MATHEMATICS

°F-GROUPOIDS, CROSSED MODULES AND THE
FUNDAMENTAL GROUPOID OF A TOPOLOGICAL GROUP

BY

RONALD BROWN AND CHRISTOPHER B. SPENCER ¹

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INTRODUCTION

By a 8-groupoid is meant a group object in the category of groupoids. By a crossed module \((A, B, \delta)\) is meant a pair \(A, B\) of groups together with an operation of the group \(B\) on the group \(A\), and a morphism \(\delta: A \to B\) of groups, satisfying: (i) \(\delta(a^b) = b^{-1}(\delta a)b\), (ii) \(a^{-1}a_1a = a_1a^b\), for \(a, a_1 \in A, b \in B\). One object of this paper is to advertise the result:

THEOREM 1. The categories of 8-groupoids and of crossed modules are equivalent.

This result was, we understand, known to Verdier in 1965; it was then used by Duskin [6]; it was discovered independently by us in 1972. The work of Verdier and Duskin is unpublished, we have found that Theorem 1 is little known, and so we hope that this account will prove useful. We shall also extend Theorem 1 to include in Theorem 2 a comparison of homotopy notions for the two categories.

As an application of Theorem 1, we consider the fundamental groupoid \(\pi X\) of a topological group \(X\). Clearly \(\pi X\) is a 8-groupoid; its associated crossed module has (as does any crossed module) an obstruction class or \(k\)-invariant which in this case lies in \(H^3(\pi_0 X, \pi_1(X, e))\). We prove in Theorem 3 that this \(k\)-invariant is the first Postnikov invariant of the classifying space \(BSX\) of the singular complex \(SX\) of \(X\). An example of the use of Theorem 3 is the (possibly well known) result that the first Postnikov invariant of \(BSG\) is zero.

Duskin was led to his application of Theorem 1 in the theory of group extensions by an interest in Isbell's principle (a \(B\) in an \(A\) is an \(A\) in a \(B\)). We were led to Theorem 1 as part of a programme for exploiting double groupoids (that is, groupoid objects in the category of groupoids) in homotopy theory. Basic results on and applications of double groupoids are given in [3], [2] and [4].

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We would like to thank J. Duskin for sending us a copy of [6], and for informing us of Verdier's work.

1. \(\mathcal{G}\)-GROUPOIDS

We start with apparently greater generality by considering \(\mathcal{G}\)-categories. Thus let \(\text{Cat}\) be the category of small categories. A group object \(G\) in \(\text{Cat}\) is then a small category \(G\) equipped with functors \(\cdot: G \times G \to G\), \(e: \ast \to G\) (where \(\ast\) is a singleton), \(u: G \to G\), called respectively product, unit, and inverse, and satisfying the usual axioms for a group. The product of \(a, b\) in \(G\) is also written \(ab\), \(e(\ast)\) is written \(e\), and \(u(a)\) is written \(a^{-1}\), while the composition in \(G\) of arrows \(a: x \to y, b: y \to z\) is written \(a \circ b\) and the inverse for \(\circ\) of \(a\) (if it exists) is written \(\bar{a}\).

That \(\cdot\) is a functor gives the usual interchange law
\[
(b \circ a)(d \circ c) = (bd) \circ (ac)
\]
whenever \(b \circ a, d \circ c\) are defined.

It is easy to prove that \(1_\ast\), the identity at \(e \in Ob(G)\), is equal to \(e\). Further, composition in \(G\) can be expressed in terms of the group operation, since if \(a: x \to y, b: y \to z\) then

\[
\begin{align*}
 a \circ b &= (1_y(1^{-1}_y a)) \circ (be) \\
 &= (1_y \circ b)(1^{-1}_y a \circ e) \\
 &= b1^{-1}_y a.
\end{align*}
\]

