Exponentially Stabilizing Finite-Dimensional Controllers for Linear Distributed Parameter Systems: Galerkin Approximation of Infinite Dimensional Controllers

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The Galerkin method is presented as a way to develop finite-dimensional controllers for linear distributed parameter systems (DPS). The direct approach approximates the open-loop DPS and then generates the controller from this approximation; the indirect approach approximates the infinite-dimensional stabilizing controller. The indirect approach is shown to converge to the stable closed-loop system consisting of DPS and infinite-dimensional controller; conditions are presented on the behavior of the Galerkin method for the open-loop DPS which guarantee closed-loop stability for large enough finite-dimensional approximations. © 1986 Academic Press, Inc.

1.0. INTRODUCTION

Many engineering systems are best modeled by partial differential equations or delay differential equations. These are examples of distributed-parameter systems (DPS) which require a dynamical realization on an infinite-dimensional state-space to properly describe their behavior; this is in contrast to lumped parameter systems where the better known finite-dimensional state-space descriptions can be used. Certainly the most fundamental constraint for feedback control of DPS is that the controller algorithm must be finite-dimensional in order to be realized with an on-line computer and a finite (small) number of control actuators and sensors. This is a very serious issue for DPS control since there is no guarantee that a finite-dimensional controller can even produce closed-loop stability with an infinite-dimensional DPS. Previous work on this problem includes [1, 5]; the focus of [1, 2] is on parabolic systems, i.e., ones involving analytic semigroups.
The most obvious approach to the design of a finite-dimensional controller for a DPS is to make a finite-dimensional approximation, i.e., reduced-order model (ROM), of the open-loop DPS and then design the controller directly from the ROM. This approach is used throughout the engineering community when DPS are encountered; however, there is no reason to expect that such a controller will stabilize the DPS in closed-loop, and it often does not, e.g., [7]. Moreover, the assumption usually made to facilitate any stability analysis is that the exact "modes," i.e., eigenfunctions, of the open-loop system are known; at best, such modes can be approximated for practical engineering systems.

In [4], we consider the question of closed-loop exponential stability where the controller is obtained by any Galerkin approximation of the original DPS. For stabilizable and detectable linear DPS, there exist exponentially stabilizing infinite-dimensional controllers, e.g., [6]. The Galerkin approximation of such infinite-dimensional controllers is used in [5] to generate finite-dimensional controllers. Under rather mild conditions on the behavior of the Galerkin schemes on the open-loop DPS, it is shown that the sequence of closed-loop systems with the DPS plus an approximated controller converges to the stable closed-loop system with the infinite-dimensional controller. This result gives some hope that at least some Galerkin schemes will yield a finite-dimensional stabilizing controller. However, in [5] we are only able to show this is true for DPS where the approximate controllers themselves become uniformly exponentially stable. Also, in that reference, we give conditions under which the direct approach of [4] and the indirect approach of [5] yield equivalent stabilizing controllers.

In this paper, we return to the question of whether Galerkin approximation of the infinite-dimensional stabilizing controller for a linear DPS can produce an exponentially stabilizing finite-dimensional controller. We concentrate on exponential stability, rather than strong or weak stability, of the closed-loop because of its robustness to bounded perturbations (which the others lack); such robustness is essential when dealing with engineering systems where errors are always present in the DPS model.

In Section 2, DPS preliminaries, the basic hypotheses for the class of linear DPS are presented; we do not confine ourselves to parabolic or modal systems. Galerkin approximation of DPS is presented in Section 3.0; results from [5] are summarized and extended there. We do not place any restrictions on the Galerkin schemes except that they converge to the open-loop DPS. Our main results on closed-loop stability with the approximate controllers are given in Section 4.0; further restrictions on the acceptable Galerkin schemes for closed-loop stability are needed there. Finally, in Section 5.0 we look at the special case of modal control where the original
open-loop DPS has compact normal resolvent with only a finite number of unstable eigenvalues. Under these assumptions, our results reproduce and generalize those of Sakawa in [1].

2.0. DPS Preliminaries

The linear DPS of interest will be modeled by the following state-space form:

$$\frac{dv(t)}{dt} = Av(t) + Bf(t); \quad v(0) = v_0$$

$$y(t) = Cv(t)$$

(2.1)

where the state $v(t)$ is in an infinite-dimensional Hilbert space $H$ with inner product $(\cdot, \cdot)$ and corresponding norm $\| \cdot \|$. The input-output operators $B$ and $C$ are bounded and have finite ranks $M$ and $P$, respectively, and $f(t)$, $y(t)$ represent the inputs for $M$ linear actuators and the outputs from $P$ linear sensors, respectively. Thus,

$$Bf(t) = \sum_{i=1}^{M} b_i f_i(t)$$

(2.2)

and

$$y(t) = [y_1(t), \ldots, y_P(t)]^T$$

with

$$y_j(t) = (c_j, v(t)); \quad 1 \leq j \leq P,$$

(2.3)

where $b_i$ and $c_j$ belong to $H$. In finite-dimensional theory, $A$ would be a matrix, but here the operator $A$ is a closed, linear, unbounded differential operator with domain $D(A)$ dense in $H$. Furthermore (2.1)–(2.3) represent some well-posed physical system, which in mathematical terms is the weak formulation of (2.1):

$$v(t) = U(t) v_0 + \int_0^t U(t - \tau) Bf(\tau) d\tau$$

$$y(t) = Cv(t); \quad t \geq 0$$

(2.4)

where $v_0$ is any initial state in $H$ and $U(t)$ is the $C_0$-semigroup of bounded operators generated on $H$ by $A$. This latter means:
Finite-dimensional controllers

\[ U(t + \tau) = U(t) U(\tau); \quad t \geq 0, \ \tau \geq 0 \]  
(2.5a)

\[ U(0) = I \]  
(2.5b)

\[ \lim_{t \to 0^+} [U(t) - I] v = 0; \quad v \in H \]  
(2.5c)

\[ Av = \lim_{t \to 0^+} \left[ \frac{U(t) - I}{t} \right] v; \quad v \in D(A) \]  
(2.5d)

Note that the semigroup \( U(t) \) evolves the initial conditions \( v_0 \) forward in time. When \( v_0 \) is in \( D(A) \) and \( f(t) \) has continuous first derivative, \( v(t) \) also is differentiable, lies in \( D(A) \) for \( t \geq 0 \), and satisfies (2.1). However, any \( v_0 \) in \( H \) and any square-integrable \( f(t) \) will satisfy the weak formulation (2.4) and yield states \( v(t) \) in \( H \) for all \( t \geq 0 \). Consequently, (2.4) is much easier to work with in infinite-dimensions and is more likely to represent the actual physical system being modeled by (2.1).

This form (2.1) or (2.4) models most practical interior control problems for linear DPS, where the actuator and sensor influence functions are given by \( b_i \) and \( c_j \), respectively. Linear boundary control problems for DPS have a somewhat different form from (2.1); however, they can usually be converted to equivalent interior control problems which do look like (2.1) [3]. Therefore, we will focus on the form (2.1) without any loss of generality for linear DPS problems.

The Hille–Yosida theorem provides conditions under which a closed operator \( A \) generates a \( C_0 \)-semigroup \( U(t) \) satisfying

\[ \| U(t) \| \leq Ke^{-\sigma t}, \quad t \geq 0 \]  
(2.6)

where \( K \geq 1 \) and \( \sigma \) real. The necessary and sufficient conditions are given for the resolvent operator \( R(\lambda, A) \equiv (\lambda I - A)^{-1} \):

\[ \| R(\lambda, A)^n \| \leq \frac{K}{(\lambda + \sigma)^n}; \quad n = 1, 2, \ldots \]  
(2.7)

for all real \( \lambda > -\sigma \) in the resolvent set of \( A \), \( \rho(A) = \{ \lambda \text{ complex} \mid R(\lambda, A) \text{ is a bounded operator on } H \} \). The spectrum of \( A \), \( \sigma(A) = \rho(A) \}, \) is much more complicated in infinite-dimensions, but, in finite-dimensions, it consists only of the (finite number of) eigenvalues of \( A \). Recall that the DPS (2.1) or \( A \) is exponentially stable when \( \sigma > 0 \) in (2.6), i.e., the semigroup \( U(t) \) generated by \( A \) decays exponentially at the rate. There are many other types of stability in infinite-dimensions, but no others provide the safety of a stability margin \( \sigma \); therefore, this seems like the kind of stability of most practical interest for engineering applications where there is always some uncertainty in the model of the DPS. Henceforth, when we refer to stability, we shall mean exponential stability.
We say that the pair \((A, B)\) in (2.1) is (exponentially) stabilizable if there is a linear gain operator \(G: H \to \mathbb{R}^m\) such that \(A + BG\) generates an exponentially stable \(C_0\)-semigroup, i.e., the semigroup satisfies (2.6) with \(\sigma > 0\). Similarly, the pair \((A, C)\) in (2.1) is (exponentially) detectable if \((A^*, C^*)\) is stabilizable where \(A^*\) is the adjoint operator associated with \(A\).

