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Abstract

In this paper we introduce a notion of counting problems over the real numbers. We follow the approaches of Blum et al. (1998) for computability over $\mathbb{R}$ and of Grädel and Meer (1996) for descriptive complexity theory in this setting and give a complete characterization of such problems by logical means. The main emphasis of our results is model-theoretic.

Keywords: Counting problem; Real number model; $\mathbb{R}$-structures; Model theory

1. Introduction

Many important problems in mathematics and computer science appear in the form of counting problems. As a classical example let us mention the question of counting the (real) zeros of polynomials, analyzed already in the last century (f.e. theorems by Bezout and Sturm, cf. [3]) or counting the dimension of a (semi-) algebraic set. Another example closely related to real zero counting is given by evaluating the number of sign changes in a sequence of reals. The latter has caused increasing attention again during the last years because of its extreme importance for dealing with the existential theory over real closed fields (see for example [19, 17]).

Even though an extensive theory of counting problems on finite domains has been developed in discrete computer science along the counting class #P introduced by Valiant [22], for the above-mentioned problems (and many more) this approach fails; it refers to the Turing machine as underlying computational device and thus problems appear when we wish to compute exactly with real (or complex) numbers. In order to develop a similar theory but for real number problems we will use a different machine model. Such a model was introduced in 1989 by Blum et al. [5]. It turns

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out to provide an appropriate framework for dealing with counting problems as those mentioned above.

Recall that over finite alphabets #P is defined as class of problems for which the counting function is given by the number of accepting paths of a nondeterministic polynomial time Turing machine. Equivalently, this class could be modelled also by counting the number of guesses leading a NP-machine into an accepting state. This observation causes an ambiguity concerning the task to define #P within the Blum–Shub–Smale (henceforth: BSS) model. Counting accepting paths will always give a finite number whereas counting guesses can also yield infinitely many answers. Having in mind the zero-counting problem we consider the second approach the more natural. In Section 2 we will shortly recall the basics of BSS-computability; next we introduce the real analogue $\#P_{\mathbb{R}}$ of #P. This definition is substantiated because all $\#P_{\mathbb{R}}$-functions turn out to be computable.

The main scope of this paper is to give a logical characterization of $\#P_{\mathbb{R}}$. Starting point for such a characterization is a branch in complexity theory called descriptive complexity. It aims to capture complexity classes by logical means only. A cornerstone in this area with respect to the Turing model was a paper by Fagin [9]. There he described the class of NP problems as those sets of finite structures being definable in existential second-order logic. Along these lines Saluja et al. [20] where able to classify #P-functions on ordered structures into a hierarchy of five distinct levels. Compton and Grädel [6] extended the former work to arbitrary structures. A descriptive complexity theory for the BSS model was developed in [11] and further investigated in [7]. Most real number complexity classes can be captured by certain logics defined over the so-called $\mathbb{R}$-structures. In Section 3 we make available the basic ingredients of this area needed later on and express $\#P_{\mathbb{R}}$ functions via first-order logic on $\mathbb{R}$-structures. The main results are then presented in Section 4. We show that on ordered $\mathbb{R}$-structures $\#P_{\mathbb{R}}$ again can be decomposed into five distinct levels. Each of these levels is given by a certain restriction on the number and type of quantifiers appearing in first-order formulas. The most powerful class of formulas with respect to counting is already obtained by considering those of type $\forall x \exists y \psi$, where ψ is quantifier free.

Contrary to the discrete situation where only the first level of the according hierarchy is known to be included in the class of polynomial time computable functions we show, furthermore, that over the reals the latter holds true for the two lowest levels. In a certain sense this is a negative result; there are $NP_{\mathbb{R}}$-complete problems which can be represented by all those $\mathbb{R}$-structures being a model for a specific formula belonging to the third level of our hierarchy. And there are polynomially computable problems related to a formula in the highest level of the hierarchy. Thus, our results are of minor significance from a complexity theoretic point of view. The main focus is on the model theoretic side of $\mathbb{R}$-structures.

Let us point out that one could follow also a different approach towards counting over the reals by regarding the problem to determine the measure of a semi-algebraic set. Such an approach is dealt with by Gross (see [12]) who also obtains a logical hierarchy similar to ours.

2. The Blum–Shub–Smale model; the class $\#P_{\mathbb{R}}$

We start this section by summarizing very briefly the main ideas of real number complexity theory. For a more intensive presentation see [5, 4], and also [15] for a survey on current results in this area.

Essentially, a (real) BSS-machine can be considered as a random access machine over $\mathbb{R}$ which is able to perform the basic arithmetic operations at unit cost and which registers can hold arbitrary real numbers.

Definition 1 (Blum et al. [5]). (a) Let $Y \subset \mathbb{R}^\infty := \bigoplus_{k\in\mathbb{N}} \mathbb{R}^k$, i.e. the set of finite sequences of real numbers. A BSS-MACHINE M OVER $\mathbb{R}$ WITH ADMISSIBLE INPUT SET Y is given by a finite set I of instructions labelled by $1,\dots,N$. A configuration of M is a quadruple $(n,i,j,x) \in I \times \mathbb{N} \times \mathbb{N} \times \mathbb{R}^\infty$. Here n denotes the currently executed instruction, i and j are used as addresses (copy-registers) and x is the actual content of the registers of M. The initial configuration of M's computation on input $y \in Y$ is $(1,1,1,y)$. If $n = N$ and the actual configuration is (N,i,j,x), the computation stops with output x .

The instructions M is allowed to perform are of the following types:

computation: $n : x_s \leftarrow x_k \circ_n x_l$, where $\circ_n \in \{+,-,*,:\}$ or
$n : x_s \leftarrow \alpha$ for some constant $\alpha \in \mathbb{R}$.
The register x_s will get the value $x_k \circ_n x_l$ or α resp. All other register-entries remain unchanged. The next instruction will be $n+1$; moreover, the copy-register i is either incremented by one, replaced by 0, or remains unchanged. The same holds for copy-register j.

branch: n: **if** $x_0 \geqslant 0$ **goto** $\beta(n)$ **else goto** $n+1$. According to the answer of the test the next instruction is determined (where $\beta(n) \in I$). All other registers are not changed.

copy: $n : x_i \leftarrow x_j$, i.e. the content of the "read"-register is copied into the "write"-register. The next instruction is $n+1$; all other registers remain unchanged.

(b) The size of an $x \in \mathbb{R}^k$ is $size_{\mathbb{R}}(x) = k$. The cost of any of the above operations is 1. The cost of a computation is the number of operations performed until the machine halts.

(c) For some $B \subset \mathbb{R}^\infty$ we call a function $f : B \to \mathbb{R}^\infty$ (BSS-) computable iff it is realized by a BSS machine over admissible input set B. Similarly, a set $A \subset B \subset \mathbb{R}^\infty$ is decidable in B iff its characteristic function is computable. (B,A) is called a decision problem

Now, it is straightforward to define polynomial time computability as well as complexity classes:

Definition 2. (i) A problem $(B,A), A \subset B \subset \mathbb{R}^\infty$ is in class $P_{\mathbb{R}}$ (decidable in deterministic polynomial time), iff there exist a polynomial p and a (deterministic) BSS machine M deciding A (in B) such that $T_M(y) \leqslant p(size_{\mathbb{R}}(y))\ \forall y \in B$.