Similarly
\[
(2) \quad a \circ b = a1^{-1}_y b,
\]
and it follows that if \(y = e\), then \(ba = ab\); thus elements of \(\text{Cost}_G e\) and \(\text{St}_G e\) commute under the group operation. Another consequence of (1) and (2) is that if \(a: x \to y\) then \(1_x a^{-1} 1_y = \bar{a}\), the inverse of \(a\) under \(\circ\); this proves that any \(\mathcal{G}\)-category is a groupoid (a remark due to J. Duskin [6]). A final remark in this context is that if \(a, a_1 \in \text{Cost}_G e\), and \(a\) has initial point \(x\), then \(a1^{-1}_x \in \text{St}_G e\) and so commutes with \(a_1\); this implies that
\[
(3) \quad a^{-1} a_1 a = 1^{-1}_x a_1 1_x.
\]

Now let \(A = \text{Cost}_G e, B = \text{Ob}(G)\). Then \(A, B\) inherit group structures from that of \(G\), and the initial point map \(\partial: A \to B\) is a morphism of groups. Further we have an operation \((a, x) \mapsto a^x\) of \(B\) on the group \(A\) given by \(a^x = 1^{-1}_x a1_x\), \(x \in B, a \in A\), and we clearly have
\[
\partial(a^x) = x^{-1}(\partial a)x,
\]
\[
a^{-1} a_1 a = a^{\partial(a)}, \text{ by (3)},
\]
for \(a, a_1 \in A, x \in B\). Thus \((A, B, \partial)\) is a crossed module [5, 8, 9, 13] (also called a crossed group in [6]).

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We can express this more categorically. Let $\mathcal{C}$ be the category of $\mathcal{G}$-groupoids (i.e. group objects in the category of groupoids, with morphisms functors of groupoids preserving the group structure). Let $\mathcal{C}$ be the category of crossed modules where a morphism $(f_2, f_1): (A, B, \delta) \to (A', B', \delta')$ of crossed modules is a pair $f_2: A \to A'$, $f_1: B \to B'$ of morphism of groups such that $f_1 \delta = \delta' f_2$, and $f_2$ is an operator morphism over $f_1$. Then we have clearly defined a functor $\delta: \mathcal{C} \to \mathcal{G}$. The result of Verdier is then:

**Theorem 1.** The functor $\delta: \mathcal{C} \to \mathcal{G}$ is an equivalence of categories.

We sketch the proof, partly following unpublished notes of Duskin [6].

Let $M = (A, B, \delta)$ be a crossed module. A $\mathcal{G}$-groupoid $\theta(M)$ is defined as follows. The group of objects of $\theta(M)$ is the group $B$. The group of arrows of $\theta(M)$ is the semi-direct product $A \ltimes B$ with the usual group structure

$$(a', b')(a, b) = (a'b a, b'b);$$

the initial and final maps are defined respectively by $(a, b) \mapsto \delta(b(a))$, $(a, b) \mapsto b$, while composition is given by

$$(a', b') \circ (a, b) = (aa', b).$$

It is then routine to check that $\theta(M)$ is a $\mathcal{G}$-groupoid. Clearly $\theta$ extends to a functor $\mathcal{C} \to \mathcal{G}$.

We easily obtain a natural equivalence $T: 1_\mathcal{C} \to \delta \theta$, where if $M = (A, B, \delta)$ is a crossed module, then $T_M$ is the identity on $B$ and on $A$ is given by $a \mapsto (a, e)$.

To define the natural equivalence $S: \delta \theta \to 1_\mathcal{G}$, let $G$ be a $\mathcal{G}$-groupoid. A map $S_G: \delta \theta(G) \to G$ is defined to be the identity on objects and on arrows is given by $(a, y) \mapsto 1 y a$. Clearly $S_G$ is bijective on arrows so it only remains to check that $S_G$ preserves composition and the group operation. This is routine and so is omitted.

We next show how the equivalence between $\mathcal{G}$-groupoids and crossed modules given by Theorem 1 extends to an equivalence between notions of homotopy in the two categories.

