_{r=0}^{k} (-1)^r \binom{k}{r} \tau_{2^k - 1 - r}(n), \tag{1.4}$$

where

$$\tau_j(n) = \left|\left\{(d_1, \ldots, d_j) : \prod d_i = n\right\}\right|. \tag{1.5}$$

The $\tilde{m}(n,k)$ are perhaps the most natural extension of the $m(n,k)$. The $\bar{m}(n,k)$.

on the other hand, are defined purely in terms of lattice properties of the natural numbers ordered by divisibility, and thus suggest the possibility of generalization to a broader class of lattices. As for the $\hat{m}(n,k)$, we have

$$\sum_{n=1}^{\infty} \frac{\hat{m}(n,k)}{n^s} = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \zeta^{2^k-1-r}(s), \tag{1.6}$$

whereas

$$\sum_{n=1}^{\infty} \hat{m}(n,k) \frac{x^n}{n!} = \sum_{r=0}^{k} (-1)^r \binom{k}{r} e^{(2^k-1-r)x}, \tag{1.7}$$

so that the $\hat{m}(n,k)$ are a natural extension of the $m(n,k)$ from the standpoint of generating functions (see Section 4). We remark that in some cases $\hat{m}(n,k)$ is greater than the total number of sequences $(d_1,\ldots,d_k)$ of divisors of n, precluding a combinatorial interpretation of $\hat{m}(n,k)$ analogous to (1.2) and (1.3).

2. The numbers $\hat{m}(n,k)$

For

$$\hat{m}(n,k) = |\{(d_1,\ldots,d_k): \text{l.c.m.}\,(d_i) = n \text{ and } \forall t \in [1,k], \text{l.c.m.}\,(d_i)_{i\neq t} < n\}| \tag{2.1}$$

we have the following explicit formula:

Theorem 2.1. *Let $n = p_1^{n_1} \cdots p_s^{n_s}$, where the p_j are distinct primes and the n_j are positive integers. Then, $\forall k \geq 1$,*

$$\hat{m}(n,k) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \prod_{j=1}^{s} [(n_j+1)^k - n_j^k - rn_j^{k-1}]. \tag{2.2}$$

Proof. Writing each divisor d_i of n as $d_i = p_1^{x_{i1}} \cdots p_s^{x_{is}}$, it is clear that $\hat{m}(n,k) = |L|$, where L consists of all $k \times s$ matrices (x_{ij}) such that (1) $0 \leq x_{ij} \leq n_j$, (2) $\forall j \in [1,s]$, $\exists i \in [1,k]$ such that $x_{ij} = n_j$, and (3) $\forall r \in [1,k]$, $\exists j \in [1,s]$ such that $x_{ij} < n_j$, $\forall i \neq r$. Let B denote the set of all $k \times s$ matrices satisfying properties (1) and (2) above. For each $r \in [1,k]$, let B_r denote the set of matrices $(x_{ij}) \in B$ such that, $\forall_j \in [1,s]$, $\exists i \neq r$ such that $x_{ij} = n_j$. Then $L = B - (B_1 \cup \cdots \cup B_k)$ and by the principle of inclusion and exclusion

$$\hat{m}(n,k) = |L| = |B| + \sum_{r=1}^{k} (-1)^r \binom{k}{r} |B_1 \cap \cdots \cap B_r|. \tag{2.3}$$

Now the columns of a matrix in B or in $B_1 \cap \cdots \cap B_r$ may be chosen independently of each other. Hence

$$|B| = \prod_{j=1}^{s} [(n_j+1)^k - n_j^k]$$

and

$$|B_1 \cap \cdots \cap B_r| = \prod_{j=1}^{s} [(n_j+1)^k - n_j^k - r n_j^{k-1}],$$

which, with (2.3), yields (2.2).

It follows from (2.2) that $\tilde{m}(n, 1) = 1$ and $\tilde{m}(p^m, k) = 0$, $\forall k \geq 2$. Moreover

$$\tilde{m}(p_1 \cdots p_s, k) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} (2^k - 1 - r)^s = m(s, k),$$

as one would expect from (2.1) and (1.1).