We say that \((A, B)\) in (2.1) has a pair of stabilizing subspaces \((H_N, H_R)\) if the following hold:

\[
H = H_N \oplus H_R \quad (2.8a)
\]
\[
\text{dim } H_N = N < \infty, \quad H_R \text{ closed}, \quad H_N \subseteq D(A) \quad (2.8b)
\]

and \(A_0 \equiv A + BG\) generates an exponentially stable \(C_0\)-semigroup \(U_0(t)\), i.e.,

\[
\| U_0(t) \| \leq K_0 e^{-\sigma_0 t}, \quad t \geq 0 \quad (2.8c)
\]

with \(K_0 \geq 1\) and \(\sigma_0 > 0\), where

\[
G = G P_N \quad \text{(or } G P_R = 0) \quad (2.8d)
\]

with \((P_N, P_R)\) the projections defined by (2.8a). Thus, stabilizing subspaces guarantee that the projection feedback law:

\[
f(t) = GP_N v(t) \quad (2.9)
\]

can produce an exponentially stable closed-loop system (2.1) and (2.9). Usually, we assume that \(\sigma_0\) is specified; hence (2.1) may have stabilizing subspaces for some values \(\sigma_0\) but not for others (clearly, if it has them for some \(\sigma_0 > 0\) then it will have them for all smaller values \(0 < \sigma \leq \sigma_0\)). Of course, it should be noted that (2.9) is an ideal control law which cannot in general be generated from the sensor outputs (2.3). The main result in [3] shows that every finite-dimensional stabilizing controller must asymptotically reproduce (2.9) for a special pair of stabilizing subspaces, when an associated asymmetric Riccati equation is solvable.

Next we present some results for infinite-dimensional DPS controllers which are analogous to the finite-dimensional state-space controllers for lumped parameter systems. Unlike their finite-dimensional counterparts, these controllers cannot be implemented with practical computers and devices in general. Nevertheless, such results give further insight into the DPS control problem and are needed in later sections.

The first result gives conditions under which the full state \(u(t)\) of the DPS can be recovered asymptotically from the finite number of available measurements \(y(t)\) by an infinite-dimensional state estimator (Kalman filter or Luenberger observer):
**Theorem 2.1.** If \((A, C)\) is detectable then there is a bounded operator \(K\) mapping \(\mathbb{R}^p\) into \(D(A)\) such that the estimated state \(\hat{v}(t)\) generated by the state estimator:

\[
\frac{\partial \hat{v}(t)}{\partial t} = A\hat{v}(t) + Bf(t) + K(y(t) - \dot{y}(t)), \quad \hat{v}(0) = 0
\]

converges in norm to the actual state \(v(t)\) at an exponential rate (determined by \(K\)).

The second result gives conditions under which stability of the DPS may be achieved using the state-estimator (2.10):

**Theorem 2.2.** In addition to the hypothesis of Theorem 2.1, if \((A, B)\) is also stabilizable, then there is a bounded operator \(G\) from \(D(A)\) into \(\mathbb{R}^m\) such that the control law:

\[
f(t) = G\hat{v}(t),
\]

where \(\hat{v}(t)\) is generated by (2.10), produces an exponentially stable closed-loop system consisting of (2.1) and (2.10)–(2.11).

The proofs for these results are given in [6]; except for some infinite-dimensional technicalities they are the same as those for the finite-dimensional case. Note that for finite-dimensional systems \((A, B, C)\) controllable and observable would be sufficient to satisfy the hypothesis of Theorems 2.1–2.2; however, in infinite dimensions this is not the case when controllability and observability are taken in the approximate (and most reasonable) sense of [7, Chap. 4].

Therefore, under the above stabilizability–detectability conditions on \((A, B, C)\), a stabilizing controller exists, i.e., (2.10)–(2.11); however, this infinite-dimensional controller cannot be implemented. In this paper we shall be concerned with continuous-time, finite-dimensional, linear controllers for (2.1) of the form:

\[
f(t) = L_{11} y(t) + L_{12} z(t)
\]

\[
\dot{z}(t) = L_{21} y(t) + L_{22} z(t)
\]

where \(\text{dim } z = \alpha < \infty\). It is not an essential restriction that (2.12) be continuous-time; this is only done for convenience. In the next section, we describe methods to generate such controllers for DPS.
3.0. GALERKIN APPROXIMATION OF DPS: DIRECT AND INDIRECT MODEL REDUCTION

In general, a reduced-order model (ROM) of (2.1) is produced by projecting onto a finite-dimensional subspace. Suppose

$$H = H_N \oplus H_R,$$

where $H_R$ closed, $H_N \subseteq D(A)$, and $\dim H_N = N < \infty$. Let $v_N = P_N v$ and $v_R = P_R v$, where $P_N$, $P_R$ are the projections (not necessarily orthogonal) onto $H_N$, $H_R$, respectively. Then (2.1) decomposes into the following form when $v_0$ is in $D(A)$:

$$\frac{\partial v_N}{\partial t} = A_N v_N + A_{NR} v_R + B_N f; \quad v_N(0) = P_N v_0 \quad (3.2a)$$

$$\frac{\partial v_R}{\partial t} = A_{RN} v_N + A_R v_R + B_R f; \quad v_R(0) = P_R v_0 \quad (3.2b)$$

$$y = C_N v_N + C_R v_R, \quad (3.2c)$$

where $v = v_N + v_R$, $A_N = P_N A P_N$, $B_N = P_N B$, $C_N = C P_N$, $A_{NR} = P_N A P_R$, etc. All parameters except $A_R$, are bounded operators since $P_N$ is bounded and has finite rank. The ROM is produced by ignoring the residuals $v_R$ in (3.2):

$$\frac{\partial v_N}{\partial t} = A_N v_N + B_N f$$

$$y = C_N v_N. \quad (3.3)$$

This is a finite-dimensional approximation of (2.1) and the parameters $(A_N, B_N, C_N)$ may be identified with their corresponding matrices in any appropriate basis of $H_N$. Note that $A_N$ is defined on all of $D(A)$, but we shall usually think of its restriction to $H_N$.

In the special case [9 Theorem 6.17, p. 178], where the spectrum of $A$ may be separated into two parts $\sigma(A_N)$ and $\sigma(A_R)$, where $\sigma(A_N)$ consists of $N$ isolated eigenvalues of $A$ which can be separated from the rest of the spectrum $\sigma(A_R)$ by a smooth closed curve in the complex plane, there exist reducing subspaces $H_N$ and $H_R$ such that $A_N$ has the spectrum $\sigma(A_N)$, $A_R$ has the spectrum $\sigma(A_R)$, and these subspaces are $A$-invariant:

$$A_{NR} = 0 \quad \text{and} \quad A_{RN} = 0. \quad (3.4)$$

These are also called modal subspaces since $H_N = \text{sp}\{\phi_1, \ldots, \phi_N\}$, where $\phi_k$ are the mode shapes or eigenfunctions of the operator $A$ which correspond to the eigenvalues $\lambda_1, \ldots, \lambda_N$ in $\sigma(A_N)$. 
Now we develop two basic procedures for synthesizing finite-dimensional controllers for the DPS (2.1):

1. **Direct model reduction**, i.e., perform a model reduction on the DPS (2.1) and synthesize the controller directly from this ROM;

2. **Indirect model reduction**, i.e., perform a model reduction on the infinite-dimensional controller (2.10)–(2.11) to obtain a finite-dimensional approximation.

We will use the Galerkin method for model reduction in both cases.

