On the diameter and girth of a zero-divisor graph

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Abstract

For a commutative ring $R$ with zero-divisors $Z(R)$, the zero-divisor graph of $R$ is $\Gamma(R) = Z(R) - \{0\}$, with distinct vertices $x$ and $y$ adjacent if and only if $xy = 0$. In this paper, we characterize when either $\text{diam}(\Gamma(R)) \leq 2$ or $\text{gr}(\Gamma(R)) \geq 4$. We then use these results to investigate the diameter and girth for the zero-divisor graphs of polynomial rings, power series rings, and idealizations. © 2006 Elsevier B.V. All rights reserved.

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1. Introduction

Let $R$ be a commutative ring with 1, and let $Z(R)$ be its set of zero-divisors. The zero-divisor graph of $R$, denoted by $\Gamma(R)$, is the (undirected) graph with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of $R$, and for distinct $x, y \in Z(R)^*$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$. Note that $\Gamma(R)$ is the empty graph if and only if $R$ is an integral domain. Moreover, a nonempty $\Gamma(R)$ is finite if and only if $R$ is finite and not a field [3, Theorem 2.2].

The concept of a zero-divisor graph was introduced by Beck [6], and then further studied in [1]. However, they let all the elements of $R$ be vertices of the graph, and they were mainly interested in colorings. Our definition of $\Gamma(R)$ and the emphasis on studying the interplay between the graph-theoretic properties of $\Gamma(R)$ and the ring-theoretic properties of $R$ are from [3].

Let $G$ be a graph. Recall that $G$ is connected if there is a path between any two distinct vertices of $G$. For vertices $x$ and $y$ of $G$, let $d(x, y)$ be the length of a shortest path from $x$ to $y$ ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The diameter of $G$ is $\text{diam}(G) = \sup\{d(x, y) \mid x$ and $y$ are vertices of $G\}$. The girth of $G$, denoted by $\text{gr}(G)$, is the length of a shortest cycle in $G$ ($\text{gr}(G) = \infty$ if $G$ contains no cycles).

It is known that $\Gamma(R)$ is connected if $\text{diam}(\Gamma(R)) \leq 3$ [3, Theorem 2.3] and that $\text{gr}(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle (this was proved for artinian rings in [3, Theorem 2.4], in general in [13, 1.4]) and [8, Theorem 1.6], and a simple proof is given in [4, Theorem 2.2]). Thus $\text{diam}(\Gamma(R)) = 0, 1, 2$, or 3; and $\text{gr}(\Gamma(R)) = 3, 4$, or $\infty$. Note that $\Gamma(R)$ is a singleton (i.e., $\text{diam}(\Gamma(R)) = 0$) if and only if $R \cong \mathbb{Z}_4$ or $\mathbb{Z}[X]/(X^2)$ [6, Proposition 2.2]. Let $A$ be a...
subring of a commutative ring $B$. Then $\Gamma(A)$ is a (induced) subgraph of $\Gamma(B)$, denoted by $\Gamma(A) \subseteq \Gamma(B)$, and hence $\text{gr}(\Gamma(A)) \geq \text{gr}(\Gamma(B))$. If $B$ is also a flat $A$-module, then $\text{diam}(\Gamma(A)) \leq \text{diam}(\Gamma(B))$.

In Section 2, we characterize when either $\text{diam}(\Gamma(R)) \leq 2$ or $\text{gr}(\Gamma(R)) \geq 4$. These results are then used in Section 3 to investigate the diameter and girth for the zero-divisor graphs of polynomial rings, power series rings, and idealizations. These zero-divisor graphs have recently been studied in [4,5,12]. We give several alternative proofs to those given in [4,5], and [12], and we answer some questions raised in [4] and [5]. Our approach is to work in $T(R)$, the total quotient ring of $R$, and then use the fact that $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic [2, Theorem 2.2].

Let $G_1, G_2$ be the graphs formed by joining the complete bipartite graph $G_1 = K_{m,n}$ with $|A| = m$ and $|B| = n$ to the star graph $G_2 = K_1,n$ by identifying the center of $G_2$ and a point of $B$. Note that $\text{gr}(K_{m,n}) = 4$ if $m, n \geq 1$, $\text{gr}(K_{m,3}) = 4$ if $m \geq 2$, and $\text{gr}(K_{1,3}) = \infty$. Also, $\text{diam}(K_{1,1}) = 1$ and $\text{diam}(K_{m,n}) = 2$ if either $m \geq 2$ or $n \geq 2$.

Throughout, $R$ will be a commutative ring with $1 \neq 0$, $\text{nil}(R)$ its set of nilpotent elements, and $T(R) = R_S$, where $S = R - Z(R)$, its total quotient ring. As usual, we assume that a subring has the same identity element as the ring. We say that $R$ is reduced if $\text{nil}(R) = \{0\}$. We let $\mathbb{Z}$ and $\mathbb{Z}_n$ be the integers and integers modulo $n$, respectively. For any undefined ring-theoretic terminology, see [9] or [10]. A general reference for graph theory is [7]. To avoid trivialities when $\Gamma(R)$ is empty, we will implicitly assume when necessary that $R$ is not an integral domain.

### 2. Diameter and Girth

In this section, we characterize when $\Gamma(R)$ has girth $\geq 4$ or diameter $\leq 2$ in terms of $T(R)$ and $\Gamma(R)$.