Replacing the variable r in (2.2) by $k - r$ yields

$$\begin{aligned}\tilde{m}(n, k) &= \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \prod_{j=1}^{s} [(n_j+1)^k - n_j^k - k n_j^{k-1} + r n_j^{k-1}] \\ &= \Delta^k \prod_{j=1}^{s} [(n_j+1)^k - n_j^k - k n_j^{k-1} + x n_j^{k-1}]\big|_{x=0}.\end{aligned}$$

Hence it is clear that $\tilde{m}(p_1^{n_1} \cdots p_s^{n_s}, k) = 0$ if $k > s$. Moreover,

$$\tilde{m}(p_1^{n_1} \cdots p_s^{n_s}, s) = \Delta^s (n_1 \cdots n_s)^{s-1} x^s |_{x=0} = s! (n_1 \cdots n_s)^{s-1},$$

which may also be derived directly from (2.1).

3. The numbers $\tilde{m}(n, k)$.

For

$$\tilde{m}(n, k) = \left| \left\{ (d_1, \ldots, d_k) : d_i \mid n,\ n \mid \prod d_i, \text{ and } \forall t \in [1, k],\ n \nmid \prod_{i \neq t} d_i \right\} \right|, \tag{3.1}$$

we have the following explicit formula:

Theorem 3.1. *Let $n = p_1^{n_1} \cdots p_s^{n_s}$ where the p_j are distinct primes and the n_j are positive integers. Then, $\forall k \geq 1$,*

$$\tilde{m}(n, k) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \prod_{j=1}^{s} \sum_{v=0}^{r} (-1)^v \binom{r}{v} s(n_j, k, v), \tag{3.2}$$

where

$$s(n_j, k, 0) = (n_j + 1)^k - \binom{n_j + k - 1}{k}, \tag{3.3}$$

$$s(n_j, k, 1) = n_j \binom{n_j + k - 2}{k - 1} - \binom{n_j + k - 2}{k}, \tag{3.4}$$

and for $v \geq 2$,

$$s(n_j, k, v) = \sum_{i=0}^{n_j - 1} \binom{n_j - 1 - (v-1)i + (k - v)}{k - 1}. \tag{3.5}$$

Proof. Writing each divisor d of n as $d = p_1^{\cdots} \cdots p_s^{\cdots}$, it is clear that $\tilde{m}(n, k)$

$|M|$, where M consists of all $k\times s$ matrices (x_{ij}) such that (1) $0\leq x_{ij}\leq n_j$, (2) $\sum_i x_{ij}\geq n_j$, and (3) $\forall r\in[1,k]$, $\exists j\in[1,s]$ such that $\sum_{i\neq r} x_{ij}<n_j$. Let S denote the set of all $k\times s$ matrices satisfying properties (1) and (2) above. For each $r\in[1,k]$, let S_r denote the set of matrices $(x_{ij})\in S$ such that $\forall j\in[1,s]$, $\sum_{i\neq r} x_{ij}\geq n_j$. Then $M=S-(S_1\cup\cdots\cup S_k)$ and so by the principle of inclusion and exclusion,

$$\bar{m}(n,k)=|M|=|S|+\sum_{r=1}^{k}(-1)^r\binom{k}{r}|S_1\cap\cdots\cap S_r|. \tag{3.6}$$

Now the columns of a matrix in S may be chosen independently of each other. Denote by $s(n_j,k,0)$ the number of possible choices for the jth column of such a matrix. Then

$$\begin{aligned} s(n_j,k,0)&=\left|\left\{(x_1,\ldots,x_k):0\leq x_i\leq n_j \text{ and } \sum x_i\geq n_j\right\}\right|\\ &=(n_j+1)^k-\sum_{r=0}^{n_j-1}\binom{r+k-1}{k-1}\\ &=(n_j+1)^k-\binom{n_j+k-1}{k}, \end{aligned} \tag{3.7}$$

and so

$$|S|=\prod_{j=1}^{s} s(n_j,k,0), \tag{3.8}$$

where $s(n_j,k,0)$ is given by (3.7).