The standard notion of homotopy in $\mathcal{C}$ is as follows [5]. Let $(f_2, f_1)$ and $(g_2, g_1): M \to M'$ be morphisms of crossed modules $M = (A, B, \delta)$, $M' = (A', B', \delta')$. A homotopy $d: (f_2, f_1) \simeq (g_2, g_1)$ is a function $d: B \to A'$ such that

1. $d(b'b) = d(b')d_1(b), \text{ all } b, b' \in B,$
2. $\delta'd(b) = g_1(b)^{-1}f_1(b), \text{ all } b \in B,$ and
3. $d\delta(a) = g_2(a)^{-1}f_2(a), \text{ all } a \in A.$

Given such a homotopy $d$, let $\theta(d): B \to A' \ltimes B'$ be the function into the semi-direct product $A' \ltimes B'$ such that $\theta(d)$ has components $d$ and $g_1$. Then condition (i) on $d$ is equivalent to $\theta(d)$ being a morphism of groups.
Now $B$ is the set of objects of $\theta(M)$, $A' \xrightarrow{\sim} B'$ is the set of arrows of $\theta(M')$, while $(f_2, f_1), (g_2, g_1)$ determine morphisms $\theta(f_2, f_1), \theta(g_2, g_1): \theta(M) \to \theta(M')$ of $\mathcal{G}$-groupoids, $i = 0, 1$. It is now straightforward to verify that $\theta(d)$ is a homotopy or natural equivalence between these morphisms of groupoids.

We therefore define a $\mathcal{G}$-homotopy $V: h \simeq k$ of morphisms $h, k: G \to H$ of $\mathcal{G}$-groupoids to be a homotopy (or natural equivalence) in the usual sense with the additional property that $V: \theta(h) \to \theta(k)$ is a morphism of groups. This notion of homotopy gives $\mathcal{G}$ the structure of a 2-category. Similarly $\mathcal{C}$ has, with its notion of homotopy, the structure of a 2-category.

**Theorem 2.** The 2-categories of $\mathcal{G}$-groupoids and of crossed modules are equivalent 2-categories.

We omit further details.

2. **The Fundamental Groupoid of a Topological Group**

Let $M = (A, B, \partial)$ be a crossed module. Then $M$ determines an obstruction class or $k$-invariant $k(M) \in H^3(\pi_0 A, A)$ where $Q = \text{Coker} \partial, A = \text{Ker} \partial$ [8, 9], which for free $B$ classifies the crossed module up to homotopy equivalences which are the identity on $A$ and on $Q$.

Thus a $\mathcal{G}$-groupoid $G$ also determines a $k$-invariant $k(G) \in H^3(\pi_0 G, G\{e\})$, namely the $k$-invariant of its associated crossed module. (Here $G\{e\}$ is the abelian group $G(e, e)$.)

A particular example of a $\mathcal{G}$-groupoid is the fundamental groupoid $\pi X$ of a topological group $X$, with group structure on $\pi X$ induced by that of $X$ (using the rule $\pi(X \times X) = \pi X \times \pi X$ [1]).

The object of this section is to prove:

**Theorem 3.** If $X$ is a topological group, then the $k$-invariant of the $\mathcal{G}$-groupoid $\pi X$ can be identified with the first Postnikov invariant of $B \Sigma X$, the classifying space of the singular complex of $X$.

In this Theorem, the singular complex $\Sigma X$ is a simplicial group, and its classifying space $K = B \Sigma X$ is the $CW$-complex which is the realisation $|L|$ of the simplicial set $L = \overline{W} \Sigma X$ [10]. We follow [9] in describing the Postnikov invariant of $K$ in terms of its cell-structure as the $k$-invariant of the crossed module $M_K = (\tau_2(K, K^1), \tau_1(K^1), \partial')$, where $K^1$ is the 1-skeleton of $K$ and $\partial'$ is the homotopy boundary. Now the $k$-invariant of $\pi X$ is that of the crossed module $M_X = (\text{Cost} \pi X e, X, \partial)$. We prove Theorem 3 by constructing a morphism $f: M_K \to M_X$ such that $f$ induces isomorphisms $f_0: \text{Coker} \partial' \to \text{Coker} \partial, f_3: \text{Ker} \partial' \to \text{Ker} \partial$. The construction of $k$-invariants [9] then implies that $f_0^* k(M_K) = f_3^* k(M_X)$ and this is what is required for Theorem 3.
Since $K = |L|$ where $L = WSX$, we may use the descriptions of the $i$-simplices of $L$ given in [10]. Thus $L_0$ is a point, and $L_1$ consists of the points of $X$. Hence $\pi_1(K)$ is the free group on generators $[u]$ for $u \in X$ with relation $[e] = 1$, and a morphism $f_1 : \pi_1(K) \to X$ is obtained by extending the identity map on generators.