In [2, Theorem 2.2], the authors showed that $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic as graphs. In particular, $\Gamma(R)$ and $\Gamma(T(R))$ have the same diameter and girth. However, these two facts may be easily proved without recourse to [2, Theorem 2.2]. We do that in our first lemma.

**Lemma 2.1.** Let $R$ be a commutative ring with total quotient ring $T(R)$. Then $\text{diam}(\Gamma(T(R))) = \text{diam}(\Gamma(R))$ and $\text{gr}(\Gamma(T(R))) = \text{gr}(\Gamma(R))$.

**Proof.** Let $T = T(R)$. Clearly $\text{diam}(\Gamma(T)) = 1$ if and only if $\text{diam}(\Gamma(R)) = 1$. Suppose that $\text{diam}(\Gamma(T)) = 2$. Then $\text{diam}(\Gamma(R)) \geq 2$. Let $a, b \in Z(R)^*$ with $a \neq b$ and $ab \neq 0$. Then $aq = 0 = bq$ for some $q \in Z(T)^* - \{a, b\}$. Let $q = c/t$ with $c \in R$ and $t \in R - Z(R)$. Then $ac = 0 = bc$. Thus $d(a, b) = 2$, and hence $\text{diam}(\Gamma(R)) = 2$. A similar argument shows that $\text{diam}(\Gamma(T)) = 2$ if $\text{diam}(\Gamma(R)) = 2$. The result for the diameter now follows since the diameter of a zero-divisor graph is at most 3 [3, Theorem 2.3].

Since $\Gamma(R)$ is a subgraph of $\Gamma(T)$, clearly $\text{gr}(\Gamma(T)) \leq \text{gr}(\Gamma(R))$. Suppose that $\text{gr}(\Gamma(T)) = 3$. Then there are distinct nonzero elements $q_1, q_2, q_3 \in T$ such that $q_1q_2 = q_2q_3 = q_3q_1 = 0$. Let each $q_i = a_i/t$ with $a_i \in R$ and $t \in R - Z(R)$. Then $a_1a_2 = a_2a_3 = a_3a_1 = 0$. Thus $a_1 - a_2 - a_3 - a_1$ is a triangle in $\Gamma(R)$; so $\text{gr}(\Gamma(T)) = 3$. Similarly, $\text{gr}(\Gamma(R)) = 4$ if $\text{gr}(\Gamma(T)) = 4$. The result for the girth now follows since the girth of a zero-divisor graph is either 3, 4, or $\infty$ [13, (1.4)].

Following [11], we say that distinct vertices $a$ and $b$ in a graph $G$ are orthogonal, written $a \perp b$, if $a$ and $b$ are adjacent and there is no vertex $c$ which is adjacent to both $a$ and $b$, i.e., the edge $a - b$ is not part of any triangle of $G$. As in [2], we say that $G$ is complemented if for each vertex $a$ of $G$, there is a vertex $b$ of $G$ such that $a \perp b$, and that $G$ is uniquely complemented if $G$ is complemented and whenever $a \perp b$ and $a \perp c$, then $b$ and $c$ are adjacent to exactly the same vertices. In [2], the authors classified the commutative rings $R$ such that $\Gamma(R)$ is complemented or uniquely complemented. For example, a reduced commutative ring $R$ is complemented if and only if $T(R)$ is von Neumann regular [2, Theorem 3.5]. Note that if $\text{gr}(\Gamma(R)) = 4$ or $\infty$ (with $\text{diam}(\Gamma(R)) \geq 2$), then $\Gamma(R)$ is complemented. However, if $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(R)$ is (uniquely) complemented and $\text{gr}(\Gamma(R)) = 3$.

We next use the above concepts and results from [2] to determine when $\text{gr}(\Gamma(R)) = 4$. We have two cases, depending on whether or not $R$ has any nonzero nilpotent elements.

**Theorem 2.2.** The following statements are equivalent for a reduced commutative ring $R$.

1. $\text{gr}(\Gamma(R)) = 4$.
2. $T(R) = K_1 \times K_2$, where each $K_i$ is a field with $|K_i| \geq 3$.
3. $\Gamma(R) = K_{m,n}$ with $m, n \geq 2$. 

**Proof.**
**Proof.** (1) $\Rightarrow$ (2) Suppose that $\text{gr}(\Gamma(R)) = 4$. Then $\Gamma(R)$ is complemented. Thus $T = T(R)$ is von Neumann regular by [2, Theorem 3.5] and not a field. Hence $T$ has a nontrivial idempotent, and thus $T = T_1 \times T_2$. Suppose that there are $0 \neq x, y \in T_1$ with $xy = 0$ (note that $x \neq y$ since $R$, and hence $T$, is reduced). Then $(x, 0) - (y, 0) = (0, 1) - (x, 0)$ is a triangle in $\Gamma(T)$, a contradiction since $\text{gr}(\Gamma(T)) = \text{gr}(\Gamma(R)) = 4$ by Lemma 2.1. Thus $T_1$ is an integral domain, in fact, a field. Similarly, $T_2$ must also be a field. Hence $T = K_1 \times K_2$ for fields $K_1$ and $K_2$. If either $K_1$ or $K_2$ has only 2 elements, then $\Gamma(T)$ is a star graph. In this case, $\text{gr}(\Gamma(T)) = \infty$, a contradiction since $\text{gr}(\Gamma(T)) = \text{gr}(\Gamma(R)) = 4$ by Lemma 2.1.