Similarly, the columns of a matrix belonging to $S_1\cap\cdots\cap S_r$ may be chosen independently of each other. The jth column of such a matrix consists of a sequence $(x_1,\ldots,x_k)$ such that $0\leq x_i\leq n_j$ and, $\forall v\in[1,r]$, $(x_1+\cdots+x_k)-x_v\geq n_j$. Let $T=\{(x_1,\cdots,x_k):0\leq x_i\leq n_j \text{ and } x_1+\cdots+x_k\geq n_j\}$, and for all $v\in[1,r]$, let $T_v=\{(x_1,\ldots,x_k)\in T:(x_1+\cdots+x_k)-x_v<n_j\}$. It follows from the principle of inclusion and exclusion that the jth column of a matrix in $S_1\cap\cdots\cap S_r$ may be chosen in

$$|T|+\sum_{v=1}^{r}(-1)^v\binom{r}{v}|T_1\cap\cdots\cap T_v|$$

ways. By (3.7), $|T|=s(n_j,k,0)$. Denote $|T_1\cap\cdots\cap T_v|$ by $s(n_j,k,v)$. Then

$$|S_1\cap\cdots\cap S_r|=\prod_{j=1}^{s}\left[\sum_{v=0}^{r}(-1)^v\binom{r}{v}s(n_j,k,v)\right], \tag{3.9}$$

and we need only evaluate the $s(n_j,k,v)$ for $v\geq 1$ to complete the proof.

Clearly, $s(n_j,k,v)=|\{(x_1,\cdots,x_k):0\leq x_i\leq n_j,\ x_1+\cdots+x_k\geq n_j$, and $(x_1+\cdots+x_k)-x_z<n_j$ for all $z\in[1,v]\}|$. We enumerate such sequences by the value w taken on by $x_1 (1\leq w\leq n_j)$. For fixed $w=x_1$, we must count all sequences $(x_2,\ldots,x_k)$ such that (1) $x_2+\cdots+x_k\geq n_j-w$, (2) $x_2+\cdots+x_k<n_j$, and (3) $(x_2+\cdots+x_k)-x_z<n_j-w$ for all $z\in[2,v]$. We count such sequences by the value n_j-w+t taken on by $x_2+\cdots+x_k$ $(0\leq t\leq w-1)$. For fixed t, we require

the number of solutions to $x_2 + \cdots x_k = n_j - w + t$ subject to $x_i > t$ for $i \in [2, v]$ and $x_i \geq 0$ for $i \in [v+1, k]$. There are

$$\binom{n_j - w + t - 1 - (v-1)t + (k-v)}{k-2}$$

such solutions. Hence

$$\begin{aligned} s(n_j, k, v) &= \sum_{w=1}^{n_j} \sum_{t=0}^{w-1} \binom{n_j - w + t - 1 - (v-1)t + (k-v)}{k-2} \\ &= \sum_{t=0}^{n_j-1} \sum_{w=t+1}^{n_j} \binom{n_j - w + t - 1 - (v-1)t + (k-v)}{k-2} \\ &= \sum_{t=0}^{n_j-1} \left[\binom{n_j - 1 - (v-1)t + (k-v)}{k-1} - \binom{(k-v) - (v-2)t - 1}{k-2} \right]. \end{aligned} \tag{3.10}$$

We note that

$$\begin{aligned} s(n_j, k, 1) &= \sum_{t=0}^{n_j-1} \left[\binom{n_j + k - 2}{k-1} - \binom{k + t - 2}{k-1} \right] \\ &= n_j \binom{n_j + k - 2}{k-1} - \binom{n_j + k - 2}{k}, \end{aligned} \tag{3.11}$$

and that for $v \geq 2$,

$$s(n_j, k, v) = \sum_{t=0}^{n_j-1} \binom{n_j - 1 - (v-1)t + (k-v)}{k-1}. \tag{3.12}$$

In particular,

$$s(n_j, k, 2) = \binom{n_j + k - 2}{k}. \tag{3.13}$$

Theorem 3.2. *Let $n = p_1^{n_1} \cdots p_s^{n_s}$, where the p_i are distinct primes and the n_i are positive integers. Then $\bar{m}(n, k) = 0$, if $k > n_1 + \cdots + n_s$. Moreover*

$$\bar{m}(p_1^{n_1} \cdots p_s^{n_s}, n_1 + \cdots + n_s) = \frac{(n_1 + \cdots + n_s)!}{n_1! \cdots n_s!} \tag{3.14}$$