(ii) (B,A) is in $\mathrm{NP}_{\mathbb{R}}$ (verifyable in non-deterministic polynomial time over $\mathbb{R}$) iff there exist a polynomial p and a BSS machine M working on input space $B \times \mathbb{R}^{\infty}$ such that

(a) $\Phi_M(y,z) \in \{0,1\}\ \forall y \in B, z \in \mathbb{R}^{\infty}$

(b) $\Phi_M(y,z) = 1 \implies y \in A$

(c) $\forall y \in A\ \exists z \in \mathbb{R}^{\infty}\ \Phi_M(y,z) = 1$ and $T_M(y,z) \leqslant p(size_{\mathbb{R}}(y))$

(iii) A problem in $\mathrm{NP}_{\mathbb{R}}$ is $\mathrm{NP}_{\mathbb{R}}$-COMPLETE iff any other problem in $\mathrm{NP}_{\mathbb{R}}$ can be reduced to (B,A) in polynomial time.

(iv) A function f belongs to the class $FP_{\mathbb{R}}$ of functions computable in polynomial time iff it is computed by a BSS machine working in polynomial time.

An example of a $\mathrm{NP}_{\mathbb{R}}$-complete problem is given by *4-FEAS* (see [5]): given a (multivariate) polynomial f with real coefficients, $deg(f) \leqslant 4$, does there exist a real zero of f?

We are now ready to define the class $\#\mathrm{P}_{\mathbb{R}}$ of real counting problems. As already mentioned in the introduction for $\mathrm{NP}_{\mathbb{R}}$-machines it is a difference whether to count accepting paths or accepting guesses. In order to deal with functions having result ∞ for certain counting tasks we prefer the second approach.

Definition 3. The class $\#\mathrm{P}_{\mathbb{R}}$ is given by all functions $f : \mathbb{R}^{\infty} \to \{0,1\}^{\infty} \cup \{\infty\}$ such that there exists a BSS-machine M working in polynomial time and a polynomial q satisfying

$$f(y) = |\{z \in \mathbb{R}^{q(size(y))} | M(y,z) \text{ is an accepting computation}\}| \ .$$

Remark 1. (a) The dependence on q is used to avoid counting all $z \in \mathbb{R}^{\infty}$ with arbitrary size. On inputs of dimension $n \in \mathbb{N}$ machine M only takes into account that part of a guess which is bounded by a certain dimension $m := q(n)$, where q denotes some polynomial only depending on M. Without limiting the dimension of z functions in $\#\mathrm{P}_{\mathbb{R}}$ would take values in $\{0,\infty\}$ only.

(b) We identify a finite value of $f(y)$ with a natural number by taking a string in $\{0,1\}^{\infty}$ as binary expansion of that number.

The following counting problems can easily be seen to belong to $\#\mathrm{P}_{\mathbb{R}}$: The “natural” nondeterministic algorithms for establishing the corresponding decision problems (i.e. deciding whether the result is $\geqslant 1$) to be in $\mathrm{NP}_{\mathbb{R}}$ guess exactly those objects which are to be counted in the problem formulation (f.e. zeros of polynomials). We collect the problems in the following definition because they will be important later on in section 4 to separate the classes within the logical hierarchy for $\#\mathrm{P}_{\mathbb{R}}$.

Definition 4. (1) *#4-FEAS*: Given a (multivariate) polynomial f with real coefficients, $deg(f) \leqslant 4$, count the number of real zeros of f.

(2) $\#PS_3$: Given a disjunction of polynomial equations

$$p_1(x) = 0 \vee p_2(x) = 0 \vee \cdots \vee p_s(x) = 0,$$

where each p_i is of the form

$$p_i(x) = \sum_{j=1}^{s_j} f_{ij}^2(x)\ ,\ deg(f_{ij})\leqslant 2\ ,\ s_j \in \mathbb{N}$$

and each f_{ij} depends on exactly three variables (among the components of $x = (x_1,\dots,x_n)$). Count the number of points in the union of the varieties $V(p_i)$, i.e. the number of solutions of at least one equation.

(3) *#SC*: Given a sequence $x_1,\dots,x_n$ of real numbers, count the number of sign changes in the sequence, i.e. the number of pairs $(i,j) \in \{1,\dots,n\}^2$ s.t. $i < j, x_i \cdot x_j < 0$ and $x_k = 0$ for every k between i and j.

(4) $\#SC_{\text{red}}$: This is a reduced form of problem *#SC*. Again given a sequence $x_1,\dots,x_n$ of real numbers, count the number of indices $i \in \{1,\dots,n\}$ such that there is at least one more sign change with respect to x_i, i.e. there is at least one $j > i$ such that $x_i \cdot x_j < 0$. Note that for all sequences we have $\#SC \leqslant \#SC_{\text{red}}$.

Note that from a computational point of view $\#PS_3$ is as hard as #4-*FEAS*. In fact, any $\text{NP}_{\mathbb{R}}$ problem can be reduced to the solvability question of a single polynomial $p = \sum f_j^2$ where the f_j satisfy the above conditions (the direct approach gives at most three variables for each f_j but this can easily be changed to exactly three variables; see proof of Theorem 5 below). From a logical point of view $\#PS_3$ gives a little more information which causes the counting problem to fall into a lower class of the hierarchy for $\#\text{P}_{\mathbb{R}}$ we are going to define.

As already mentioned above these counting tasks provide fundamental problems within the area of (semi-) algebraic geometry. Knowing the number of zeros of a (nondegenerated) polynomial system, for example, is crucial for homotopy methods used to compute approximations of zeros (see [1, 21]).

The following theorem substantiates the introduction of $\#\text{P}_{\mathbb{R}}$:

Theorem 1. *Every function in* $\#\text{P}_{\mathbb{R}}$ *is computable in simply exponential time.*

Proof. Let $f \in \#\text{P}_{\mathbb{R}}$ and M be the corresponding machine, i.e. for $y \in \mathbb{R}^\infty$ we have

$$f(y) = |\{z \in \mathbb{R}^m | M(y,z) \text{ is an accepting computation}\}|,$$

where $n, m \in \mathbb{N}$ and $m = q(n)$ for some polynomial q depending on M only. The decision problem "given $y \in \mathbb{R}^\infty$, is $f(y)\geqslant 1$?" obviously, belongs to $\text{NP}_{\mathbb{R}}$. Thus, we can find a degree 4 polynomial p_y depending on the guess z and some additional variables u such that p_y has a real zero if and only if y is accepted by M for some computation. Moreover, the proof of $\text{NP}_{\mathbb{R}}$-completeness of 4-*FEAS* in [5] shows that $p_y(z,u) = 0$ iff z is an appropriate guess for y, i.e. iff $M(y,z)$ accepts; the u denote new variables introduced during the reduction process.