The direct procedure is quite straightforward and is the most natural one to use from a practical standpoint. It requires nothing but ROM information for the controller synthesis and can be carried out even though the conditions for existence of an infinite-dimensional controller are not verified. However, it need not produce a stable closed-loop even in the modal case [7]. The indirect procedure requires the existence of an infinite-dimensional controller and some knowledge of the gain operators $G$ and $K$. When this knowledge is available, it seems reasonable to take advantage of it; the finite-dimensional approximation of the infinite-dimensional controller may perform better, and under some restrictions on the Galerkin scheme, it will yield closed-loop stability.

3.1. **The Galerkin Approximation**

Let $H_N$ be an increasing sequence of finite dimensional subspaces of the state space $H$ for (2.1):

$$H_N \subseteq H_{N+1} \subseteq \cdots \subseteq H. \tag{3.5}$$

Each subspace $H_N$ has dimension $N$. To make life easier, we assume that each $H_N$ is a subspace of $D(A)$ so that its elements satisfy the boundary conditions for $A$; however, so-called non-conforming elements may be used in the more general case. In the finite element method (FEM) each subspace $H_N$ consists of splines (i.e., piecewise-polynomial functions) of fixed degree defined over a mesh (usually, of triangles) laid out to approximately cover the spatial domain $\Omega$ of the problem (see [10, Chap. 6]). No matter how irregular the shape of the boundary of $\Omega$ such meshes can be fitted very closely; this is one of the principal assets of the FEM. To each mesh, a normalized mesh parameter $h$ (where $0 < h \leq 1$) is assigned so that the mesh is refined as $h \to 0$ and the dimension $N$ of the subspaces increases indefinitely.

Recall that a sequence $\{A_N\}_{N=1}^{\infty}$ of linear operators $A_N: H \to H$ converges strongly to $A$, i.e., $A_N \to^s A$, when

$$\lim_{N \to \infty} \| A_N v - Av \| = 0 \quad \text{for all } v \text{ in } H.$$
Let $P_N$ be the orthogonal projection from $H$ into $H_N$; this is called the Galerkin projection. The corresponding orthogonal projection onto $H_N$ is called $P_R$ (i.e., $P_R = I - P_N$). The “rate of convergence” of $H_N$ to $H$ is said to be of order $q$ when

$$\| P_R v \| \leq K h^q \quad (3.6)$$

for $v$ in $D(A)$; this rate is related to the ability of splines in $H_N$ to interpolate functions in $H$. We shall not be concerned with the rate of convergence $q$; consequently we write (3.6) as

$$\lim_{N \to \infty} \| P_R v \| = 0 \quad \text{for } v \in D(A) \quad (3.7)$$

(i.e., $P_N \to I$ or $P_R \to 0$ in $D(A)$) and suppress the dependence on $h$ henceforth.

Let $\psi_1(x), \ldots, \psi_N(x)$ form a basis in $H_N$ (i.e., they are linearly independent). These functions are called patch functions or assumed mode shapes. An approximation of the solution $v(x, t)$ of (2.1) can be formed in $H_N$ by

$$v_N(x, t) = \sum_{k=1}^N v_k(t) \psi_k(x) \quad (3.8)$$

i.e., assume separation of time and space variables with all spatial variation lumped into the patch functions $\psi_k(x)$. The choice of the coefficients $v_k(t)$ remains; these are obtained by substitution of (3.8) into (2.1):

$$\frac{\partial v_N}{\partial t} = A v_N + B f + E_N \quad (3.9)$$

where $E_N$ is the equation error, and the $v_k(t)$'s are chosen so that

$$P_N(E_N) = 0. \quad (3.10)$$

This is called the Galerkin approximation; when it is carried out with the subspaces $H_N$ described above, it produces (3.8) where the coefficients $v_k(t)$ are given by the entries of the solution vector $v_N(t) = [v_1(t), \ldots, v_N(t)]^T$ for the following system of ordinary differential equations:

$$\bar{\mathbf{M}}_N \ddot{v}_N = \bar{\mathbf{A}}_N v_N + \bar{\mathbf{B}}_N f, \quad (3.11)$$

where $\bar{\mathbf{M}}_N = [(\psi_1, \psi_k)]$, $\bar{\mathbf{A}}_N = [(\psi_1, A \psi_k)]$, and $\bar{\mathbf{B}}_N = [(\psi_1, B f)]$. The matrix $\bar{\mathbf{M}}_N$ is symmetric and positive definite because $\{\psi_k(x)\}_{k=1}^N$ are linearly independent.
Therefore, (3.11) can be solved uniquely for \( v_N(t) \) whenever \( v_N(0) \) is specified, and hence the Galerkin approximation (3.8) is obtained. It is assumed that \( v_N(0) \) is given by the vector of coefficients of

\[
v_N(0) = P_N v_0
\]  

(3.12)

expanded in the basis \( \{\psi_k(x)\}_{k=1}^N \). Note that \( v_N \neq P_N v \); however,

\[
v_N = P_N v_N.
\]  

(3.13)

The approximation (3.8) is called a *semidiscretization* of (2.1) because time \( t \) remains continuous.

It should be noted that to obtain the most analytical benefit from the Galerkin method, the approximation (3.8) should be obtained from the "weak" form of (2.1); however, we omit discussion of this technicality and refer to [10] for further details.

3.2. Feedback Controllers: Direct Model Reduction

The *Galerkin reduced-order model* associated with (2.1) is defined on \( H_N \) and given by

\[
\frac{\partial v_N}{\partial t} = A_N v_N + B_N f, \quad v_N(0) = P_N v_0
\]  

(3.14)

\[
y = C_N v_N,
\]

where \( (A_N, B_N, C_N) \) are defined from (3.9)–(3.10) using (3.13) to be \( A_N = P_N A P_N, \ B_N = P_N B, \) and \( C_N = C P_N. \) Since \( H_N \) is a finite-dimensional subspace, \( (A_N, B_N, C_N) \) may be identified with their matrices in an appropriate basis of \( H_N, \) and (3.14) is equivalent to a lumped parameter, state variable system for which a well-developed feedback control theory exists. The controllability and observability of \( (A_N, B_N, C_N) \) are easily checked. Henceforth, for the direct method \( (A_N, B_N, C_N) \) will be assumed to be stabilizable and detectable.

The *Galerkin Feedback Controller* is based on the ROM (3.14) and defined by

\[
f = \bar{G}_N \delta_N
\]  

(3.15a)

\[
\frac{\partial \delta_N}{\partial t} = A_N \delta_N + B_N f + \bar{K}_N (y - \bar{y})
\]  

(3.15b)

\[
y = C_N \delta_N; \quad \delta_N(0) = 0,
\]  

(3.15c)

where, due to the stabilizability and detectability of the ROM, we can adjust the controller gains \( \bar{G}_N \) and \( \bar{K}_N \) so that \( A_N + B_N \bar{G}_N \) and \( A_N - \bar{K}_N C_N \)
have some stability margin. The controller (3.15) has finite dimension \( N \) (where \( N = \dim H_N \)) and consists of a linear feedback control law and full-order state estimator (full-order in the sense that it is matched to the full-order ROM). Much lower order controllers than (3.15) may be developed, but we will not pursue that here. Note that (3.15) has the form (2.12), where \( \alpha = N \), \( L_{11} = 0 \), \( L_{12} = \bar{G}_N \), \( L_{21} = \bar{K}_N \), and \( L_{22} = A_N + B_N \bar{G}_N - \bar{K}_N C_N \).

We define the estimator error \( e_N = \dot{v}_N - v_N \) and, from (3.14) and (3.15), obtain

\[
\frac{\partial e_N}{\partial t} = (A_N - \bar{K}_N C_N) e_N \tag{3.16}
\]

and

\[
\frac{\partial v_N}{\partial t} = (A_N + B_N \bar{G}_N) v_N + B_N \bar{G}_N e_N. \tag{3.17}
\]

If there were no solution error (i.e., \( v = v_N \)), then (3.16) and (3.17) would be designed with some stability margin. Consequently, the controller (3.15) would stabilize the model (3.14) by design; however, our principal concern in [4] was the closed-loop stability of the actual DPS (2.1) with the controller (3.15) when \( v \neq v_N \), which is the usual case.

3.3. Feedback Controllers: Indirect Model Reduction

In the previous subsection, we outlined the direct approach, where a Galerkin approximation of the open-loop DPS (2.1) is made and a controller (3.15) based on this approximation is designed. The only requirement for doing this is that the ROM \((A_N, B_N, C_N)\) in (3.14) be stabilizable and detectable for each \( N \). Now we present the indirect approach which is to Galerkin approximate the infinite-dimensional controller (2.10)–(2.11).