According to [9] $M_K$ is isomorphic to $M_K = (\varrho_2, d_2, \varpi_1(K^1), d_2)$ where $d_2$ is induced by $d_2$ in the operator sequence [13]

$$
\varrho \xrightarrow{d_2} \varrho_2 \xrightarrow{d_2} \pi_1(K^1),
$$

and $\varrho_n = \pi_1(K^n, K^{n-1})$ is for $n = 2$ the free crossed $(\varrho_1, d_2)$-module on the $2$-cells of $K$ and for $n = 3$ is the free $\pi_1(K^2)$-module on the $3$-cells of $K$ [13]. Thus to define $f_2 : \varrho_2 \to \pi_1(K^2)$ we need only define $f_2 : \varrho_2 \to \text{Cost}_{\text{int}} e$ by specifying $f_2(\delta)$ for each $2$-cell $\delta$ of $K$ in such a way that (i) $\delta f_2(\delta) = -f_1 d_2(\delta)$ and (ii) $f_2(\delta) = 0$.

The elements of $L_2$ are pairs $(\lambda, u)$ such that $u \in X$ and $\lambda$ is a path in $X$ from $\lambda(0)$ to $\lambda(1)$. Then $d_0(\lambda, u) = u$, $d_1(\lambda, u) = \lambda(1)u^{-1}$, $d_2(\lambda, u) = \lambda(0)$ and so $d_0(\lambda, u) = [\lambda(0)][u][\lambda(1)u]^{-1}$. So we define $f_2(\lambda, u)$ to be $e$ if $(\lambda, u)$ is degenerate and otherwise to be $[\lambda](1_{x(1)})^{-1}$. It follows that

$$
f_1 d_2(\lambda, u) = \lambda(0) u u^{-1} \lambda(1)^{-1}
$$

This proves (i).

For the proof of (ii) it is enough to show that $f_2 d_0(\kappa) = 0$ for each non-degenerate $3$-simplex $\kappa$ of $L$, since these form a set of free generators of the $\pi_1(K^2)$-module $\varrho_3$, and $d_0$ is an operator morphism. For such a $\kappa$ the homotopy addition lemma gives

$$
d_0(\kappa) = [\varrho_3 \kappa][\varrho_1 \kappa][\varrho_2 \kappa]^{-1}([\varrho_0 \kappa])^{-1}
$$

where $\alpha$ is the element of $\pi_1(K^1)$ determined by $\varrho_0 \varrho_2(\kappa)$ and $[\varrho_0 \kappa]$ denotes the generator of $\varrho_2$ corresponding to $\varrho_0 \kappa$. Now $\kappa$ is a triple $(\sigma, \lambda, u)$ where $u \in X$, $\lambda$ is a path in $X$ and $\sigma$ is a singular $2$-simplex in $X$. Let $\delta_0 \sigma = \gamma$, $\delta_1 \sigma = \alpha$ (so that, in $\pi X$, $[\gamma] = [\alpha] \circ [\beta]$), and let $\alpha(0) = x, \gamma(0), x(1) = \beta(0), y, \beta(1) = \gamma(1) = z, \lambda(1) = s, \lambda(0) = t$. By the formulae 1) for $\delta_1$ in $\mathcal{W}G$, and since $f_2$ is an operator morphism

$$
f_2 d_0(\kappa) = f_2(\kappa, t) f_2(\alpha, \beta, u) f_2(\gamma, su)^{-1} f_2((\lambda, u)z)^{-1}
$$

(1) $= [\kappa][x]^1 [\beta][1_x 1_x]^{-1}([\gamma] 1_z) [1_x 1_x 1_z]^{-1} (1_x 1_x 1_z)^{-1}

(2) $= [\kappa][x]^1 [\beta]([\lambda] 1_z^{-1}) ([\beta]^{-1} 1_y) ([x] 1_x 1_x 1_z) (1_x 1_x 1_z)^{-1}