One problem occurs at this point: the number of solutions of $p_y(z,u) = 0$ might increase with respect to the number of accepting guesses. This happens if the reduction has to deal with a test instruction t: "is $x\geqslant 0$?" performed by M. In order to express the outcome of this test via a polynomial equation a new variable u_t is introduced. The reduction produces an equation

$$\begin{aligned}&\big((x \cdot u_t^2 + 1)^2 + p_1^2\big) \cdot \big((x \cdot u_t^2 - 1)^2 + p_2^2\big) \cdot \\ &\quad \big((x^2 + u_t^2) \cdot (x^2 + (u_t - 1)^2) + p_3^2\big) = 0, \qquad (*)\end{aligned}$$

where p_1, p_2, and p_3 express the machine action in case $x < 0, x > 0, x = 0$ resp. and $p_i = 0$ has a unique solution. Hence, we see that in all of these cases two choices of u_t are possible. Thus, an equation of form $(*)$ doubles the number of solutions. All other equations produced during the reduction have a unique solution. In order to free ourselves from the necessity to keep track of the number of test instructions passed by M during a computation we artificially increase the number of solutions as well for other kind of instructions. If an equation $p(z,u) = 0$ has a unique solution introducing a new variable v gives two solutions of $p(z,u)^2 + v^2 \cdot (v-1)^2 = 0$. Thus, we can guarantee the following: If the reductuion process for y takes s steps and if M accepts y for $f(y)$ many guesses then the resulting polynomial $p_y(z,u)$ has $f(y) \cdot 2^s$ many zeros. The running time of the reduction is independent of y and can be efficiently computed, thus s and 2^s are computable in polynomial time. It follows that knowing the number of points in $V(p_y) := \{(z,u) | p_y(z,u) = 0\}$ suffices to compute $f(y)$ by dividing the former by 2^s.

If the variety $V(p_y)$ is of dimension zero its number of connected components gives the value we are looking for. This number can be computed in simply exponential time (see [14]). On the other hand, $f(y) = \infty$ iff $dim\, V(p_y) \geqslant 1$. The latter is equivalent to the existence of at least one coordinate axis – say z_1 for simplicity – such that the projection of $V(p_y)$ into the direction z_1 contains an interval.

This can be expressed via the formula

$$\psi : \exists a < b \;\forall t \in (a,b)\; \exists \tilde{z}, \tilde{u}\;\; p_y(t, \tilde{z}, \tilde{u}) = 0.$$

In order to check validity of ψ we perform quantifier elimination, which in the above case can be done in simply exponential time applying for example the algorithms in [13] or [19] (note that the above formula ψ has a fixed number of three alternating quantifiers only). □

Remark 2. The above used fact that a reduction to 4-*FEAS* does not only carry over the existence of certain objects but also information about these objects is closely related to the existence of the so-called NP universal problems, see [2, 18].

Recall that the above result also holds true for #P, this time because the number of accepting guesses a priori is bounded exponentially.

Clearly, if $P_{\mathbb{R}} \neq NP_{\mathbb{R}}$ we have $\#P_{\mathbb{R}} \not\subset FP_{\mathbb{R}}$. On the other hand, any function $f : \mathbb{R}^\infty \to \{0,1\}^\infty \cup \{\infty\}$ in $FP_{\mathbb{R}}$ also belongs to $\#P_{\mathbb{R}}$.

Lemma 1. *Any function* $f : \mathbb{R}^\infty \to \{0,1\}^\infty \cup \{\infty\}$ *in* $FP_{\mathbb{R}}$ *belongs to* $\#P_{\mathbb{R}}$.

Proof. Let M be a polynomial time machine computing f. Because of the polynomial running time any function value $f(x) \neq \infty$ represents (the binary expansion of) a natural number $\leqslant 2^{q(size_{\mathbb{R}}(x))}$ (where q is a polynomial). A nondeterministic machine establishing $f \in \#P_{\mathbb{R}}$ works as follows: Take as input x as well as a guess $z \in \mathbb{R}$. Use M to compute $f(x)$. Then accept any z if $f(x) = \infty$, and if not accept z if z is an integer, $1 \leqslant z \leqslant f(x)$. This can be checked in time $O(log f(x)) \leqslant O(q(n))$. The set of accepted inputs consists of all (x,z) where $z \in \{1,\dots,f(x)\}$. $\square$

3. $\mathbb{R}$-structures and counting

In this section we first recall basic notions of $\mathbb{R}$-structures and their logics. This concept was first introduced in [11]; we refer to this paper as well as [7] for a closer study of descriptive complexity in the BSS model.

Then we relate it to counting problems.

3.1. $\mathbb{R}$-structures

We suppose the reader familiar with the main terminology of logic as well as with the concepts of vocabulary, first-order formula or sentence, interpretation and structure (see for example [8]).

Definition 5. Let L_s, L_f be finite vocabularies where L_s may contain relation and function symbols, and L_f contains function symbols only. A $\mathbb{R}$-*structure of signature* $\sigma = (L_s, L_f)$ is a pair $\mathfrak{D} = (\mathscr{A}, \mathscr{F})$ consisting of

(i) a finite structure $\mathscr{A}$ of vocabulary L_s, called the *skeleton* of $\mathfrak{D}$, whose universe A will also be said to be the *universe* of $\mathfrak{D}$, and

(ii) a finite set $\mathscr{F}$ of functions $X : A^k \to \mathbb{R}$ interpreting the function symbols in L_f.

Definition 6. Let $\mathfrak{D}$ be a $\mathbb{R}$-structure with skeleton $\mathscr{A}$. We denote by $|A|$ and also by $|\mathfrak{D}|$ resp. the cardinality of the universe A of $\mathscr{A}$. This number is called the size of the structure $\mathfrak{D}$. A $\mathbb{R}$-structure $\mathfrak{D} = (\mathscr{A}, \mathscr{F})$ is *ranked* if there is a unary function symbol $r \in L_f$ whose interpretation ρ in $\mathscr{F}$ bijects A with $\{0, 1, \dots, |A| - 1\}$. The function ρ is called *ranking*. We will write $i < j$ for $i, j \in A$ iff $\rho(i) < \rho(j)$. A k-ranking on A is a bijection between A^k and $\{0, 1, \dots, |A|^k - 1\}$. It can easily be defined if a ranking is available. We denote by ρ^k the interpretation of the k-ranking induced by ρ.

Throughout this paper we suppose all $\mathbb{R}$-structures to be ranked. We therefore notationally suppress the symbol $\leqslant$ in the sets $\mathscr{F}$ considered. The basic logic important for our work is first-order logic which we are going to define now.

Fix a countable set $V = \{v_0, v_1, \dots,\}$ of variables. These variables range only over the skeleton; we do not use element variables taking values in $\mathbb{R}$.

Definition 7. The language $FO_{\mathbb{R}}$ contains, for each signature $\sigma = (L_s, L_f)$ a set of formulas and terms. Each term t takes, when interpreted in some $\mathbb{R}$-structure, values in either the skeleton, in which case we call it an *index term*, or in $\mathbb{R}$, in which case we call it a *number term*. Terms are defined inductively as follows

(i) The set of index terms is the closure of the set V of variables under applications of function symbols of L_s.

(ii) Any real number is a number term.