(2) $\Rightarrow$ (3) This follows since the graphs $\Gamma(R)$ and $\Gamma(T)$ are isomorphic [2, Theorem 2.2] and $\Gamma(K_1 \times K_2) = K^{m,n}$, where $m = |K_1| - 1$ and $n = |K_2| - 1$.

(3) $\Rightarrow$ (1) This is clear. □

**Theorem 2.3.** The following statements are equivalent for a commutative ring $R$ with $\text{nil}(R)$ nonzero.

1. $\text{gr}(\Gamma(R)) = 4$.
2. $R \cong D \times B$, where $D$ is an integral domain with $|D| \geq 3$ and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. (Thus $T(R) \cong T(D) \times B$.)
3. $\Gamma(R) = \overline{K}^{m,3}$ with $m \geq 2$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $\text{gr}(\Gamma(R)) = 4$. Then $\Gamma(R)$ is complemented. If $\Gamma(R)$ is uniquely complemented, then $\Gamma(R)$ is a star graph [2, Theorem 3.9], and hence $\text{gr}(\Gamma(R)) = \infty$, a contradiction. Thus $R \cong D \times B$, where $D$ is an integral domain and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by [2, Theorem 3.14]. Hence $\Gamma(R) = \overline{K}^{m,3}$, where $m = |D| - 1$. We must have $|D| \geq 3$ since otherwise $\text{gr}(\Gamma(R)) = \infty$, a contradiction.

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are both clear. □

Next we determine when $\text{gr}(\Gamma(R)) = \infty$ using ideas from [2]. Similar results have also been obtained in [8, Theorems 1.7 and 1.12] and [13] (cf. Remark 2.6). Again, we have two cases, depending on whether or not $R$ is reduced. Since $\text{gr}(\Gamma(R)) = 3, 4$ or $\infty$, we have thus, in some sense, also characterized when $\text{gr}(\Gamma(R)) = 3$.

**Theorem 2.4.** The following statements are equivalent for a reduced commutative ring $R$.

1. $\Gamma(R)$ is nonempty with $\text{gr}(\Gamma(R)) = \infty$.
2. $T(R) = \mathbb{Z}_2 \times K$, where $K$ is a field.
3. $\Gamma(R) = K^{1,n}$ for some $n \geq 1$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $\text{gr}(\Gamma(R)) = \infty$ and $\Gamma(R) \neq \emptyset$. Then $|\Gamma(R)| \geq 2$ since $R$ is reduced, and thus $\Gamma(R)$ is complemented. As in the proof of (1) $\Rightarrow$ (2) of Theorem 2.2, we have $T(R) = K_1 \times K_2$ for fields $K_1$ and $K_2$. If each field has at least three elements, then $\text{gr}(\Gamma(R)) = 4$ by Theorem 2.2, a contradiction. Hence we may assume that $K_1$ has 2 elements; so $K_1 = \mathbb{Z}_2$.

(2) $\Rightarrow$ (3) This follows since the graphs $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic [2, Theorem 2.2] and $\Gamma(\mathbb{Z}_2 \times K) = K^{1,n}$, where $n = |K| - 1$.

(3) $\Rightarrow$ (1) This is clear. □

**Theorem 2.5.** The following statements are equivalent for a commutative ring $R$ with $\text{nil}(R)$ nonzero.

1. $\text{gr}(\Gamma(R)) = \infty$.
2. $R \cong B$ or $R \cong \mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, or $\Gamma(R)$ is a star graph.
3. $\Gamma(R)$ is a singleton, a $\overline{K}^{1,3}$, or a $K^{1,n}$ for some $n \geq 1$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $\text{gr}(\Gamma(R)) = \infty$. If $\Gamma(R)$ is a point, then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. So assume that $\Gamma(R)$ has at least 2 elements. Then $\Gamma(R)$ is complemented. If $\Gamma(R)$ is uniquely complemented, then $\Gamma(R)$ is a star graph by [2, Theorem 3.9]. If $\Gamma(R)$ is not uniquely complemented, then $R \cong D \times B$, where $D$ is an integral domain and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$ by [2, Theorem 3.14]. If $|D| \geq 3$, then $\text{gr}(\Gamma(R)) = 4$ as in Theorem 2.3, a contradiction. Thus $|D| = 2$; so $D = \mathbb{Z}_2$.

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are both clear. □
Theorems 2.2–2.5

Theorem 2.7

Lemma 2.1

Remark 2.6.

Proof. Let $T = T(R)$. Note that (1) holds if and only if $T$ has a unique maximal ideal. So suppose that $\text{diam}(\Gamma(R)) \leq 2$ and that $Z(R)$ is not a prime ideal of $R$. Then there are distinct maximal ideals $M$ and $N$ of $T$. Thus $x + y = 1$ for some $x \in M$ and $y \in N$, and hence $\text{ann}(x) \cap \text{ann}(y) = \{0\}$. Since $\text{diam}(\Gamma(T)) = \text{diam}(\Gamma(R)) \leq 2$ by Lemma 2.1, we must have $xy = 0$, and thus $x$ and $y$ are idempotent. Hence $T = T_1 \times T_2$. Suppose that there is a $c \in Z(T_1)^*$. Then $a = (1, 1)$ and $b = (1, 0)$ are in $Z(T)^*$ and $d(a, b) \geq 3$, a contradiction. Thus $T_1$ must be an integral domain, in fact, a field. Similarly, $T_2$ is a field. Hence $T(R) = K_1 \times K_2$ with each $K_i$ a field. Thus (2) holds. □

Theorem 2.7. Let $R$ be a commutative ring with $\text{diam}(\Gamma(R)) \leq 2$. Then exactly one of the following holds.