We rewrite (2.10)–(2.11) as the following:

\[
\dot{f}(t) = G\tilde{v}(t) \tag{3.18a}
\]

\[
\frac{\partial \tilde{v}(t)}{\partial t} = L\tilde{v}(t) + K\psi(t) \tag{3.18b}
\]

\[
\tilde{v}(0) = 0, \tag{3.18c}
\]

where \( L = A + BG - KC \) is a closed operator with domain \( D(L) = D(A) \) due to the fact that \( BG \) and \( KC \) are bounded (finite rank) perturbations of the closed operator \( A \).

The Galerkin approximation of (3.18) is straightforward. We let

\[
\dot{v}_N(t) = \sum_{k=1}^{N} \tilde{v}_k(t) \psi_k \tag{3.19}
\]
where $\psi_k$ are in $H_N$ and consider

$$\frac{\partial \dot{\varphi}_N}{\partial t} = L\dot{\varphi}_N + Ky + \dot{E}_N$$

(3.20)

$$\dot{\varphi}_N(0) = 0.$$  

We choose the coefficients $\dot{\varphi}_k$ so that

$$P_N(\dot{E}_N) = 0.$$  

(3.21)

Although $\dot{\varphi}_N \neq P_N\dot{\varphi}$, we do have

$$\dot{\varphi}_N = P_N\dot{\varphi}_N.$$  

(3.22)

From (3.20) and (3.22), we obtain the Galerkin feedback controller:

$$f(t) = G_N\dot{\varphi}_N(t)$$

(3.23a)

$$\frac{\partial \dot{\varphi}_N(t)}{\partial t} = L_N\dot{\varphi}_N(t) + K_N y(t)$$

(3.23b)

$$\dot{\varphi}_N(0) = 0.$$  

(3.23c)

where $G_N \equiv GP_N$, $L_N \equiv P_N L_P N$, and $K_N \equiv P_N K$. This is also a finite-dimensional controller of the form (2.12) with $\alpha = N$, $L_{11} = 0$, $L_{12} = G_N$, $L_{21} = K_N$, and $L_{22} = L_N$. The difference between this controller (3.23) obtained via the indirect method and the one in (3.15) obtained by the direct method is the way the gains are obtained; the ones in (3.23) come from the infinite-dimensional stabilizing controller (3.18) [or (2.10)–(2.11)], but the ones in (3.15) are calculated directly from the ROM $(A_N, B_N, C_N)$ at each $N$.

Consider the closed-loop systems (2.1) and (3.15):

$$\frac{\partial v}{\partial t} = Av + B\hat{G}_N\dot{\varphi}_N$$

(3.24a)

$$\frac{\partial \dot{\varphi}_N}{\partial t} = \bar{K}_N Cv + \bar{L}_N\dot{\varphi}_N,$$  

(3.24b)

where $\bar{L}_N \equiv A_N + B_N\hat{G}_N - \bar{K}_N C_N$ and (2.1) and (3.23):

$$\frac{\partial \hat{\varphi}}{\partial t} = Av + BG_N\dot{\varphi}_N$$

(3.25a)

$$\frac{\partial \dot{\varphi}_N}{\partial t} = K_NCv + L_N\dot{\varphi}_N.$$  

(3.25b)
We define the operators $\tilde{A}_N$ and $\bar{A}_N$ by

$$
\tilde{A}_N = \begin{bmatrix}
    A & B \tilde{G}_N \\
    \tilde{K}_N C & \tilde{L}_N
\end{bmatrix}
$$

(3.26)

and

$$
\bar{A}_N = \begin{bmatrix}
    A & B G_N \\
    K_N C & L_N
\end{bmatrix}.
$$

(3.27)

We say that the controllers (3.15) and (3.23) are equivalent if both exist and exponential stability of either (3.26) or (3.27) implies it for both. This is the case under the conditions given in [5].

3.4. Closed-Loop Convergence of the Indirect Model Reduction Method

In [4, 5] we developed convergence and closed-loop stability results for the direct and indirect model reduction approaches with Galerkin's method. With the indirect approach there will actually be a limiting stabilizing controller which is approached as our approximation improves. In this section, we review and extend the convergence results of [5].

In [4], we made the following two assumptions about the Galerkin method:

$$
P_N \xrightarrow{s} I \text{ on } D(A) \text{ or } H \tag{3.28a}
$$

$$
A_{RN} = P_R A P_N \xrightarrow{s} 0 \text{ on } H \tag{3.28b}
$$

as $N \to \infty$. We have discussed (3.28a) already in Section 3.1; however, we note that, since $\|P_N\| = 1$ by orthogonality, we can assume (3.28a) is true on $H$, even though defined only on $D(A)$, due to [9, Lemma 3.5, p. 151]. Assumption (3.28b) says that the Galerkin method consistently approximates the operator $A$; note that $A_{RN}$ is defined on all of $H$ since $H_N \subseteq D(A)$. These assumptions are typical of the ones made whenever the Galerkin method is used to approximate partial differential equations; they yield the following convergence result for the open-loop DPS:

**Theorem 3.1.** If $v_0$ is in $D(A)$ and (3.28) holds, then, for any $t$ such that $0 \leq t \leq T < \infty$,

$$
\lim_{N \to \infty} \| \tilde{e}_N(t) \| = 0, \tag{3.29a}
$$

where

$$
\tilde{e}_N(t) \equiv v(t) - v_N(t) \tag{3.29b}
$$
which is the error between the actual solution of the open-loop DPS (2.1) and its Galerkin approximation (3.8) and (3.14).

The proof in [4] uses the fact that $A_{RN}$ is uniformly bounded due to (3.28b) and the uniform boundedness principle [9, pp. 136–137]. Also, note that (3.28b) does not necessarily imply

$$A_N \xrightarrow{s} A$$ on $D(A)$ \hspace{1cm} (3.30a)

which is equivalent to

$$A_{NR} \xrightarrow{s} 0$$ on $D(A)$ \hspace{1cm} (3.30b)

due to the fact that

$$A_N - A = AP_R + A_{RN} = P_RA + A_{NR}. \hspace{1cm} (3.31)$$

The difficulty arises because (3.28a) guarantees $P_RA \xrightarrow{s} 0$ on $D(A)$ but not necessarily the same for $AP_R$ since $A$ is closed (not bounded) and $AP_R$ is defined only on $D(A)$.

Now we will change the hypothesis (3.28) somewhat. In the rest of this paper, we assume the following open-loop convergence hypothesis for the Galerkin method on the DPS (2.1):

$$P_N \xrightarrow{s} I$$ on $D(A)$ on $H$ \hspace{1cm} (3.32a)

$$A_{NR} \equiv P_NAP_R \xrightarrow{s} 0$$ on $D(A)$ \hspace{1cm} (3.32b)

$$\|U_N(t)v\| \leq Ke^{\beta t}\|v\|$$ for $v$ in $D(A)$ \hspace{1cm} (3.32c)

where $(K, \beta)$ are real constants independent of $N$ and $v$. Of course, (3.32a) is the same as (3.28a), but the other two parts are different; they alter the consistency requirements of the Galerkin method. However, they are not terribly strange since (3.32b) implies (3.30a) as seen before, and (3.32c) is a condition on the "numerical stability" of the open-loop approximation; (3.32c) would not be necessary if (3.32b) held on all of $H$, but this seems unlikely for most Galerkin schemes.

In (3.32c), the meaning of $U_N(t)$ is that it is the $C_0$-semigroup generated on $H_N$ by the bounded operator $A_N \equiv P_NAP_N$ restricted to $H_N$. However, in what follows we would like to speak of $U_N(t)$ on all of $H$; therefore, we define

$$U_N(t) \equiv P_R$$ on $H_R \equiv H_N^+ \hspace{1cm} (3.33a)$$

and thus obtain

$$P_NU_N(t)P_N = U_N(t). \hspace{1cm} (3.33b)$$
The open-loop convergence condition yields the following convergence result.