(iii) If $h_1, \dots, h_k$ are index terms and X is a k-ary function symbol of L_f then $X(h_1, \dots, h_k)$ is a number term.

(iv) If t, t' are number terms, then so are $t + t'$, $t - t'$, $t \times t'$, t/t' and $\operatorname{sign}(t)$.

Atomic formulas are equalities $h_1 = h_2$ of index terms, equalities $t_1 = t_2$ and inequalities $t_1 < t_2$ of number terms, and expressions $P(h_1, \dots, h_k)$ where P is a k-ary predicate symbol in L_s and $h_1, \dots, h_k$ are index terms.

The set of formulas of $FO_{\mathbb{R}}$ is the smallest set containing all atomic formulas and which is closed under Boolean connectives and quantification $(\exists v)\psi$ and $(\forall v)\psi$. Note that we do *not* consider formulas $(\exists x)\psi$ where x ranges over $\mathbb{R}$.

Example 1 (*cf. Cucker and Meer* [7] *and Grädel and Meer* [11]). Let L_s be the empty set and L_f be $\{r, X\}$ where both function symbols have arity 1. Then, a simple class of ranked $\mathbb{R}$-structures with signature (L_s, L_f) is obtained by letting $\mathscr{A}$ be a finite set A, $r^{\mathfrak{D}}$ any ranking on A and $X^{\mathfrak{D}}$ any unary function $X^{\mathfrak{D}} : A \to \mathbb{R}$. Since $r^{\mathfrak{D}}$ bijects A with $\{0, 1, \dots, n-1\}$ where $n = |A|$, this $\mathbb{R}$-structure is a point $x_{\mathfrak{D}}$ in $\mathbb{R}^\infty$. Conversely, for each point $x \in \mathbb{R}^\infty$ there is an $\mathbb{R}$-structure $\mathfrak{D}$ such that $x = x_{\mathfrak{D}}$. Thus, this class of structures models $\mathbb{R}^\infty$.

On the other hand, any $\mathbb{R}$-structure $\mathfrak{D} = (\mathscr{A}, \mathscr{F})$ can be identified with a vector $e(\mathfrak{D}) \in \mathbb{R}^\infty$ using a natural encoding. To this aim choose a ranking on A. Without loss of generality, the skeleton of $\mathfrak{D}$ can be assumed to consist of the plain set A only by replacing all functions and relations in L_s by their corresponding characteristic functions – the latter being considered as elements of the set $\mathscr{F}$. Now, using the ranking each of the functions X in $\mathscr{F}$ can be represented by a vector $v_X \in \mathbb{R}^m$ for some appropriate m. The concatenation of all these v_X yields the encoding $e(\mathfrak{D}) \in \mathbb{R}^\infty$. Note that the length of $e(\mathfrak{D})$ is polynomially bounded in $|A|$; moreover, for all $\mathbb{R}$-structures $\mathfrak{D}$, all rankings E on A and all functions $X : A^k \to \mathbb{R}$ the property that X represents the encoding $e(\mathfrak{D})$ of $\mathfrak{D}$ with respect to E is first-order expressible (see [11]).

Example 1 allows us to speak about complexity classes among $\mathbb{R}$-structures. If S is a set of $\mathbb{R}$-structures closed under isomorphisms, we say that S belongs to a complexity class $\mathscr{C}$ over the reals if the set $\{e(\mathfrak{D}) \mid \mathfrak{D} \in S\}$ belongs to $\mathscr{C}$.

Remark 3. If ρ is a ranking on A and $|A| = n$ then, there are elements $\mathtt{o}, \mathtt{l} \in A$ such that $\rho(\mathtt{o}) = 0$ and $\rho(\mathtt{l}) = n - 1$. Note that these two elements are first-order definable.

In order to describe the class $NP_{\mathbb{R}}$ it turns out to be fruitful also considering an extension of first-order logic.

Definition 8. We say that ψ is an *existential second-order sentence* (of signature $\sigma = (L_s, L_f)$) if $\psi = \exists Y_1 \ldots \exists Y_r \phi$ where ϕ is a first-order sentence in $FO_{\mathbb{R}}$ of signature $(L_s, L_f \cup \{Y_1, \ldots, Y_r\})$. The symbols $Y_1, \ldots, Y_r$ will be called *function variables*. The sentence ψ is true in a $\mathbb{R}$-structure $\mathfrak{D}$ of signature σ when there exist interpretations of $Y_1, \ldots, Y_r$ such that ϕ holds true on $\mathfrak{D}$. The set of existential second-order sentences will be denoted by $\exists SO_{\mathbb{R}}$. Together with the interpretation above it constitutes *existential second-order logic*.

The following example already gives the right idea to relate counting problems with logics on $\mathbb{R}$-structures.

Example 2 (*Grädel and Meer* [11]). Let us see how to describe 4-*FEAS* with an existential second-order sentence. Consider the signature $(\emptyset, \{r, c\})$ where the arities of r and c are 1 and 4, respectively, and require that r is interpreted as a ranking.

Let $\mathfrak{D} = (\mathscr{A}, \mathscr{F})$ be any $\mathbb{R}$-structure where $\mathscr{F}$ consists of interpretations $C : A^4 \to \mathbb{R}$ and $\rho : A \to \mathbb{R}$ of c and r. Let $n = |A| - 1$ so that ρ bijects A with $\{0, 1, \ldots, n\}$. Then $\mathfrak{D}$ defines a homogeneous polynomial $\widehat{g} \in \mathbb{R}[X_0, \ldots, X_n]$ of degree four, namely

$$\widehat{g} = \sum_{(i,j,k,\ell) \in A^4} C(i,j,k,\ell) X_i X_j X_k X_\ell .$$

We obtain an arbitrary, that is, not necessarily homogeneous, polynomial $g \in \mathbb{R}[X_1, \ldots, X_n]$ of degree four by setting $X_0 = 1$ in $\widehat{g}$. We also say that $\mathfrak{D}$ defines g. Note that for every polynomial g of degree four in n variables there is a $\mathbb{R}$-structure $\mathfrak{D}$ of size $n+1$ such that $\mathfrak{D}$ defines g.

Denote by $\mathtt{o}, \mathtt{1}, \bar{\mathtt{o}}$ and $\bar{\mathtt{1}}$ the first and last elements of A and A^4 with respect to ρ and ρ^4 respectively. The following sentence quantifies two functions $X : A \to \mathbb{R}$ and $Y : A^4 \to \mathbb{R}$

$$\begin{aligned}
\psi \equiv (\exists X)(\exists Y) \Big(& Y(\bar{\mathtt{o}}) = C(\bar{\mathtt{o}}) \ \&\ Y(\bar{\mathtt{1}}) = 0 \ \&\ X(\mathtt{o}) = 1 \\
& \&\ \forall u_1 \cdots \forall u_4 \, [u \neq \bar{\mathtt{o}} \Rightarrow \ \exists v_1 \cdots \exists v_4 \, (\rho^4(u) = \rho^4(v) + 1) \\
& \&\ Y(u) = Y(v) + C(u) X(u_1) X(u_2) X(u_3) X(u_4)] \Big).
\end{aligned}$$

Here, if $a_i = \rho^{-1}(i)$ for $i = 1, \ldots, n$ then, $(X(a_1), \ldots, X(a_n)) \in \mathbb{R}^n$ describes the zero of g and $Y(u)$ is the partial sum of all its monomials up to $u = (u_1, \ldots, u_4) \in A^4$ evaluated at the point $(X(a_1), \ldots, X(a_n))$.