1. $Z(R)$ is an (prime) ideal of $R$.
2. $T(R) = K_1 \times K_2$, where each $K_i$ is a field.

Proof. Let $T = T(R)$. Note that (1) holds if and only if $T$ has a unique maximal ideal. So suppose that $\text{diam}(\Gamma(R)) \leq 2$ and that $Z(R)$ is not a prime ideal of $R$. Then there are distinct maximal ideals $M$ and $N$ of $T$. Thus $x + y = 1$ for some $x \in M$ and $y \in N$, and hence $\text{ann}(x) \cap \text{ann}(y) = \{0\}$. Since $\text{diam}(\Gamma(T)) = \text{diam}(\Gamma(R)) \leq 2$ by Lemma 2.1, we must have $xy = 0$, and thus $x$ and $y$ are idempotent. Hence $T = T_1 \times T_2$. Suppose that there is a $c \in Z(T_1)^*$. Then $a = (1, 1)$ and $b = (1, 0)$ are in $Z(T)^*$ and $d(a, b) \geq 3$, a contradiction. Thus $T_1$ must be an integral domain, in fact, a field. Similarly, $T_2$ is a field. Hence $T(R) = K_1 \times K_2$ with each $K_i$ a field. Thus (2) holds. □

Theorem 2.8. Let $R$ be a (reduced) commutative ring which is not an integral domain such that $R$ is a subring of $D_1 \times D_2$, where each $D_i$ is an integral domain. Then either $R \cong Z_2 \times Z_2$ (and hence $\text{diam}(\Gamma(R)) = 1$) or $\text{diam}(\Gamma(R)) = 2$.

Proof. If $\text{diam}(\Gamma(R)) = 1$, then $R \cong Z_2 \times Z_2$ [3, Theorem 2.8]. So suppose that $\text{diam}(\Gamma(R)) \geq 2$. Let $x, y \in Z(R)^*$ be distinct with $xy \neq 0$. Then we may assume that $x, y \in D_1 \times \{0\}$. Since $x \in Z(R)^*$ and $R$ is reduced, there is a $z \in Z(R)^* \setminus \{x, y\}$ such that $xz = 0$. But then $z \in \{0\} \times D_2$; so $xz = yz = 0$, and hence $d(x, y) = 2$. Thus $\text{diam}(\Gamma(R)) = 2$. □

Remark 2.9. (a) We have $\text{diam}(\Gamma(R)) = 0$ if and only if $\Gamma(R)$ is a point, i.e., $R \cong Z_4$ or $Z_2[X]/(X^2)$. We have $\text{diam}(\Gamma(R)) = 1$ if and only if $|\Gamma(R)| \geq 2$ and $\Gamma(R)$ is complete. This happens if and only if either $R \cong Z_2 \times Z_2$ or $Z(R)$ is a prime ideal of $|Z(R)| \geq 3$ and $|Z(R)^2| = \{0\}$ [3, Theorem 2.8].

(b) As a converse to Theorem 2.7(2), let $T(R) = K_1 \times K_2$ be the product of two fields. If $K_1 = K_2 = Z_2$, then (by Lemma 2.1) $\text{diam}(\Gamma(R)) = 1$; otherwise, $\text{diam}(\Gamma(R)) = 2$. As a converse to part (1) of Theorem 2.7, note that if $R$ is noetherian, then $Z(R)$ is an annihilator ideal if and only if it is an (prime) ideal [10, Theorems 6 and 82], and in this case $\text{diam}(\Gamma(R)) \leq 2$. However, it is possible to have $Z(R)$ an ideal of a reduced ring $R$, yet $\text{diam}(\Gamma(R)) = 3$ (see the remarks after [12, Example 5.1]).

(c) A nice ideal-theoretic characterization of $\text{diam}(\Gamma(R))$ is given in [12, Theorem 2.6]. In particular, $\text{diam}(\Gamma(R)) = 3$ if and only if there are distinct $a, b \in Z(R)^*$ with $\text{ann}(a) \cap \text{ann}(b) = \{0\}$ and either (i) $R$ is reduced with at least three minimal prime ideals, or (ii) $R$ is not reduced. □
3. Applications

In this section, we apply the results about diameter and girth obtained in the previous section to certain classes of commutative rings. In [4, Section 4], the authors investigated the girth and diameter of $\Gamma(R[[X]])$ and $\Gamma(R[X])$. In this section, we first completely characterize $\text{gr}(\Gamma(R[X]))$ and $\text{gr}(\Gamma(R[[X]]))$ in terms of $\text{gr}(\Gamma(R))$. As a first step, we show that $\Gamma(R[X])$ and $\Gamma(R[[X]])$ always contain a cycle.

Lemma 3.1. Let $R$ be a commutative ring which is not an integral domain. Then $\text{gr}(\Gamma(R[X]))$ and $\text{gr}(\Gamma(R[[X]]))$ are either 3 or 4. If $R$ is not reduced, then both zero-divisor graphs have girth 3.