**Theorem 3.2.** When (3.32) holds and \( v_0 \) is in \( D(A) \), we have, for any \( 0 \leq t \leq T < \infty \),

\[
\lim_{N \to \infty} \| \bar{e}_N(t) \| = 0 \tag{3.34a}
\]

where

\[
\bar{e}_N(t) \equiv P_N v(t) - v_N(t). \tag{3.34b}
\]

Also, (3.29) holds.

The proof is given in Appendix I. This result shows that the Galerkin schemes satisfying (3.32) converge to the open-loop DPS (2.1) solutions.

Next we want to show that (3.32) also implies the closed-loop system converges as well. We shall need the following version of the Trotter–Kato Theorem.

**Theorem 3.3.** Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of closed operators defined on \( D(A) \) dense in \( H \) with "generalized" limit \( A \) (also closed). I.e.,

\[
R(\lambda, A_n) \xrightarrow{s} R(\lambda, A) \tag{3.35}
\]

for some \( \lambda > \beta \), and \( A_n, A \) generate \( C_0 \)-semigroups \( U_n(t), U(t) \), respectively, satisfying for each \( n \):

\[
\max(\| U_n(t) \|, \| U(t) \|) \leq Ke^{\beta t}, \quad t \geq 0 \tag{3.36}
\]

with \( K \geq 1 \) and \( \beta \) real (both constants independent of \( n \)). Then

\[
U_n(t) \xrightarrow{s} U(t) \tag{3.37}
\]

uniformly on any finite interval of \( t \geq 0 \).

The proof of this is given in [9, Theorem 2.16, p. 502]. Note that \( \beta \) need not be negative, but (3.36) does require a uniform exponential bound on \( \| U_n(t) \| \) that is independent of \( n \).

The following result gives conditions under which the indirect method (2.1) and (3.23) converge to the closed-loop system (2.1) and (3.18) which is stable.

**Theorem 3.4.** Assume (3.32) and \((A, B, C)\) stabilizable and detectable. If \( \bar{U}_N(t) \) is the \( C_0 \)-semigroup generated by \( \bar{A}_N \) in (3.27), then it is uniformly exponentially bounded, i.e.,

\[
\| \bar{U}_N(t) \| \leq K e^{\beta t}, \quad t \geq 0, \tag{3.28}
\]
where $\bar{K} \geq 1$ and $\beta$ real (both independent of $N$) and $\bar{U}_N(t) \to \bar{U}(t)$ uniformly on any finite interval of $t \geq 0$, where $\bar{U}(t)$ is the $C_0$-semigroup generated by

$$\bar{A} = \begin{bmatrix} A & BG \\ KC & L \end{bmatrix},$$

(3.39)

where $A + BG$ and $A - KC$ are (exponentially) stable.

This result shows that the closed-loop solutions with the approximated controller converge to the closed-loop solutions with the infinite-dimensional controller.

The proof of the above uses Theorem 3.3 and is given in Appendix II; it is essentially the same as Theorem 4.2 in [5] but is given here for completeness. It depends on noting the following:

$$\bar{A}_N = \bar{P}_N \bar{A} \bar{P}_N \to \bar{A}, \text{ as } N \to \infty,$$

(3.40)

where

$$\bar{P}_N = \begin{bmatrix} I & 0 \\ 0 & P_N \end{bmatrix},$$

(3.41)

is an orthogonal projection on the Hilbert space $\bar{H} = H \times H$ with corresponding complementary projection:

$$\bar{P}_R = \begin{bmatrix} 0 & 0 \\ 0 & P_R \end{bmatrix}.\]

(3.42)

From (3.32a), we have as $N \to \infty$:

$$\bar{P}_N \to I \quad (3.43a)$$

$$\bar{P}_R \to 0. \quad (3.43b)$$

Also, it is clear that $\bar{A}$ generates $\bar{U}(t)$ exponentially stable because

$$\bar{Q}^{-1} \bar{A} \bar{Q} = \begin{bmatrix} A + BG & BG \\ 0 & A - KC \end{bmatrix}$$

is (exp.) stable by the choice of the infinite-dimensional controller gains $G$ and $K$, where

$$\bar{Q} = \begin{bmatrix} I & 0 \\ +I & I \end{bmatrix} \quad \text{and} \quad \bar{Q}^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}.\]

(3.45)

Thus,

$$\| \bar{U}(t) \| \leq \bar{K} e^{-\bar{\sigma}t}, t \geq 0$$

(3.46)

where $\bar{\sigma} > 0$ and we use the same $\bar{K}$ as in (3.38) without loss of generality.
Note that (3.38) does not require uniform exponential stability for \( \tilde{A}_N \). Yet, we are most interested in the question: When does \( \tilde{A}_N \) become exponentially stable for sufficiently large \( N \)? The results of Theorem 3.4 cannot answer this question because they are only valid on finite intervals of \( t \geq 0 \). This question of stability for \( N \) sufficiently large is addressed in the next section.

### 4.0. Main Results: Closed-Loop Stability

In [5], we defined \( \tilde{e}_N = \hat{e}_N - P_N \hat{v} \), the error between the approximate controller's estimate and the projection of the infinite-dimensional controller's estimate of the state of the DPS (2.1). The closed-loop system (2.1) and (3.23) can be written (for \( v_0 \) in \( D(A) \)):

\[
\frac{\partial \omega(t)}{\partial t} = \tilde{A}_N \omega(t),
\]

where \( \omega(t) \equiv \begin{bmatrix} v(t) \\ \hat{v}(t) \\ \hat{e}_N(t) \end{bmatrix} \)

and

\[
\tilde{A}_N \equiv \begin{bmatrix}
A & B_{G_N} & B_{G_N} \\
Kc & L & 0 \\
0 & -L_{NR} & L_N
\end{bmatrix}.
\]

The following result gives conditions under which the closed-loop system consisting of (2.1) and (3.23) are stable for sufficiently large \( N \):

**THEOREM 4.1.** Assume (3.32a) and \((A, B, C)\) stabilizable and detectable (with \( A + B_{G} \) and \( A - KC \) stable). If

(a) \( \|A_N v\| \leq \alpha_N \|v\| \) for all \( v \) in \( D(A) \) with \( \lim_{N} \alpha_N = 0 \). \hfill (4.2)

(b) \( L_N \) uniformly exponentially stable for \( N \) sufficiently large, i.e. \( \| V_N(t) \| \leq K_0 e^{-\sigma_0 t}, \ t \geq 0 \). \hfill (4.3)

where \( V_N(t) \) is the \( C_0 \)-semigroup generated by \( L_N \) and \( K_0 \geq 1, \ \sigma_0 > 0 \) (independent on \( N \)), then, for \( N \) sufficiently large, the \( C_0 \)-semigroup \( \tilde{U}_N(t) \) generated by \( \tilde{A}_N \) in (4.1) is uniformly exponentially stable, i.e.,

\[
\| \tilde{U}_N(t) \| \leq K e^{-\tilde{\sigma}_N t}, \quad t \geq 0,
\]
where $\tilde{K} \geq 1$ (independent of $N$), $\tilde{\sigma}_N \leq \tilde{\sigma}$, and $\lim_{N} \tilde{\sigma}_N = \tilde{\sigma} = \min (\tilde{\sigma}, \sigma_0)$ with $\tilde{\sigma}$ and $\sigma_0$ given in (3.46) and (4.3), respectively.

The proof is given in [5]. Note that $A_{NR} \equiv P_N A P_R$ is a bounded operator on $D(A)$ for each $N$ because $A P_R$ is closed on $D(A) \cap H_R$ and $P_N$ has finite rank (see [9, p. 166]). However, (4.2) says that those bounds $\alpha_N$ converge to zero; this is a very uniform consistency requirement for the Galerkin method on the differential operator $A$ in (2.1). Also, note that the above result says that $\delta_N \rightarrow \delta$ for the controller (3.23).

If reducing (modal) subspaces for $A$ are used for the Galerkin method, then $A_{NR} = 0$; hence, (a) is satisfied. Thus, we have closed-loop stability for $N$ large when (b) is satisfied. However, note that for reducing subspaces: $A P_N = P_N A$; also, $P_N$ commute with the $C_0$-semigroup generated by $A$. Suppose $(H_N, H_R)$ reducing subspaces are also stabilizing subspaces for $(A, B)$ and $(A^*, C^*)$, then for (b) to hold we need that $L = A + B G - K C$ is exponentially stable, i.e., we have a stable, infinite-dimensional controller, and $P_N$ commutes with $B G - K C$.