(3) $= e

1) The formula for $\delta_{t+1}$ in $\mathcal{W}G$ on p. 87 of [10] should interchange $\varrho_0 \varrho_{n-1}$ and $\varrho_{n-1}$.  


where to deduce (2) from (1) we use the substitution $\gamma = [\alpha]_1 \gamma^{-1}[\beta]$, and to deduce (3) from (2) we note that each term in round brackets lies in $\text{Cost}_{\alpha X} e$ and so by §1 commutes with elements of $\text{St}_{\alpha X} e$, and in particular with each term in curly brackets.

This completes the proof of (ii) and so gives the construction of $f' : M_K \to M_X$. Now $\pi_1(K) = \text{Coker} (d_2 : g_2 \to g_1)$ is the free group on generators $[u], u \in X$, with relations $[e] = 1, [u \cdot \lambda(0)] = [u][\lambda(1)]$, for $u \in X, \lambda$ a path in $X$. It follows easily that $f'_0 : \text{Coker} d_2 \to \text{Coker} \delta$ is an isomorphism $\pi_1(K) \to \pi_0(X)$. We also need to prove that $f'_1 : \text{Ker} d_2 \to \text{Ker} \delta$ is an isomorphism, where $\text{Ker} d_2 = \pi_2(K)$ and $\text{Ker} \delta = \pi_1(X, e)$. But $\pi_2(K) = \pi_2(L)$ and $L$ is a Kan complex ([7] Proposition 10.4); hence elements of $\pi_2(L)$ are represented by elements of $L \mathbin{\times} e$ with faces at the base point, i.e. by pairs $(\lambda, e)$ where $\lambda$ is a loop at $e$. Then $f'_2$ is given by $[(\lambda, e)] \to [\lambda]$, and this is the standard map $\pi_2(L) \to \pi_1(SX)$ which is known to be an isomorphism.

This completes the proof of Theorem 3.

There should be a better proof of Theorem 3.

What would seem to be required is either a description of the first $k$-invariant of a topological space $K$ in terms of some crossed module defined using fibrations, or else a description of the first $k$-invariant of a simplicial set $L$ in terms of a crossed module defined directly by the simplicial structure of $L$. Neither such description is known to us.

Our final result gives an application of Theorem 3.

**Proposition 4.** Let $X$ be a topological group which is a split extension of a discrete group by a path-connected group. Then the first $k$-invariant of $BSX$ is zero.

**Proof.** According to [9] p. 43 the $k$-invariant of the crossed module $M_X = (\text{Cost}_{\alpha X} e, X, \delta)$ is determined from the exact sequence

$$e \to \pi_1(X, e) \to \text{Cost}_{\alpha X} e \to X \to F \to e$$

(where in our case $F = \pi_0 X$ is discrete) by considering first the deviation from being a morphism of a section $s$ of $\nu$. However $\nu : X \to F$ has a section which is a morphism, since $X$ is a split extension. Therefore the $k$-invariant of $M_X$ is zero. The result follows from Theorem 3.

**Corollary 5.** The first $k$-invariant of the classifying space $B_X$ is zero if $X$ is any quotient of $O(n)$ by a normal subgroup.

**Proof.** If $X$ is connected the result is clear. Otherwise it is well-known that the determinant map $0(n) \to \mathbb{Z}_2$ has a section $s$ which is a morphism,
and so a section for \( X \to \pi_0 X \) is the composite \( Z_2 \xrightarrow{s} \pi_0(n) \xrightarrow{p} X \) where \( p \) is the projection.

There are examples of topological groups \( X \) such that \( M_X \) has non-trivial \( k \)-invariant—such an \( X \) is \( G(Y) \) where \( Y \) is a connected finite simplicial complex with non-trivial first Postnikov invariant, and \( G(Y) \) is Milnor's group model of the loop-space of \( Y \) [11].

R. Brown, 
School of Mathematics and Computer Science, University College of North Wales, Bangor, Gwynedd, LL57 2UW

C. B. Spencer, 
Department of Mathematics, The University, Hong Kong

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