Proof. If $ab = 0$ for distinct $a, b \in Z(R)^*$, then $a - b - aX - bX - a$ is a 4-cycle; and if $a^2 = 0$ for $a \in Z(R)^*$, then $a - aX - aX^2 - a$ is a 3-cycle.

Parts of our next theorem are given in [4, Section 4]. Specifically, they showed that $\text{gr}(\Gamma(R)) \geq \text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]]))$, and that equality holds if $R$ is reduced and $\Gamma(R)$ contains a cycle [4, Theorem 4.3]. However, our methods are very different.

Theorem 3.2. Let $R$ be a commutative ring.

(1) Suppose that $\Gamma(R)$ is nonempty with $\text{gr}(\Gamma(R)) = \infty$.

(a) If $R$ is reduced, then $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 4$.
(b) If $R$ is not reduced, then $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 3$.

(2) If $\text{gr}(\Gamma(R)) = 3$, then $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 3$.

(3) Suppose that $\text{gr}(\Gamma(R)) = 4$.

(a) If $R$ is reduced, then $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 4$.
(b) If $R$ is not reduced, then $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 3$.

Proof. We have already observed in Lemma 3.1 that $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 3$ if $R$ is not reduced. Thus 1(b) and 3(b) hold. Clearly (2) holds since $\Gamma'(R) \subseteq \Gamma'(R[X]) \subseteq \Gamma'(R[[X]])$. Hence we may assume that $R$ is reduced. First, suppose that $\text{gr}(\Gamma(R)) = 4$. Then $\Gamma'(R) \subseteq \Gamma'(R[X])$; so $\text{gr}(\Gamma(R[X])) \leq 4$. Also, $R \subseteq T(R) = K_1 \times K_2$ by Theorem 2.2, and thus $R[X] \subseteq K_1[X] \times K_2[X]$. Hence $\Gamma'(R[X]) \subseteq \Gamma'(K_1[X] \times K_2[X])$ contains no triangles; so $\text{gr}(\Gamma(R[X])) = 4$. Similarly, $\text{gr}(\Gamma'(R[[X]])) = 4$. So 3(a) holds. Finally, suppose that $\text{gr}(\Gamma(R)) = \infty$. By Lemma 3.1, $\text{gr}(\Gamma'(R[X])) \leq 4$. Moreover, $R \subseteq T(R) = \mathbb{Z}_2 \times K$ for some field $K$ by Theorem 2.4. Thus $R[X] \subseteq \mathbb{Z}_2[X] \times K[X]$; and hence $\Gamma'(R[X]) \subseteq \Gamma'(\mathbb{Z}_2[X] \times K[X])$ contains no triangles. Thus $\text{gr}(\Gamma'(R[X])) = 4$. Similarly, $\text{gr}(\Gamma'(R[[X]])) = 4$. So 1(a) holds. The “in particular” statement follows directly from (1)–(3).

We next consider the diameter of the zero-divisor graph of a polynomial ring or power series ring. The problem of computing the diameter for the zero-divisor graphs of polynomial and power series rings was first considered in [4]. Some cases for $R$ a non-noetherian commutative ring left open in [4] were then resolved in [12]. Note that $R[X] \subseteq T(R)[X] \subseteq T(R)$; so $\text{diam}(\Gamma'(R[X])) = \text{diam}(\Gamma'(T(R)[X])) = \text{diam}(\Gamma'(T(R)))$ by Lemma 2.1. Always $\text{diam}(\Gamma(R)) \leq \text{diam}(\Gamma'(R))$ and $\text{diam}(\Gamma'(R)) \leq \text{diam}(\Gamma(R'[R]))$. Thus $\text{diam}(\Gamma'(R)) = \text{diam}(\Gamma(R'[R])) = \text{diam}(\Gamma(R[[X]]))$ when $\text{diam}(\Gamma(R)) = 3$. If $\text{diam}(\Gamma(R)) = 0$, then $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[Y]/(Y^2)$, and hence $\Gamma'(R)$ and $\Gamma(R[[X]])$ are both complete graphs with $\text{diam}(\Gamma'(R)) = \text{diam}(\Gamma(R[[X]])) = 1$. Our next theorem handles the case when $T(R)$ is von Neumann regular; it is similar to (and extends to power series rings) [12, Corollary 3.5], but our proof is in the spirit of Section 2. However, if $R$ is reduced, but $T(R)$ is not von Neumann regular, then it is possible to have either $2 = \text{diam}(\Gamma(R)) < \text{diam}(\Gamma'(R[[X]])) = \text{diam}(\Gamma(R[[X]])) = 3$ or $2 = \text{diam}(\Gamma(R)) = \text{diam}(\Gamma'(R[[X]])) < \text{diam}(\Gamma'(R[[X]])) = 3$ [12, Theorem 4.9].

Theorem 3.3. Let $R$ be a (reduced) commutative ring with $T(R)$ von Neumann regular. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $1 = \text{diam}(\Gamma'(R)) < \text{diam}(\Gamma'(R[[X]])) = \text{diam}(\Gamma'(R[[X]])) = 2$. Otherwise, $\text{diam}(\Gamma'(R)) = \text{diam}(\Gamma'(R[[X]])) = \text{diam}(\Gamma'(R[[X]]))$.

(1) If $R$ has exactly 2 minimal prime ideals, and 3 if $R$ has at least 3 minimal prime ideals.