In an effort to remove the hypothesis (b) in Theorem 4.1 of uniform stability for the controller alone, we introduce the error term: $e_N \equiv \delta_N - P_N v$. When $v_0$ is in $D(A)$, we have the closed-loop system (2.1) and (3.23) given by

\[
\frac{\partial v(t)}{\partial t} = (A + B G_N) v(t) + B G_N e_N(t) \tag{4.5a}
\]

\[
\frac{\partial e_N(t)}{\partial t} = A_{NR} v(t) + (A_N - K_N C_N) e_N(t) \tag{4.5b}
\]

with $v(0) = v_0$ and $e_N(0) = -P_N v_0$ and $\Delta_{NR} = K_N C_R - A_{NR}$. This can be written as

\[
\frac{\partial \omega(t)}{\partial t} = A_N \omega(t), \tag{4.6}
\]

where

\[
\omega(t) = \begin{bmatrix} v(t) \\ e_N(t) \end{bmatrix} \quad \text{in} \quad \bar{H} = H \times H
\]

and

\[
A_N = \begin{bmatrix} A + B G_N & B G_N \\ A_{NR} & A_N - K_N C_N \end{bmatrix}
\]

define on $D(A) \times D(A)$. 
The following is our main result on closed-loop stability:

**Theorem 4.2.** Assume the open-loop convergence hypothesis (3.32) and the uniform consistency (4.2) hold. If there exists \( N^* \) such that the \( C_0 \)-semigroup \( \hat{U}_N(t) \) generated by \( A_N - K_N C_N = P_N (A - KC) P_N \) restricted to \( H_N \) is uniformly exponentially stable, i.e., for all \( N \geq N^* \),

\[
\| U_N(t) \| \leq \hat{K} e^{-\hat{\sigma} t},
\]

where \( \hat{K} \geq 1 \) and \( \hat{\sigma} > 0 \) do not depend on \( N \); then for \( u_0 \) in \( D(A) \) the closed-loop system (2.1) and (3.23), or equivalently (4.5), is exponentially stable for all \( N \) sufficiently large.

This result shows that a finite-dimensional stabilizing controller exists by Galerkin approximating the stabilizing infinite-dimensional controller; the Galerkin schemes that will work must satisfy (3.32), (4.2), and (4.7). Note again that modal or reducing subspaces eliminate (4.2). Furthermore, (4.7) does not require that the controller alone become uniformly stable. For modal subspaces we have the following corollary.

**Theorem 4.3.** If modal subspaces \( (H_N, H_R) \), i.e., \( A_{NR} = 0 \) and \( A_{RN} = 0 \), are used in the Galerkin scheme such that (3.28a) holds, then the conclusions of Theorem 4.2 hold whenever, for some \( N \), there exist stabilizing modal subspaces for \( (A^*, C^*) \), i.e., \( A - K N C \) generates an exponentially stable \( C_0 \)-semigroup for \( K_N \equiv P_N K = K \) (or \( P_R K = 0 \)).

The proofs of Theorems 4.2 and 4.3 are given in Appendix III. The latter uses the fact that, since \( H_N \subseteq H_{N+1} \) for Galerkin schemes,

\[
P_{N+1} P_N = P_N.
\]

### 5.0. Special Case: Modal Control

In [1], Sakawa developed modal control for a generalized heat equation with a finite number of unstable modes and showed that a finite-dimensional controller of the form

\[
f(t) = G_1 Z_1(t)
\]

\[
Z_1(t) = (A_1 - K_1 C_1) Z_1(t) + K_1 (y(t) - C_2 Z_2(t)) + B_1 f(t)
\]

\[
Z_2(t) = A_2 Z_2(t) + B_2 f(t)
\]

could stabilize the DPS. The dim \( Z_1 \) equaled the number of unstable modes, which were assumed controllable and observable, and the dim \( Z_2 \) was allowed to increase until it was sufficiently large to produce closed-loop stability. In this section we show that such a result is a special case of...
Theorem 4.3 and need not be restricted to generalized heat equations, i.e., $A$ in (2.1) symmetric with compact resolvent.

Henceforth, assume that $A$ has compact resolvent and the corresponding modal subspaces $H_N$ form an increasing sequence of subspaces ($H_N \subseteq H_{N+1}$) with the property (3.32a), i.e.,

$$ P_N \xrightarrow{s} I, $$(5.1)

where $P_N$ is orthogonal projection onto $H_N$; in particular, this is true when $A$ is also normal, (i.e., commutes with $A^*$) as shown in [9, p. 277]. Furthermore, the $H_N$ subspaces reduce $A$ in the sense that for all $N$.

$$ A_{NR} = 0 \quad \text{and} \quad A_{RN} = 0 $$ (5.2)

and the corresponding projections $P_N$ satisfy

$$ P_k P_l = P_l P_k = P_k \quad \text{for} \quad l \geq k. $$ (5.3)

In this modal case the projections commute with the resolvent operator for $A$ and hence the $C_0$-semigroup $U(t)$ generated by $A$, i.e., $A_N \equiv P_N A P_N$ generates

$$ U_N(t) = P_N U(t) P_N. $$ (5.4)

Therefore, $\| U_N(t) \| \leq \| U(t) \|$ since the $P_N$ are orthogonal projections; consequently the open-loop convergence hypothesis (3.32) is satisfied.

Suppose that there are only a finite-number of instabilities in the spectrum of $A$, i.e., the eigenvalues of $A_{N_0} \equiv P_{N_0} A P_{N_0}$ are the unstable or closed right half-plane (point) spectrum of $A$, where $H_{N_0}$ is the eigenspace associated with these instabilities. We assume $(A, B, C)$ stabilizable and detectable which is the same as $(A_{N_0}, B_{N_0}, C_{N_0})$ controllable and observable because $(H_{N_0}, H_{R_0} \equiv H_{N_0}^\perp)$ form a pair of stabilizing subspaces for $(A, B)$ and $(A^*, C^*)$. In fact, we can choose $(G_{N_0}, K_{N_0})$ such that

$$ G_{N_0} P_N = G_{N_0} \quad \text{and} \quad P_N K_{N_0} = K_{N_0} \quad \text{for} \quad N \geq N_0 $$ (5.5a)

and

$$ A_{N_0} + B_{N_0} G_{N_0} \quad \text{and} \quad A_{N_0} - K_{N_0} C_{N_0} \text{ are stable} $$ (5.5b)

and this leads to the choice

$$ G \equiv G_{N_0} \quad \text{and} \quad K \equiv K_{N_0} $$ (5.6)

for the infinite-dimensional controller (3.18).

Also, note that the spectrum of $A_{R_0} \equiv P_{R_0} A P_{R_0}$, where $P_{R_0} \equiv I - P_{N_0}$, is all point spectrum and is contained entirely in the open left-half plane.
Thus, since $A$ has compact resolvent, the \textit{spectrum determined growth condition} applies (e.g. [11]), and $A_{R_0}$ generates the $C_0$-semigroup $U_{R_0}(t) = P_{R_0} U(t) P_{R_0}$ with
\[
\|U_{R_0}(t)\| \leq K_0 e^{-\sigma_0 t},
\]
where $K_0 \geq 1$ and $\sigma_0 > 1$. Therefore, take $N = N_0$ in Theorem 4.3, and consider
\[
A - K_{N_0} C = \begin{bmatrix}
A_{N_0} - K_{N_0} C & -K_{N_0} C_{R_0} \\
0 & A_{R_0}
\end{bmatrix}
\]
which is exponentially stable due to (5.5)-(5.7); so, there must exist $N$ sufficiently large that the controller (3.23) stabilizes the closed-loop.