Proof. It is clear from (3.5) that if $v > n_j$, then $s(n_j, k, v) = 0$. Hence we may write

$$\bar{m}(n, k) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \prod_{j=1}^{s} \sum_{v=0}^{n_j} (-1)^v \binom{r}{v} s(n_j, k, v). \tag{3.15}$$

Replacing the variable r in (3.15) by $k - r$ yields

$$\begin{aligned} \bar{m}(n, k) &= \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \prod_{j=1}^{s} \sum_{v=0}^{n_j} (-1)^v \binom{k-r}{v} s(n_j, k, v) \\ &= \Delta^k \prod_{j=1}^{s} \sum_{v=0}^{n_j} (-1)^v \binom{k-x}{v} s(n_j, k, v)\Big|_{x=0}. \end{aligned} \tag{3.16}$$

It follows from (3.4) and (3.5) that $s(n_j, k, n_j)=1$, $\forall k \geqslant 1$. Hence $\bar{m}(n,k)=\Delta^k f(x)|_{x=0}$, where $\deg f(x)=n_1+\cdots n_s$, and so $\bar{m}(n,k)=0$ if $k>n_1+\cdots+n_s$. Moreover,

$$\bar{m}(n, n_1+\cdots+n_s) = \Delta^{n_1+\cdots+n_s} \frac{x^{n_1+\cdots+n_s}}{n_1!\cdots n_s!}\bigg|_{x=0} = \frac{(n_1+\cdots+n_s)!}{n_1!\cdots n_s!}. \tag{3.17}$$

We conclude this section by noting some special cases of (3.2). We have $\bar{m}(n,1)=1$, $\forall n \geqslant 2$, and

$$\bar{m}(n,2)=\prod_{j=1}^{s}\binom{n_j+2}{2}-2\prod_{j=1}^{s}(r_j+1)+1. \tag{3.18}$$

Moreover,

$$\bar{m}(p_1\cdots p_s, k)=\sum_{r=0}^{k}(-1)^r\binom{k}{r}(2^k-1-r)^s=m(s,k), \tag{3.19}$$

as one would expect from (1.1) and (3.1).

For $n=p^m$, we have

$$\begin{aligned}\bar{m}(p^m,k) &= \sum_{r=1}^{k}\sum_{v=1}^{r}(-1)^{r+v}\binom{k}{r}\binom{r}{v}s(m,k,v) \\ &= \sum_{v=1}^{k}s(m,k,v)\sum_{r=v}^{k}(-1)^{r+v}\binom{k}{r}\binom{r}{v} \\ &= s(m,k,k),\end{aligned} \tag{3.20}$$

as one would expect. Hence $\bar{m}(p^m,1)=s(m,1,1)=1$, and for $k \geqslant 2$.

$$\bar{m}(p^m,k)=\sum_{t=0}^{m-1}\binom{m-1-(k-1)t}{k-1}=\sum_{t=0}^{[m-k/k-1]}\binom{m-1-(k-1)t}{k-1}, \tag{3.21}$$

since $(m-1)-(k-1)t<k-1$ if $t>[(m-k)/(k-1)]$. In particular,

$$\begin{aligned}&\bar{m}(p^m,2)=\binom{m}{2}, \\ &\bar{m}(p^m,m)=1, \\ &\bar{m}(p^m,m-1)=(m-1)+\binom{1}{m-2}, \quad m \geqslant 3.\end{aligned} \tag{3.22}$$

4. The numbers $\hat{m}(n,k)$

Let $\sigma_k(n)$ denote the number of ordered partitions of an n-set, with k blocks. As is well-known,

$$\sigma_k(n)=\Delta^k x^n|_{x=0}=\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}r^n. \tag{4.1}$$