The sentence ψ describes 4-*FEAS* in the sense that for any $\mathbb{R}$-structure $\mathfrak{D}$ it holds $\mathfrak{D} \models \psi$ if and only if the polynomial g of degree four defined by $\mathfrak{D}$ has a real zero.

The fact that existential second-order logic describes a $NP_{\mathbb{R}}$-complete problem is not fortuitous.

Theorem 2 (Grädel and Meer [11]). *Let (F, F^+) be a decision problem of $\mathbb{R}$-structures as explained in Example* 1. *Then $(F, F^+) \in \mathrm{NP}_{\mathbb{R}}$ if and only if there exists an $\exists\mathrm{SO}_{\mathbb{R}}$-formula ψ such that $F^+ = \{\mathfrak{D} \in F | \mathfrak{D} \models \psi\}$.*

Here two remarks are essential for what will follow: The above example shows that the number of choices for (X, Y) which cause the sentence ψ to hold true equals the number of real zeros of the given polynomial (note that Y is unique as soon as X is determined). This leads to the idea of counting the satisfying assignments for first-order formulas. On the other hand, first-order formula with two quantifier alternations suffice to represent a $\mathrm{NP}_{\mathbb{R}}$ computation. Thus the hierarchy we will define collapses after at most five levels.

Definition 9. (a) Let $\sigma = (L_s, L_f)$ be a vocabulary and let $\mathbf{D}$ be a family of $\mathbb{R}$-structures over vocabulary σ. A function $f : \mathbf{D} \to \{0,1\}^\infty \cup \{\infty\}$ belongs to class $\#FO_{\mathbb{R}}$ if the following holds: there exists a first-order formula $\psi_{\mathbf{D}}$ over vocabulary $(L_s, L_f \cup \{X_1, \ldots, X_l\})$ such that

$$f(\mathfrak{D}) = |\{(X_1, \ldots, X_l, z_1, \ldots, z_m) : \mathfrak{D} \models \psi_{\mathbf{D}}(X, z)\}|.$$

Here $X := (X_1, \ldots, X_l)$ denotes a sequence of functions from some A^{k_i} to $\mathbb{R}$, $1 \leqslant i \leqslant l$, and $z := (z_1, \ldots, z_m)$ denotes a sequence of first-order variables. Moreover, l as well as m are natural numbers with $l + m > 0$ depending on $\psi_{\mathbf{D}}$ only.

(b) The subclasses $\#\Sigma_{n\mathbb{R}}$ and $\#\Pi_{n\mathbb{R}}, n \in \mathbb{N}$ of $\#FO_{\mathbb{R}}$ are defined similarly by restricting $\psi_{\mathbf{D}}$ to be a $\Sigma_{n\mathbb{R}}$ resp. $\Pi_{n\mathbb{R}}$ formula. Here $\Sigma_{n\mathbb{R}}$ resp. $\Pi_{n\mathbb{R}}$ consist of those formulas in prenex normal form with n alternating blocks of first-order quantifiers beginning with $\exists$ resp. $\forall$.

Remark 4. In many situations it turns out to be more natural splitting the counting objects into X and z, the reason why we took the above definition instead of counting X solely.

In the following example, the problems of Definition 4 are classified into some of these classes. To this aim we use straightforward representations of the according problem instances as $\mathbb{R}$-structures.

Example 3. (1) #4-*FEAS*: We have already seen how to describe the 4-*FEAS* problem via an existential second-order formula in Example 2. The according counting function is then given by

$$|\{(X, Y) | \psi_{4\text{-}FEAS}(X, Y)\}| \ ,$$

where

$$\psi_{4\text{-}FEAS}(X, Y) \equiv \Big(Y(\bar{o}) = C(\bar{o}) \ \& \ Y(\bar{1}) = 0 \ \& \ X(o) = 1 \ \&$$

$$\& \ \forall u \in A^4 \ \forall v \in A^4 \ \exists w \in A^4 \ \ [u \neq \bar{o} \Rightarrow$$

$$\{(u > v \wedge w > v) \rightarrow w \geqslant u\} \,\&$$

$$\& \; Y(u) = Y(v) + C(u)X(u_1)X(u_2)X(u_3)X(u_4)]\Big),$$

which shows #4-*FEAS* to belong to $\#\Pi_{2\mathbb{R}}$.

(2) $\#PS_3$: Our representation of a polynomial disjunction $\{p_1,\ldots,p_s\}$ as $\mathbb{R}$-structure on the first glance may seem a little artificial. However we need to proceed in this way for our later purposes.

The size of such a system depends on the number n of variables and and the number t of different quadratic terms f_{ij}^2 appearing in the description of the system. We thus take as universe the set $A := A_1 \uplus A_2$ where $A_1 = \{0,\ldots,n\}$ and $A_2 = \{1,\ldots,t\}$. We use a constant function K from A^0 to $\mathbb{R}$ which is interpreted as the number n of variables. This also enables us to deal with A_1 and A_2 separately. Our intention is to identify each $u \in A_2$ with a polynomial f_{ij}. To this aim two other constant functions $eq : A_2 \rightarrow \mathbb{R}$ and $pol : A_2 \rightarrow \mathbb{R}$ belong to the set L_f of the signature. More explicitely, for $u \in A_2$ we interpret $eq(u) = i$ as an equation number and $pol(u) = j$ as index of the polynomial in equation i (note that here we use i both as element in the universe and as real. This is possible because of the ranking). Finally, a function $C : A_2 \times A_1^2 \rightarrow \mathbb{R}$ interprets the coefficients of the polynomials: $C(u,v_1,v_2)$ gives the coefficient of $x_{v_1} \cdot x_{v_2}$ in f_{ij}, where $eq(u) = i, pol(u) = j$.

Note that the image of eq is not necessarily an initial part of $\mathbb{N}$, i.e. we do not force the enumeration of the equations in the system to be of the shape $1,2,\ldots,s$ for some s. This will be useful later on.

Now, we count the assignments $X : A \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned}
\psi_{PS_3}(X) \equiv \exists i \in \{1,\ldots,t\} \; \forall u \in A_2 \; \Bigg[\; & eq(u) = i \text{ and } pol(u) = j \\
& \Rightarrow \left\{ \forall v_1, v_2, v_3 \in A_1 \quad \left(\bigwedge_{1 \leqslant l \leqslant 3} \; \bigvee_{0 \leqslant k \leqslant 3} C(u, v_l, v_k) \neq 0 \right) \right. \\
& \left. \Rightarrow \sum_{0 \leqslant k \leqslant l \leqslant 3} C(u, v_l, v_k) \cdot X(v_l) \cdot X(v_k) = 0 \right\} \Bigg].
\end{aligned}$$

Here we interpret v_0 as $\mathrm{o} \in A_1$. Note that in the above formula the left side of the second implication holds true iff v_1, v_2, v_3 are those variables f_{ij} depends on. The sum appearing in the right side has a constant number of terms and thus is expressible with a quantifier free formula.