Take $Z_1 \equiv P_{N_0} \hat{v}_N$ and $Z_2 \equiv (P_N - P_{N_0}) \hat{v}_N$ and consider (3.23) for $N \geq N_0$; using (5.3) and (5.5)-(5.6), we have
\[
f = G_N \hat{v}_N = GP_N \hat{v}_N = G_{N_0} \hat{v}_N = G_{N_0} [P_{N_0} \hat{v}_N] = G_{N_0} Z_1
\]
(5.8a)
\[
\frac{\partial Z_1}{\partial t} = \frac{\partial P_{N_0} v_N}{\partial t} = P_{N_0} L_N (P_{N_0} \hat{v}_N + P_{R_0} \hat{v}_N) + P_{N_0} K_N y(t)
\]
\[
= P_{N_0} L_N P_{N_0} Z_1 + P_{N_0} L_N P_{R_0} \hat{v}_N + P_{N_0} P_N K_{N_0} y
\]
\[
= L_{N_0} Z_1 + K_{N_0} y + P_{N_0} L Z_2
\]
\[
= L_{N_0} Z_1 + K_{N_0} y + P_{N_0} L P_{R_0} Z_2
\]
\[
= (A_{N_0} + B_{N_0} G_{N_0} - K_{N_0} C_{N_0}) Z_1 + K_{N_0} y - K_{N_0} C_{R_0} Z_2
\]
\[
= (A_{N_0} + B_{N_0} G_{N_0} - K_{N_0} C_{N_0}) Z_1 + K_{N_0} y - K_{N_0} C (P_N - P_{N_0}) Z_2
\]
\[
\therefore \frac{\partial Z_1}{\partial t} = (A_{N_0} - K_{N_0} C_{N_0}) Z_1 + K_{N_0} (y - C (P_N - P_{N_0}) Z_2) + B_{N_0} f.
\]
(5.8b)

Also, since $Z_2 = P_{R_0} Z_2$, we have
\[
\frac{\partial Z_2}{\partial t} = P_{R_0} \frac{\partial Z_2}{\partial t} = P_{R_0} (P_N - P_{N_0}) [L_N \hat{v}_N + K_N y]
\]
\[
= P_{R_0} (P_N - P_{N_0}) L_N (Z_1 + P_{R_0} \hat{v}_N) + P_{R_0} (P_N - P_{N_0}) K_{N_0} y
\]
\[
= (P_N - P_{N_0}) P_{R_0} [L (P_N Z_1 + P_N P_{R_0} \hat{v}_N)] + (P_N - P_{N_0}) P_{R_0} K_{N_0} y
\]
\[
= (P_N - P_{N_0}) [L_{R_0} N_0 Z_1 + L_{R_0} Z_2] + O y
\]
\[
= (P_N - P_{N_0}) [B_{R_0} G_{N_0} Z_1 + A_{R_0} Z_2]
\]
\[
\therefore \frac{\partial Z_2}{\partial t} = (P_N - P_{N_0}) B f + (P_N - P_{N_0}) A (P_N - P_{N_0}) Z_2.
\]
(5.8c)
In the above we have used the following identities which are easily obtained from (5.3):

\[(P_N - P_{N_0})^2 = P_N - P_{N_0}\]  
\[(P_N - P_{N_0})P_{R_0} = P_{R_0}(P_N - P_{N_0}) = P_N - P_{N_0}\]

i.e., \(P_N - P_{N_0}\) is an orthogonal projection.

Now, if we identify \(A_1 = P_{N_0}A_{P_{N_0}}, K_1 = K_{N_0}, C_1 = CP_{N_0}, G_1 = G_{N_0}, C_2 = C(P_N - P_{N_0}), B_1 = P_{N_0}B, A_2 = (P_N - P_{N_0})A(P_N - P_{N_0}),\) and \(B_2 = (P_N - P_{N_0})B,\) then (5.8) is the same as (5.1) which is Sakawa's controller. But, (5.8) is just the modal Galerkin approximation of the infinite-dimensional stabilizing controller for (2.1); hence, by Theorem 4.3, for \(N\) sufficiently large, (5.8) produces exponential closed-loop stability. Consequently, this includes the result of [1]. Note that the controller (5.8) is not just the \(N_0\)-dimensional modal controller for the unstable system \((A_{N_0}, B_{N_0}, C_{N_0});\) it has an additional \(N - N_0\) state, where \(N\) may turn out to be quite large (even though finite) in some applications.

APPENDIX I: PROOF OF THEOREM 3.2

Since \(v_0 \in D(A),\) we have \(\tilde{e}_N\) differentiable and

\[\frac{\partial \tilde{e}_N}{\partial t} = P_N(Av + Bf) - (A_Nv_N + B_Nf)\]

\[= P_NAv - A_Nv_N\]

\[= (P_NAP_N)P_Nv + (P_NAP_{R_0})v - A_Nv_N\]

\[\begin{align*}
\therefore \frac{\partial \tilde{e}_N}{\partial t} &= A_N\tilde{e}_N + A_{NR}v. 
\end{align*}\]

(A.I.1)

Also,

\[\tilde{e}_N(0) = P_Nv_0 - v_N(0) = P_Nv_0 - P_Nv_0 = 0.\]  
(A.I.2)

\[\therefore \tilde{e}_N(t) = U_N(t)\tilde{e}_N(0) + \int_0^t U_N(t - \tau) A_{NR}v(\tau) d\tau\]

\[= \int_0^t U_N(t - \tau) A_{NR}v(\tau) d\tau.\]

Fix \(t\) in \([0, T]\) and obtain from (3.32c)

\[\|\tilde{e}_N(t)\| \leq Ke^{\beta T} \int_0^T e^{-\beta \tau} \|A_{NR}v(\tau)\| d\tau\]
Since \( v_0 \in D(A) \), we have \( v(\tau) \in D(A) \) for all \( 0 \leq \tau \leq T \) and so, from (3.32b),
\[
\lim_{N} \| \tilde{e}_N(t) \| \leq K e^{\beta T} \int_{0}^{T} e^{-\beta T} \lim_{N} \| A_{NR} v(\tau) \| \, d\tau
\]
\[
= 0 \quad \text{as desired.}
\]
Furthermore,
\[
\| \tilde{e}_N(t) \| \leq \| v(t) - P_N v(t) \| + \| \tilde{e}_N(t) \|
\]
\[
= \| P_R v(t) \| + \| \tilde{e}_N(t) \|. \quad (A.I.3)
\]
This yields (3.29) due to (3.32a).

**APPENDIX II: PROOF OF THEOREM 3.4**

Since
\[
\tilde{A}_N = \begin{bmatrix} A & BG_N \\ K_N C & L_N \end{bmatrix} = \bar{P}_N \bar{A} \bar{P}_N,
\]
we can rewrite it as
\[
\tilde{A}_N = \tilde{A} + \tilde{A}_N, \quad (A.II.1)
\]
where
\[
\tilde{A}_N = \begin{bmatrix} 0 & B(G_N - G) \\ (K_N - K) C & L_N - L \end{bmatrix}.
\]
From (3.32a and b), we have \( A_N \to^s A \) on \( D(A) \); therefore,
\[
\tilde{A}_N \to^s 0 \text{ on } D(A) \quad (A.II.2)
\]
because \( G_N \equiv G P_N \to^s G \) and \( K_N \equiv P_N K \to^s K \). Consequently,
\[
\tilde{A}_N \to^s \tilde{A} \text{ on } D(\tilde{A}) = D(A) \times D(A) \quad (A.II.3)
\]
Furthermore, since both \( B \) and \( C \) are bounded and have finite rank, as \( N \to \infty \),
\[
B_N G_N \to B G; \quad B G_N \to B G \quad (A.II.4)
\]
\[
K_N C_N \to K C; \quad K_N C \to K C,
\]
where the convergence is uniform. Now, we write

$$\bar{A}_N = \begin{bmatrix} A & 0 \\ 0 & \bar{A}_N \end{bmatrix} + \bar{W}_N,$$

(A.II.5)

where, from (A.II.4), we have

$$\bar{W}_N \equiv \begin{bmatrix} 0 & BG_N \\ K_NC & B_NC - K_NC_N \end{bmatrix} \to \bar{W} = \begin{bmatrix} 0 & BG \\ KC & BG - KC \end{bmatrix},$$

(A.II.6)

uniformly as \(N \to \infty\). From (3.32c) (which can be extended to all of \(H\) since \(D(A)\) is dense in \(H\)) (2.6) and [12 Theorem 10.9], we have \(A_N\) generates \(\bar{U}_N(t)\) such that

$$\|\bar{U}_N(t)\| \leq K e^{\beta t},$$

(A.II.7)

where \(\beta_N \equiv \max (-\sigma_0, \beta) + \bar{K} \|\bar{W}_N\| \leq \beta\) with \(\beta = \max (-\sigma_0, \beta) + \bar{K} \bar{M}\) due to (A.II.6) which yields \(\|\bar{W}_N\| \leq \bar{M}\) for all \(N\). Thus, (3.38) holds with \((\bar{K}, \beta)\) real constants independent of \(N\).