Proof. Suppose that $T = T(R)$ is von Neumann regular. Thus $R, R[X]$, and $R[[X]]$ are all reduced. If $\text{diam}(\Gamma'(R)) = 1$, then $R = T(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ [3, Theorem 2.8], and hence $R[X] \cong \mathbb{Z}_2[X] \times \mathbb{Z}_2[X]$ and $R[[X]] \cong \mathbb{Z}_2[[X]] \times \mathbb{Z}_2[[X]]$. Thus $\text{diam}(\Gamma(R[X])) = \text{diam}(\Gamma(R[[X]])) = 2$. Next suppose that $\text{diam}(\Gamma(R)) = 2$. Then $R$ has exactly 2 minimal
prime ideals and $T = K_1 \times K_2$ for fields $K_1$ and $K_2$ by Theorem 2.7. Hence $R[X] \subseteq K_1[X] \times K_2[X]$ and $R[[X]] \subseteq K_1[[X]] \times K_2[[X]]$. Thus $\dim(\Gamma(R[X])) = \dim(\Gamma(R[[X]])) = 2$ by Theorem 2.8. If $R$ has at least 3 minimal prime ideals, then $\dim(\Gamma(R)) = 3$ by Theorem 2.7, and hence also $\dim(\Gamma(R[[X]])) = \dim(\Gamma(R[X])) = 3$. \qed

Next we investigate the girth and diameter of the zero-divisor graph of a ring formed by idealization. First, we briefly recall the idealization construction. Let $R$ be a commutative ring and $M$ an (unitary) $R$-module. The idealization of $R$ and $M$, denoted by $R(+)M$, is defined to be the ring $R \times M$ with addition and multiplication given by $(a, m) + (b, n) = (a + b, m + n)$ and $(a, m)(b, n) = (ab, an + bm)$, respectively (see [9]). It is known that $Z(R(+)M) = (Z(R) \cup Z(M))(+)M$ [9, Theorem 25.3], where $Z(M) = \{r \in R \mid rm = 0$ for some $0 \neq m \in M\}$. As usual, $\text{ann}(M) = \{r \in R \mid rm = 0$ for all $m \in M\}$. We will always assume that $M$ is nonzero, and hence $R(+)M$ is not reduced since $(0)(+)M \subseteq \text{nil}(R(+)M)$.

In [5, Theorem 2.8], the authors explicitly determined when $\gamma(\Gamma(R(+)M)) = 3$ or $\infty$. Thus, in some sense, the $\gamma(\Gamma(R(+)M)) = 4$ case is also determined. However, we next determine precisely when $\gamma(\Gamma(R(+)M)) = 4$.

**Theorem 3.4.** Let $R$ be a commutative ring and $M$ a nonzero $R$-module. Then $\gamma(\Gamma(R(+)M)) = 4$ if and only if $R \cong D \times \mathbb{Z}_2$, where $D$ is an integral domain with $|D| \geq 3$ and $M \cong \mathbb{Z}_2$. (So $\gamma(\Gamma(R)) = \infty$.)

**Proof.** By Theorem 2.3, $S = R(+)M \cong D \times B = T$, where $D$ is an integral domain with $|D| \geq 3$ and $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Thus $\text{nil}(S) = \text{nil}(T) = 2$; so $R$ must be reduced and $M \cong \mathbb{Z}_2$. Hence $\text{nil}(S) = (0)(+)M$ and $\text{nil}(T) = \{0\} \times C$, where $C \cong \mathbb{Z}_2$. Thus $R \cong R(+)M/(0)(+)M \cong D \times B/(0) \times C \cong D \times \mathbb{Z}_2$. An explicit isomorphism is given by $\varphi : (D \times \mathbb{Z}_2)(+)\mathbb{Z}_2 \to D \times \mathbb{Z}_2[X]/(X^2)$ with $\varphi((d, a), b) = (d, a + bX)$. \qed

In [5, Theorem 2.8(ii)], the authors determined when $\gamma(\Gamma(R(+)M)) = \infty$. The following theorem recovers their result using ideas from Section 2.

**Theorem 3.5.** Let $R$ be a commutative ring and $M$ a nonzero $R$-module. Then $\gamma(\Gamma(R(+)M)) = \infty$ if and only if exactly one of the following hold.

1. $R \cong \mathbb{Z}_3$ and $M \cong \mathbb{Z}_3$.
2. Either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $R$ is an integral domain, and $M \cong \mathbb{Z}_2$.

**Proof.** Let $A = R(+)M$. From Theorem 2.5, we have $\gamma(\Gamma(A)) = \infty$ if and only $A \cong B$ or $A \cong \mathbb{Z}_2 \times B$, where $B = \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, or $\Gamma(A)$ is a star graph. If $A \cong B$, then $|A| = 4$; so $A$ must be $\mathbb{Z}_2(+)\mathbb{Z}_2 \cong \mathbb{Z}_2[X]/(X^2)$. If $A \cong \mathbb{Z}_2 \times B$, then $A$ must be $\mathbb{Z}_2(+)\mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$. Now suppose that $\Gamma(A)$ is a star graph. By Remark 2.6(c), $\text{nil}(A) \leq 4$. First suppose that $\text{nil}(A) = 2$. In this case, $M \cong \mathbb{Z}_2$, and either $R$ is an integral domain or $R$ is reduced (and not a domain) with $\gamma(\Gamma(R)) = \infty$. In the second case, $R \cong \mathbb{Z}_2 \times K$ for some field $K$ by Theorem 2.4. We may assume that $R$ acts on $M$ by $(a, b)m = am$. If $|K| \geq 3$, then $((0, 1), 0) - ((0, 0), 1) = ((0, 0), 0) - ((0, 1), 0)$ is a $4$-cycle in $\Gamma(A)$ for $a \in K - \{0, 1\}$; so $\gamma(\Gamma(A)) \leq 4$, a contradiction. If $\text{nil}(A) = 3$, then $A \cong \mathbb{Z}_3$ or $A \cong \mathbb{Z}_3[X]/(X^2)$. In this case, $A$ must be $\mathbb{Z}_3(+)\mathbb{Z}_3 \cong \mathbb{Z}_3[X]/(X^2)$. If $\text{nil}(A) = 4$, then $A \cong \mathbb{Z}_4$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$. Thus $|R| = 4$ and $M \cong \mathbb{Z}_2$ (if $|M| = 4$, then we would have $\gamma(\Gamma(A)) = 3$, a contradiction). In this case, either $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, and in each case $\gamma(\Gamma(A)) = 3$, again, a contradiction. \qed