It follows $\#PS_3 \in \#\Sigma_{2\mathbb{R}}$.

Let us mention that restricting the $\#PS_3$ problem to a single equation $p = 0$, where $p = \sum f_j^2$, gives a counting problem in $\#\Pi_{1\mathbb{R}}$ (just remove the $\exists i$ quantifier in ψ_{PS_3}).

(3) $\#SC$: A sequence of real numbers is represented as $\mathbb{R}$-structure $\mathfrak{D} = (\mathscr{A}, \mathfrak{S})$, where $\mathfrak{S} : A \to \mathbb{R}$. The counting function for $\#SC$ is given as

$$|\{(i,j) \in A^2 \mid \mathfrak{D} \models i < j \wedge \operatorname{sign} \mathfrak{S}(i) \cdot \operatorname{sign} \mathfrak{S}(j) \\ = -1 \wedge \forall k\, (i < k < j \;\Rightarrow\; \mathfrak{S}(k) = 0)| \,.$$

We thus have $\#SC \in \#\Pi_{1\mathbb{R}}$.

(4) $\#SC_{\text{red}}$: For the reduced sign change problem we consider the same structure as in 2). The corresponding counting function now is

$$|\{i \in A \mid \mathfrak{D} \models \exists j \in A \; j > i \wedge \operatorname{sign} \mathfrak{S}(i) \cdot \operatorname{sign} \mathfrak{S}(j) = -1\}| \,,$$

showing $\#SC_{\text{red}} \in \#\Sigma_{1\mathbb{R}}$.

The above problems are not chosen arbitrarily. In the next section some of them serve to show the distinctness of the corresponding classes they belong to.

The justification for defining class $\#FO_{\mathbb{R}}$ with respect to $\#\mathrm{P}_{\mathbb{R}}$ is given in the following

Theorem 3. $\#\mathrm{P}_{\mathbb{R}} = \#FO_{\mathbb{R}} = \#\Pi_{2\mathbb{R}}$.

Proof. The inclusion $\#FO_{\mathbb{R}} \subset \#\Pi_{2\mathbb{R}}$ can either be obtained from the proof of Theorem 2 in [11] or – more basically – from a fundamental argument (already present in [10] [1]): if a formula ρ with more than two alternating quantifiers is given one can reduce this number (with respect to the task of counting satisfying assignments) by defining new predicates for the subformulas of ρ. Seperating the latter definition in a new formula the shape of quantifiers can be reduced up to $\forall\exists$. It follows $\#FO_{\mathbb{R}} = \#\Pi_{2\mathbb{R}}$ (cf. Example 3).

Any problem in $\#FO_{\mathbb{R}}$ can be proven to belong to $\#\mathrm{P}_{\mathbb{R}}$ by taking a $\mathrm{NP}_{\mathbb{R}}$ machine which guesses a satisfying assignment. The reverse inclusion follows by noting that in the proof of Theorem 2 every accepting computation on a $\mathrm{NP}_{\mathbb{R}}$ machine corresponds to a satisfying assignment of the related existential second-order formula. □

4. The five distinct levels

It is known that for ordered finite structures in the discrete setting the first level $\#\Sigma_0$ of the hierarchy for #P consists of polynomial time computable functions only, whereas problems in the second level $\#\Sigma_1$ at least allow a fully polynomial time randomized approximation scheme (see [20]). Over the reals we can say more because of the possibly uncountable number of satisfying assignments for a function from A^k to $\mathbb{R}$.

Theorem 4. $\#\Sigma_{1\mathbb{R}} \subset FP_{\mathbb{R}}$.

[1] Thanks to the referee for pointing out this reference.

Proof. Let a function $f:\mathbf{D} \to \{0,1\}^\infty \cup \{\infty\}$ in $\#\Sigma_{1\mathbb{R}}$ be given by

$$f(\mathfrak{D}) = |\{(X_1,\dots,X_l,z_1\dots,z_m)|\mathfrak{D} \models \exists x \in A^k \ \psi(X,x,z)\}|,$$

where ψ is quantifier free. Fix a $z^* \in A^m$ and an $x^* \in A^k$. Consider all terms $X_i(y)$ appearing in this formula (where $X = (X_1,\dots,X_l)$ as usual). For each one among them (i.e. different i or different argument y) we introduce a real variable. Note that there will be at most a constant number s of variables where s only depends on ψ but not on the size n of $\mathfrak{D}$. (Up to this point the proof closely followed the one in [20] for showing $\#\Sigma_0 \in FP$). Thus for every fixed (x^*,z^*) we obtain a quantifier-free formula in the ordered fields language $\rho(t_1,\dots,t_s)$ with s free real variables.

There are three cases to be analyzed: First assume at least one of the X_i to have an arity $\geqslant 1$. There exists a dimension n_0 such that for every structure $\mathfrak{D}$ of size $n \geqslant n_0$ the satisfyability of the above formula in $\mathfrak{D}$ implies the existence of infinitely many X s.t. $\psi(X,x^*,z^*)$ holds. This is true because in that case X_i must be defined on more than s arguments showing that for large enough structures some of these arguments do not appear as arguments of X_i in ψ. Hence, their values do not affect validity of ψ. Now if one of the sentences $\exists t_1,\dots,t_s \ \rho(t_1,\dots,t_s)$ holds in $\mathbb{R}$ then $f(\mathfrak{D}) = \infty$ for any $\mathfrak{D}$ of size at least n_0; if not $f(\mathfrak{D}) = 0$ for those $\mathfrak{D}$. For structures of size less than n_0 we can compute the counting function using a brute algorithm (see Theorem 1) working in constant time. Due to the fact that there are at most polynomially many different tuples (x^*,z^*) the first case is solvable in polynomial time.

Secondly, assume all X_i to be nullary functions. Then the corresponding first-order formula over the reals has the fixed number l of free real variables (not depending on $|\mathfrak{D}|$). By adding dummy variables we can reduce the problem in constant time to counting the zeros of a polynomial which has constant degree and a constant number of variables (with respect to n). Again this can be done in constant time, and thus once more all together in polynomial time for all choices of (x^*,z^*).

Finally, if ψ does not depend on X (i.e. we only count z) there are only polynomially many assignments for z which have to be checked. $\square$

We observe that the finite values of a function in $\#\Sigma_{1\mathbb{R}}$ are bounded by a polynomial in the size of $\mathfrak{D}$.

The above result shows that the different levels of our hierarchy do not bear much significance with respect to complexity issues. Whereas problems expressible in the two lowest levels are polynomial time computable there is a direct jump to $\mathrm{NP}_\mathbb{R}$-complete problems expressible in $\#\Pi_{1\mathbb{R}}$. The latter can be seen by considering again Example 3,2. If the question is changed to ask for common solutions of all polynomials in the system one obtains a $\#\Pi_{1\mathbb{R}}$ formula. The corresponding decision problem nevertheless is $\mathrm{NP}_\mathbb{R}$-complete. And there are problems in $\#\Pi_{2\mathbb{R}} \setminus \#\Sigma_{2\mathbb{R}}$ related to polynomial time solvable decision problems: the corrsponding part of the proof of Theorem 5 below similarly could be used for say counting the solutions of a system of linear equations.