In [1], Carlitz considered the numbers

$$\tau'_k(n) = \sum_{r=0}^{k} (-1)^{k-r}\binom{k}{r}\tau_r(n), \tag{4.2}$$

where

$$\tau_r(n) = |\{(d_1, \ldots, d_r) : \prod d_i = n\}|. \tag{4.3}$$

It follows that

$$\tau'_k(n) = |\{d_1, \ldots, d_k) : d_i > 1 \quad \text{and} \quad \prod d_i = n\}| \tag{4.4}$$

and

$$\tau'_k(p_1 \cdots p_s) = \sigma_k(s) \tag{4.5}$$

for all sequences $(p_1, \ldots, p_s)$ of distinct primes so that in Carlitz's terminology, the $\tau'_k(n)$ *extend* the $\sigma_k(n)$. In addition, it is easy to see that

$$\sum_{n=1}^{\infty} \sigma_k(n)\frac{x^n}{n!} = P(e^x) \tag{4.6}$$

and

$$\sum_{n=1}^{\infty} \frac{\tau'_k(n)}{n^s} = P(\zeta(s)), \tag{4.7}$$

where

$$P(z) = (z-1)^k. \tag{4.8}$$

Since

$$m(n, k) = \sum_{r=0}^{k} (-1)^r\binom{k}{r}(2^k - 1 - r)^n, \tag{4.9}$$

the foregoing remarks suggest that we consider the array $\hat{m}(n, k)$ given by

$$\hat{m}(n, k) = \sum_{r=0}^{k} (-1)^r\binom{k}{r}\tau_{2^k-1-r}(n). \tag{4.10}$$

It is clear that the $\hat{m}(n, k)$ extend the $m(n, k)$. Moreover,

$$\sum_{n=1}^{\infty} m(n, k)\frac{x^n}{n!} = M(e^x) \tag{4.11}$$

and

$$\sum_{n=1}^{\infty} \frac{\hat{m}(n, k)}{n^s} = M(\zeta(s)), \tag{4.12}$$

where

$$M(z) = \sum_{r=0}^{k} (-1)^r\binom{k}{r} z^{2^k-1-r} = z^{2^k-1}(1 - z^{-1})^k. \tag{4.13}$$

For $n = p_1^{n_1} \cdots p_s^{n_s}$ we have the expanded formula

$$\hat{m}(n, k) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} \prod_{j=1}^{s} \binom{n_j + 2^k - 2 - r}{n_j}. \tag{4.14}$$

We may employ finite difference methods on (4.14), as we did with $\bar{m}(n, k)$ and $\tilde{m}(n, k)$, to show that $\hat{m}(n, k) = 0$ if $k > n_1 + \cdots + n_s$ and that

$$\hat{m}(n, n_1 + \cdots + n_s) = \frac{(n_1 + \cdots + n_s)!}{n_1! \cdots n_s!} = \bar{m}(r, n_1 + \cdots + n_s). \tag{4.15}$$

Moreover, it is easy to check that $\hat{m}(n, 2) = \bar{m}(n, 2)$, (see (3.18)). For $n = p^m$, we have

$$\begin{aligned} \hat{m}(p^m, k) &= \sum_{r=0}^{k} (-1)^r \binom{k}{r} \binom{m + 2^k - 2 - r}{m} \\ &= \Delta^k \binom{m - k + 2^k - 2 + x}{m} \Bigg|_{x=0} \\ &= \binom{m - k + 2^k - 2}{m - k}. \end{aligned} \tag{4.16}$$

In particular,

$$\hat{m}(p^m, 3) = \binom{m + 3}{6}. \tag{4.17}$$

On the other hand, the total number of sequences (d_1, d_2, d_3) of divisors of p^m is $(m+1)^3$, and since, for example, $\hat{m}(p^{10}, 3) > 11^3$, there is no possibility of furnishing a combinatorial interpretation of the $\hat{m}(n, k)$ analogous to those of $\bar{m}(n, k)$ and $\tilde{m}(n, k)$. However, it is clear from (4.10) that

$$\hat{m}(n, k) = \left|\{(d_1, \cdots, d_{2^k - 1}) : \textstyle\prod d_i = n \quad \text{and} \quad d_i > 1, \forall i \in [1, k]\}\right|, \tag{4.18}$$

and so the $\hat{m}(n, k)$, like the $\bar{m}(n, k)$ and $\tilde{m}(n, k)$, count divisor sequences (albeit with length $2^k - 1$, rather than k) having a certain minimality property.

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References

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