**Remark 3.6.** (a) By Theorem 3.5 and its proof, the idealizations $A = R(+)M$ with $\gamma(\Gamma(A)) = \infty$ are as follows: (1) $\Gamma(A)$ is a point if and only if $A = \mathbb{Z}_2(+)\mathbb{Z}_2 \cong \mathbb{Z}_2[X]/(X^2)$, (2) $\Gamma(A) = \mathbb{K}^{1,1}$ if and only if $A = (\mathbb{Z}_2 \times \mathbb{Z}_2)(+)\mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$, (3) $\Gamma(A) = \mathbb{K}^{1,3}$ if and only if $A = \mathbb{Z}_3(+)\mathbb{Z}_3 \cong \mathbb{Z}_3[X]/(X^2)$, and (4) $\Gamma(A) = \mathbb{K}^{1,n}$ with $n \geq 2$ if and only if $A = R(+)\mathbb{Z}_2$ for $R$ an infinite integral domain. In this last case, $R$ is an integral domain with maximal ideal $N$ such that $R/N \cong \mathbb{Z}_2$ (so $R$ must be infinite and not a field) and $R(+)\mathbb{Z}_2 \cong R[X]/(NX, X^2)$.

(b) Let $R$ be an infinite commutative ring. If $R$ is reduced, then $\Gamma(R)$ is a star graph if and only if $R \cong D \times \mathbb{Z}_2$ for some infinite integral domain $D$ [3, Theorem 2.5]. By (a) above, an idealization $R$ has $\Gamma(R)$ a star graph if and only if $R \cong D(+)\mathbb{Z}_2$ for some infinite integral domain $D$. However, not all infinite commutative rings with $\text{nil}(R)$ nonzero and $\Gamma(R)$ a star graph are idealizations. For example, let $R = \mathbb{Z}_2[X, Y, Z]/I$, where $I = fN$ for $f = XZ + Y^2$ and $N = (X, Y, Z)$, and let $w = f + I$. Then $\text{nil}(R) = \{0, w\}$ and $\Gamma(R)$ is an infinite star graph with center $w$, but $R$ is not an idealization. (If $R$ were an idealization, then $R \cong D(+)\mathbb{Z}_2$ for some infinite integral domain $D$ by Theorem 3.5.)
Thus there would be a derivation \( \delta : R \to \{0, w\} \) with \( \delta(w) = w \). But one can easily show that \( \delta(w) = 0 \) for any such derivation \( \delta \). Hence \( R \) is not an idealization. \( \square \)

We end this paper with some results on the diameter of \( \Gamma(R(+)M) \). If \( M \) is nonzero, then \( R(+)M \) is not reduced, and thus \( \text{diam}(\Gamma(R(+)M)) \leq 2 \) implies that \( Z(R(+)M) = (Z(R) \cup Z(M))(+)M \) is an ideal of \( R(+)M \) (and hence \( Z(R) \cup Z(M) \) is an ideal of \( R \)) by Theorem 2.7. It is easy to see that \( \text{diam}(\Gamma(R(+)M)) = 0 \) if and only if \( R(+)M \cong \mathbb{Z}_2(+)\mathbb{Z}_2 \cong \mathbb{Z}_2[X]/(X^2) \) (see the proof of Theorem 3.5).

First we determine when \( \text{diam}(\Gamma(R(+)M)) \leq 1 \). An equivalent characterization is given in [5, Theorem 3.3 and Corollary 3.4] (their conditions (b) and (c) are equivalent to \( Z(M) \subseteq Z(R) \) and \( Z(R) \subseteq \text{ann}(M) \), respectively).

**Theorem 3.7.** The following statements are equivalent for a commutative ring \( R \) and a nonzero \( R \)-module \( M \).

1. \( \Gamma(R(+)M) \) is a complete graph.
2. \( \text{diam}(\Gamma(R(+)M)) \leq 1 \).
3. \( Z(R) = Z(M) = \text{ann}(M) \) and \( Z(R)^2 = \{0\} \).
4. \( Z(R(+)M)^2 = \{0\} \).