Thus, the major importance of our hierarchy is on the model-theoretic side.

We have already seen in the last section that $\#\mathrm{P}_{\mathbb{R}}$ splits into at most five levels. The main goal of this section is to show that all these levels are distinct. To this aim we will further exploit the properties of the examples introduced in Definition 4.

Theorem 5. *The class* $\#\mathrm{P}_{\mathbb{R}}$ *splits into five different levels*

$$\#\Sigma_{0\mathbb{R}} = \#\Pi_{0\mathbb{R}} \subsetneq \#\Sigma_{1\mathbb{R}} \subsetneq \#\Pi_{1\mathbb{R}} \subsetneq \#\Sigma_{2\mathbb{R}} \subsetneq \#\Pi_{2\mathbb{R}} = \#\mathrm{P}_{\mathbb{R}} \ .$$

Proof. The proof is done by showing that every problem of Definition 4 will separate the class it belongs to from the next lower one (see Example 3).

$\#\Sigma_{0\mathbb{R}} \subsetneq \#\Sigma_{1\mathbb{R}}$: Assume $\#SC_{\mathrm{red}} \in \#\Sigma_{0\mathbb{R}}$; let ψ be the corresponding $\Sigma_{0\mathbb{R}}$ formula, i.e.

$$|\{(X,z)|\mathfrak{D} \models \psi(X,z)\}|$$

counts the number of indices causing another sign change in the sequence given by structure $\mathfrak{D}$.

Because of the proof of Theorem 4 the above ψ must be independent of such X_i's having arity $\geqslant 1$. Thus, we count $|\{(X,z)|\mathfrak{D} \models \psi(X,z)\}|$, where all X_i in X are nullary. Let m be the arity of z; consider a family of $\mathbb{R}$-structures

$$\mathfrak{D}^{(i)} = (\mathscr{A}, \mathfrak{S}^{(i)}) \ , \ \ \mathfrak{S}^{(i)} = (1,0,\ldots,0,\underbrace{-1}_{i},0,\ldots,0)$$

for i varying between 2 and $n = |A|$. Here the size n of A is assumed to be large enough to satisfy the conditions necessary below. For all $2 \leqslant i \leqslant n$ it is $\#SC_{\mathrm{red}}(\mathfrak{D}^{(i)}) = 1$. Let $(X^{(i)},z^{(i)})$ be the unique assignment with $\mathfrak{D}^{(i)} \models \psi(X^{(i)},z^{(i)})$. Obviously, $z^{(i)}$ must depend on i, that is at least one of the m components of $z^{(i)}$ must be the element $i \in A$. Otherwise ψ would not 'realize' a change of the value $\mathfrak{S}^{(i_0)}(i)$ in the input structure from -1 to 0, though this would change the outcome of the counting process to 0.

If n is large enough there exist i_0 and j_0 , $i_0 < j_0$ such that neither $z^{(i_0)}$ depends on j_0 nor $z^{(j_0)}$ depends on i_0 in the above sense (this follows by a simple combinatorial argument). Consider the new structure $\mathfrak{D}^{(i_0,j_0)} = (\mathscr{A}, \mathfrak{S}^{(i_0,j_0)})$ where

$$\mathfrak{S}^{(i_0,j_0)} = (1,0,\ldots,0,\underbrace{-1}_{i_0},0,\ldots,0,\underbrace{-1}_{j_0},0,\ldots,0) \ .$$

The above conditions on i_0,j_0 imply $\mathfrak{D}^{(i_0,j_0)} \models \psi(X^{(i_0)},z^{(i_0)})$ as well as $\mathfrak{D}^{(i_0,j_0)} \models \psi(X^{(j_0)},z^{(j_0)})$ (note that $X^{(i_0)}$ and $X^{(j_0)}$ do not depend on any element of A). However, $z^{(i_0)} \neq z^{(j_0)}$ and $\#SC_{\mathrm{red}}(\mathfrak{D}^{(i_0,j_0)}) = 1$ gives a contradiction.

$\#\Sigma_{1\mathbb{R}} \subsetneq \#\Pi_{1\mathbb{R}}$: The inclusion $\#\Sigma_{1\mathbb{R}} \subset \#\Pi_{1\mathbb{R}}$ follows exactly as in [20], noting that instead of counting tuples (X,z) satisfying a sentence $\exists x \ \psi(X,x,z)$ it suffices to count those tuples (X,x^*,z) satisfying $\psi(X,x^*,z)$, where x^* denotes the lexicographic smallest among those x with $\psi(X,x,z)$. The latter condition is $\Pi_{1\mathbb{R}}$ definable.

To show distinctness of both classes one could prove $\#SC \notin \#\Sigma_{1\mathbb{R}}$ using a similar argument as before. However, one can proceed easier. Consider the $\#\Pi_{1\mathbb{R}}$-formula

$$\psi(X) : \forall y\ X(y) = 0 \ \vee \ X(y) = 1 \ .$$

The value of the corresponding counting problem is 2^n. Thus, according to the observation following the proof of Theorem 4 this counting problem cannot be expressed via a $\#\Sigma_{1\mathbb{R}}$ formula.

$\#\Pi_{1\mathbb{R}} \subsetneqq \#\Sigma_{2\mathbb{R}}$: We are going to show $\#PS_3 \notin \#\Pi_{1\mathbb{R}}$. For sake of notational simplicity we will consider below polynomial systems with at most 3 variables present in the according f_{ij}'s. However, note that these systems easily could be enlarged by introducing additional variables such that the proof remains valid and the arising systems contain exactly 3 variables for every f_{ij}. For example, change the polynomial t^2 to the system $(t-u-v)^2 = (t+u+v)^2 = (t-u+v)^2 = 0$, where u, v are new variables and $t = u = v = 0$ is the only solution; other cases are treated similarly.

Now, assume $\#PS_3 \in \#\Pi_{1\mathbb{R}}$ and let the witnessing formula be

$$|\{(X,z)|\mathfrak{D} \models \forall y\ \psi(X,y,z)\}| \ .$$

For $1 \leqslant i \leqslant n$ let $p_i(x) := \sum_{j=1}^{n+1} f_{ij}^2$, where $f_{ij} = x_j$, $1 \leqslant j \leqslant n$ and $f_{i,n+1} = x_i - 1$; let $\mathfrak{D}^{(i)}$ be the structure representing the polynomial disjunction

$$p_1 = 0 \ \vee \ \cdots \ \vee \ p_{i-1} = 0 \ \vee \ \sum_{k=1}^{n} x_k^2 = 0 \ \vee \ p_{i+1} = 0 \ \vee \ \cdots \ \vee \ p_n = 0$$

(i.e. the term $(x_i - 1)^2$ is removed in p_i).