For any \(I \in \rho(\bar{A}_N) \cap \rho(\bar{A})\):

$$R(\lambda, \bar{A}_N) - R(\lambda, \bar{A}) = R(\lambda, \bar{A}_N)[\bar{A}_N - \bar{A}] R(\lambda, \bar{A})$$

(A.II.8)

which is obtained by multiplying the above on the left by \(\lambda I - \bar{A}_N\) and on the right by \(\lambda I - \bar{A}\). From the previous discussion, \(\beta > -\dot{\sigma}\) in (3.38) and (3.46) and we choose \(\lambda\) real with \(\lambda > \beta\), then \(\lambda\) is in \(\rho(\bar{A}_N) \cap \rho(\bar{A})\). From (3.38) and the Hille–Yosida theorem (see (2.6)–(2.7)):

$$\|R(\lambda, \bar{A}_N)\| \leq K/(\lambda - \beta).$$

(A.II.9)

Therefore, due to (A.II.8) and (A.II.9),

$$\| [R(\lambda, \bar{A}_N) \ - \ R(\lambda, \bar{A})] \| \leq \frac{K}{(\lambda - \beta)} \| (\bar{A}_N - \bar{A}) \omega \|,$$

(A.II.10)

where \(\nu\) is in \(\bar{H}\) and \(\omega \equiv R(\lambda, \bar{A}) \nu\). Because \(\omega\) is in \(D(\bar{A})\) and \(A_N \to \bar{A}\) there (A.II.3), this gives us: \(R(\lambda, \bar{A}_N) \to \bar{R}(\lambda, \bar{A})\). Also, since \(\beta > -\dot{\sigma}\), \(\|\bar{U}(t)\| \leq K e^{-\sigma t} \leq K e^{\beta t}\). Thus, the hypotheses of Theorem 3.3 are satisfied and the desired result holds.

**APPENDIX III: PROOFS OF THEOREMS 4.2 AND 4.3**

**THEOREM 4.2.** From (4.6), we write

$$\bar{A}_N = A_N^0 + A_N^0,$$

(A.III.1)
where

\[ A_N^0 = \begin{bmatrix} A + BG & BG \\ 0 & P_N(A - KC) P_N \end{bmatrix} \quad \text{and} \quad A_N^0 = \begin{bmatrix} BG(I - P_N) & BG(I - P_N) \\ A_{NR} & 0 \end{bmatrix} \]

with \( A_{NR} = P_N KC(I - P_N) - A_{NR} \). From (A.II.4) and (3.32a) we have

\[ \begin{align*}
BG(I - P_N) & \to 0 \\
\lim_{N \to \infty} P_N KC(I - P_N) & \to 0,
\end{align*} \]

(A.III.2)

where convergence is uniform on \( H \). Also, from (4.2), which can be extended to all of \( H \) due to the \( D(A) \) being dense, we have

\[ A_{NR} \to 0 \]

(A.III.3)

uniformly on \( H \).

Therefore, (A.III.2)–(A.III.3) imply

\[ \lim_{N \to \infty} \| A_N^0 \| = 0. \]

(A.III.4)

Since \( N^* \) exists such that, for all \( N \geq N^* \), we have (4.7), choose \( N \geq N^* \) and use the fact that \( A + BG \) is stable to obtain the \( C_0 \)-semigroup \( U_N^0(t) \) generated by \( A_N^0 \) satisfies

\[ \| U_N^0(t) \| \leq K_0 e^{-\sigma_0 t}, \]

(A.III.5)

where \( K_0 \geq 1, \sigma_0 > 0 \) are independent of \( N \). From [12 Thereom 10.9], the \( C_0 \)-semigroup \( \bar{U}_N(t) \) generated by \( \bar{A}_N \) must satisfy

\[ \| \bar{U}_N(t) \| \leq K_0 e^{-\tilde{\sigma}_N t}, \]

(A.III.6)

where \( \tilde{\sigma}_N = \sigma_0 - K_0 \| A_N^0 \| \). From (A.III.4), we can find \( \tilde{N} \) such that, for all \( N \geq \tilde{N}, \tilde{\sigma}_N > 0 \) (i.e., \( \| A_N^0 \| < \sigma_0 / K_0 \)); hence, the closed-loop system is exp. stable and Theorem 4.2 is proved.

**Theorem 4.3.** When modal subspaces are used, (4.2) is automatically satisfied since \( A_{NR} = 0 \). Also, \( A_N = P_N AP_N \) generates the \( C_0 \)-semigroup \( U_N(t) = P_N U(t) P_N \) because \( P_N \) and \( R(z, A) \) and, hence, with \( U(t) \); consequently, since \( P_N \) is orthogonal, we have

\[ \| U_N(t) \| = \| P_N U(t) P_N \| \leq \| P_N \|^2 \| U(t) \| = \| U(t) \|. \]
From (2.6) and the above, we have
\[ \| U_N(t) \| \leq Ke^{\sigma t} \]
which satisfies (3.32c); thus, (3.32) is satisfied when (3.32a) alone is satisfied in the modal case.

Suppose, for some \( \overline{N} \), there exist modal stabilizing subspaces for \((A^*, C^*)\), then \( A - K_R C \) generates an exp. stable \( C_0\)-semigroup with \( K_R \equiv P_R K = K \). Let \( N \geq \overline{N} \), then \( K_N = P_N K = P_N P_R K = P_R K = \overline{K}_N = K \) since \( H_R \subseteq H_N \). Moreover, for all \( N \geq \overline{N} \), \( A - K_N C = A - K_R C \) is exp. stable. But

\[
A - K_N C = \begin{bmatrix} P_N(A - K_N C) & P_N(A - K_N C) P_R \\ P_R(A - K_N C) & P_R(A - K_N C) P_R \end{bmatrix}
\]

\[
= \begin{bmatrix} P_N(A - K_C) & P_N - K_N C_R \\ 0 & P_R A P_R \end{bmatrix}.
\]

Therefore, \( P_N(A - K_C) P_N \) (and \( P_R A P_R \)) must be stable for all \( N \geq \overline{N} \). Furthermore, \( A - K_C C \) generates the \( C_0\)-semigroup \( \bar{U}_R(t) \) such that
\[
\| \bar{U}_R(t) \| \leq \bar{K}_R e^{-\sigma_R t}
\]
where \( \bar{K}_R \geq 1 \) and \( \sigma_R > 0 \). But \( A - K_C C = A - K_R C \); so, the semigroup is the same for all \( N \geq \overline{N} \) and is bounded by (A.III.8). Now, we write

\[
\begin{bmatrix} P_N(A - K_C) & 0 \\ 0 & P_R A P_R \end{bmatrix} = (A - K_C) + \begin{bmatrix} 0 & K_N C_R \\ 0 & 0 \end{bmatrix}
\]

\[
= (A - K_R C) + \begin{bmatrix} 0 & K_N C_R \\ 0 & 0 \end{bmatrix}.
\]

Note, the second term in (A.III.9) converges uniformly to zero as \( N \to \infty \). Thus, if we define \( \bar{U}_N(t) \) and \( \bar{U}_R(t) \) as the \( C_0\)-semigroups generated by \( P_N(A - K_C) P_N \) and \( P_R A P_R \) respectively, then we have (using A.III.8):
\[
\max ( \| \bar{U}_N(t) \|, \| \bar{U}_R(t) \| ) \leq \bar{K}_R e^{-\sigma_R t}
\]
where \( \bar{\sigma}_N \equiv \sigma_R - \bar{K}_R \| K_RC_R \| \) due to [12, Theorem 10.9]. However, since \( \lim_{N \to \infty} \| K_RC_R \| = 0 \), we can choose \( N^* (N^* \geq \overline{N}) \) sufficiently large that, for all \( N \geq N^* \),
\[
\bar{\sigma}_N \geq \sigma_R/2 > 0
\]
and, from (A.III.10),
\[
\| \bar{U}_N(t) \| \leq K_R e^{-\sigma_R/2 t}
\]
where \(K_R\) and \(\sigma_R\) are independent of \(N\). This means we could take \(\tilde{K} = K_R\) and \(\tilde{\sigma} = \sigma_R/2\) in (4.7). Therefore, we have satisfied the hypotheses of Theorem 4.2 and this yields Theorem 4.3.

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**REFERENCES**