**Proof.** (1) \( \Rightarrow \) (2) This is clear.

(2) \( \Rightarrow \) (3) Suppose that \( \text{diam}(\Gamma(R(+)M)) \leq 1 \). Let \( a \in Z(R)^* \), \( b \in Z(M)^* \), and \( 0 \neq m \in M \). Then \( (a,0),(b,m) \in (Z(R) \cup Z(M))(+)M = Z(R(+)M) \) are distinct, and hence adjacent in \( \Gamma(R(+)M) \). Thus \( (a,0),(b,m) = (0,0) \). This shows that \( Z(R) \subseteq \text{ann}(M) \), \( Z(M) \subseteq Z(R) \), and \( Z(R)Z(M) = \{0\} \). Since \( \text{ann}(M) \subseteq Z(M) \) always holds, we have \( Z(R) = Z(M) = \text{ann}(M) \) and \( Z(R)^2 = \{0\} \).

(3) \( \Rightarrow \) (4) Suppose that \( Z(R) = Z(M) = \text{ann}(M) \) and \( Z(R)^2 = \{0\} \). Then it is easy to check that \( Z(R(+)M) = Z(R(+)M)^2 \) satisfies \( Z(R(+)M)^2 = \{0\} \).

(4) \( \Rightarrow \) (1) This is clear. \( \square \)

**Remark 3.8.** In [3, Theorem 2.8], the authors showed that \( \Gamma(R) \) is complete if and only if either \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( Z(R)^2 = \{0\} \). So if \( R \) is not reduced, then \( xy = 0 \) for all \( x, y \in Z(R) \) with \( x \neq y \) actually also implies that \( x^2 = 0 \) for all \( x \in Z(R) \). This fact was not used in the proof of Theorem 3.7. \( \square \)

Our final theorem characterizes when \( \text{diam}(\Gamma(R(+)M)) \leq 2 \). This is an open question from [5]. Combining this with Theorem 3.7, thus, in some sense, characterizes when \( \text{diam}(\Gamma(R(+)M)) = 3 \).

**Theorem 3.9.** Let \( R \) be a commutative ring and \( M \) a nonzero \( R \)-module. Then \( \text{diam}(\Gamma(R(+)M)) \leq 2 \) if and only for all \( x, y \in Z(R) \cup Z(M) \), either (1) there is a \( 0 \neq z \in \text{ann}(M) \) such that \( xz = yz = 0 \), or (2) there is a \( 0 \neq m \in M \) such that \( xm = ym = 0 \).

**Proof.** Suppose that \( \text{diam}(\Gamma(R(+)M)) \leq 2 \). Let \( x, y \in Z(R) \cup Z(M) \), and suppose that (2) fails. Then \( xm = ym = 0 \) for \( m \in M \) implies \( m = 0 \). First suppose that \( x \in Z(M) \). Then \( xm = 0 \) for some \( 0 \neq m \in M \), and thus \( ym \neq 0 \). Then \( (x,m), (y,0) \in Z(R(+)M)^* \) are distinct and not adjacent; so there is a \( (z,b) \in Z(R(+)M)^* \) adjacent to both. Hence \( xz = yz = 0 \) implies \( xzn = yzn = 0 \) for all \( n \in M \). By the above comments, \( zn = 0 \); so \( z \in \text{ann}(M) \). Thus \( xb = yb = 0 \) implies \( b = 0 \); so \( z \neq 0 \). If \( x \in Z(R) - Z(M) \), then just repeat the above argument for \( (x,0),(y,m) \) with any \( 0 \neq m \in M \). Thus (1) holds.

Conversely, suppose that for all \( x, y \in Z(R) \cup Z(M) \), either (1) or (2) holds. Let \( (x,a), (y,b) \in Z(R(+)M)^* \) be distinct and not adjacent; then \( x, y \in Z(R) \cup Z(M) \). If (1) holds, then there is a nonzero \( (0,m) \) adjacent to both; and if (2) holds, then there is a nonzero \( (0,m) \) adjacent to both. Thus \( \text{diam}(\Gamma(R(+)M)) \leq 2 \). \( \square \)

**Remark 3.10.** (a) Note that either \( Z(R) \subseteq Z(M) \) or \( Z(M) \subseteq Z(R) \) if \( \text{diam}(\Gamma(R(+)M)) \leq 2 \). For suppose that \( Z(R) \not\subseteq Z(M) \) and \( Z(M) \not\subseteq Z(R) \). Choose \( x \in Z(R) - Z(M) \) and \( y \in Z(M) - Z(R) \). Then \( x, y \in Z(R) \cup Z(M) \) do not satisfy either (1) or (2) of Theorem 3.9.

(b) In [5, Theorems 3.9 and 3.10], two sufficient conditions are given to have \( \text{diam}(\Gamma(R(+)M)) = 2 \). Since their conditions (b) and (c) are equivalent to \( Z(M) \subseteq Z(R) \) and \( Z(R) \subseteq \text{ann}(M) \), respectively, [5, Theorem 3.9] translates to \( \text{diam}(\Gamma(R(+)M)) = 2 \) if \( Z(R) = Z(M) = \text{ann}(M) \) and \( Z(R)^2 \neq \{0\} \) (this implies (2) of our Theorem 3.9), and [5, Theorem 3.10] translates to \( \text{diam}(\Gamma(R(+)M)) = 2 \) if \( Z(R) = Z(M) \), \( Z(R) \not\subseteq \text{ann}(M) \), and \( Z(R)^2 = \{0\} \) (note that \( Z(R) \subseteq Z(M) \) if \( Z(R)^2 = \{0\} \)), which implies (1) of our Theorem 3.9. \( \square \)
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References