Similarly, for $1 \leqslant i \neq l \leqslant n$ let $\mathfrak{D}^{(i,l)}$ be the structure for the disjunction

$$\begin{aligned} &p_1 = 0 \ \vee \ \cdots \ \vee \ p_{i-1} = 0 \ \vee \ \sum_{k=1}^{n} x_k^2 = 0 \\ &\vee \ p_{i+1} = 0 \ \vee \ \ldots \ \vee \ p_{l-1} = 0 \ \vee \ \sum_{k=1}^{n} x_k^2 = 0 \ \vee \ p_{l+1} = 0 \ \vee \ \cdots \ \vee \ p_n = 0. \end{aligned}$$

The counting function gives $\#PS_3(\mathfrak{D}^{(i)}) = \#PS_3(\mathfrak{D}^{(i,l)}) = 1\ \forall 1 \leqslant i < l \leqslant n$ (the only zero of at least one equation is $\underline{0}$). Let $(X^{(i)}, z^{(i)})$ be the according assignment for $\mathfrak{D}^{(i)} \models \forall y\ \psi(X^{(i)}, y, z^{(i)})$. Note that we present the above polynomial system in such a way that the term $f_{i,n+1}^2 = (x_i - 1)^2$ in each p_i is an own part of the structure (and not comprised with the monomial x_i^2). Now, $z^{(i)}$ must depend on the coefficients of at least one polynomial f_{ij} in equation $p_i = 0$. For if not we could pass to that substructure of $\mathfrak{D}^{(i)}$ consisting of all former equations exept $p_i = 0$. The latter structure represents an unsolvable disjunction, even though it would still be a model for $\forall y \psi(X^{(i)}, y, z^{(i)})$, a contradiction. If the size of the initial structure is large enough there must exist indices i_0, k_0 such that $z^{(i_0)}$ does not depend on any of the coefficients of the $f_{k_0,j}$ in the system $\mathfrak{D}^{(k_0)}$ and $z^{(k_0)}$ does not depend on any of the coefficients of $f_{i_0,j}$ in $\mathfrak{D}^{(i_0)}$.

Let us analyze the substructure $\mathfrak{D}^{(i_0,k_0)}$ which results from $\mathfrak{D}^{(i_0)}$ by removing $f_{k_0,n+1}$ or from $\mathfrak{D}^{(k_0)}$ by removing $f_{i_0,n+1}$. Note that $\mathfrak{D}^{(i_0,k_0)}$ and $\mathfrak{D}^{(k_0,i_0)}$ are the same structures.

Denote by $(\tilde{X}^{(i_0)}, \tilde{z}^{(i_0)})$ resp. $(\tilde{X}^{(k_0)}, \tilde{z}^{(k_0)})$ the assignments obtained from $(X^{(k_0)}, z^{(k_0)})$ by deleting all occurences of the above polynomials. The special choice of i_0, k_0 guarantees $\tilde{z}^{(i_0)} \neq \tilde{z}^{(k_0)}$ as well as

$$\mathfrak{D}^{(i_0,k_0)} \models \forall y \ \psi(\tilde{X}^{(i_0)}, y, \tilde{z}^{(i_0)})$$

and

$$\mathfrak{D}^{(i_0,k_0)} \models \forall y \ \psi(\tilde{X}^{(k_0)}, y, \tilde{z}^{(k_0)})$$

But this would imply $\#PS_3(\mathfrak{D}^{(i_0,k_0)}) \geqslant 2$, a contradiction.

$\#\Sigma_{2\mathbb{R}} \subsetneq \#\Pi_{2\mathbb{R}}$: Suppose $\#4\text{-}FEAS \in \#\Sigma_{2\mathbb{R}}$ and let

$$|\{(X,z) | \mathfrak{D} \models \exists x \ \forall y \ \psi(X,x,y,z)\}| \ , \forall \mathfrak{D} \in \mathbf{D}$$

be the according counting formula in $\Sigma_{2\mathbb{R}}$. Here $\mathbf{D}$ is the set of $\mathbb{R}$-structures representing a degree 4 polynomial as in Example 2. Consider a particular polynomial f in n variables given by

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n} (a_i \cdot x_i - 1)^2 \ .$$

Here the a_i are taken as nonvanishing real coefficients. Thus, f has exactly one real zero. Let (X^*, z^*) be the unique assignment such that

$$\mathfrak{D} \models \exists x \ \forall y \ \psi(X^*, x, y, z^*)$$

holds for the structure $\mathfrak{D}$ representing f. Moreover, fix an x^* with

$$\mathfrak{D} \models \ \forall y \ \psi(X^*, x^*, y, z^*) \ .$$

Now, if in the above situation n is sufficiently large we can find an $i_0 \in A$ such that both x^* and z^* do not depend on i_0. Note that according to our representation of f as $\mathbb{R}$-structure this implies the validity of $\forall y \ \psi(X^*, x^*, y, z^*)$ to be independent of the value for a_{i_0}. Consider a substructure $\tilde{\mathfrak{D}}$ of $\mathfrak{D}$ by taking $\tilde{A} = A \setminus \{i_0\}$ and $\tilde{X}$ being obtained from X^* by deleting all occurences of those tuples involving i_0. The polynomial $\tilde{f}$ related with $\tilde{\mathfrak{D}}$ is of the form

$$\tilde{f} = 1 + \sum_{\substack{j=1 \\ j \neq i_0}}^{n} (a_j \cdot x_j - 1)^2$$

and thus has no zero. On the other hand, $\tilde{\mathfrak{D}} \models \forall y \ \psi(\tilde{X}, x^*, y, z^*)$ still holds because x^* and z^* do not depend on i_0 and universal formulas remain valid by passing to substructures. Hence, the above $\Sigma_{2\mathbb{R}}$ formula does not count the zeros of $\tilde{f}$ which gives the claim. $\square$

Remark 5. We point out once more that the reason for the above separations is not hidden behind the computational (!) difficulty of problems like $\#4\text{-}FEAS$ or $\#PS_3$.

For example, the separation between $\#\Pi_{1\mathbb{R}}$ and $\#\Sigma_{2\mathbb{R}}$ can be shown as well by using almost the same proof as above for analyzing the formula

$$\psi(t):\ (\forall x \forall y \forall z\ X(x,y,z) = 0\ \vee\ X(x,y,z) = 1) \wedge\ \exists y \forall x X(x,y,t) = 0$$

for $\mathbb{R}$-structures $\mathfrak{D} = (\mathscr{A}, X), X : A^3 \to \mathbb{R}$. As can be checked easily the corresponding counting problem is solvable in polynomial time.

Let us conclude with some final remarks.

An obvious direction for extending the results is given by analyzing unordered $\mathbb{R}$-structures. In the discrete setting this has been done in [6].

Another interesting topic is that of $\mathrm{NP}_{\mathbb{R}}$ optimization problems and its logical definability (see [16] for an approach in the disrete setting). Problems here already start when trying to investigate a reasonable notion of optimization problems over the reals because of the possible nonexistence of optimal values or the noncomputability of such optima within the BSS-model.

Finally, we want to refer the interested reader once more to another approach on counting over the reals using measure theory [12].

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