MATHEMATICS

$\mathscr{G}$-GROUPOIDS, CROSSED MODULES AND THE FUNDAMENTAL GROUPOID OF A TOPOLOGICAL GROUP

BY

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INTRODUCTION

By a *$\mathscr{G}$-groupoid* is meant a group object in the category of groupoids. By a *crossed module* $(A, B, \eth)$ is meant a pair A, B of groups together with an operation of the group B on the group A, and a morphism $\eth$: $A \to B$ of groups, satisfying: (i) $\eth(a^b) = b^{-1}(\eth a)b$, (ii) $a^{-1}a_1 a = a_1{}^{\eth a}$, for $a, a_1 \in A$, $b \in B$. One object of this paper is to advertise the result:

THEOREM 1. The categories of $\mathscr{G}$-groupoids and of crossed modules are equivalent.

This result was, we understand, known to Verdier in 1965; it was then used by Duskin [6]; it was discovered independently by us in 1972. The work of Verdier and Duskin is unpublished, we have found that Theorem 1 is little known, and so we hope that this account will prove useful. We shall also extend Theorem 1 to include in Theorem 2 a comparison of homotopy notions for the two categories.

As an application of Theorem 1, we consider the fundamental groupoid πX of a topological group X. Clearly πX is a $\mathscr{G}$-groupoid; its associated crossed module has (as does any crossed module) an obstruction class or k-invariant which in this case lies in $H^3(\pi_0 X, \pi_1(X, e))$. We prove in Theorem 3 that this k-invariant is the first Postnikov invariant of the classifying space B_{SX} of the singular complex SX of X. An example of the use of Theorem 3 is the (possibly well known) result that the first Postnikov invariant of $B_{0(n)}$ is zero.

Duskin was led to his application of Theorem 1 in the theory of group extensions by an interest in Isbell's principle (a $\mathscr{B}$ in an $\mathscr{A}$ is an $\mathscr{A}$ in a $\mathscr{B}$). We were led to Theorem 1 as part of a programme for exploiting double groupoids (that is, groupoid objects in the category of groupoids) in homotopy theory. Basic results on and applications of double groupoids are given in [3], [2] and [4].

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We would like to thank J. Duskin for sending us a copy of [6], and for informing us of Verdier's work.

1. $\mathcal{G}$-GROUPOIDS

We start with apparently greater generality by considering $\mathcal{G}$-categories. Thus let *Cat* be the category of small categories. A *group object* G in *Cat* is then a small category G equipped with functors $\cdot: G \times G \to G$, $e: * \to G$ (where $*$ is a singleton), $u: G \to G$, called respectively product, unit, and inverse, and satisfying the usual axioms for a group. The product of a, b in G is also written ab, $e(*)$ is written e, and $u(a)$ is written a^{-1}, while the composition in G of arrows $a: x \to y$, $b: y \to z$ is written $a \circ b$ and the inverse for $\circ$ of a (if it exists) is written $\bar{a}$.

That $\cdot$ is a functor gives the usual interchange law

$$(b \circ a)(d \circ c) = (bd) \circ (ac)$$

whenever $b \circ a$, $d \circ c$ are defined.

It is easy to prove that 1_e, the identity at $e \in Ob(G)$, is equal to e. Further, composition in G can be expressed in terms of the group operation, since if $a: x \to y$, $b: y \to z$ then

$$(1) \qquad \left\{ \begin{aligned} a \circ b &= (1_y(1_y^{-1} a)) \circ (be) \\ &= (1_y \circ b)(1_y^{-1} a \circ e) \\ &= b 1_y^{-1} a. \end{aligned} \right.$$

Similarly

$$(2) \qquad a \circ b = a 1_y^{-1} b,$$

and it follows that if $y = e$, then $ba = ab$; thus elements of $\mathrm{Cost}_G\, e$ and $\mathrm{St}_G\, e$ commute under the group operation. Another consequence of (1) and (2) is that if $a: x \to y$ then $1_x a^{-1} 1_y = \bar{a}$, the inverse of a under $\circ$; this proves that any $\mathcal{G}$-category is a groupoid (a remark due to J. Duskin [6]). A final remark in this context is that if $a, a_1 \in \mathrm{Cost}_G\, e$, and a has initial point x, then $a 1_x^{-1} \in \mathrm{St}_G\, e$ and so commutes with a_1; this implies that

$$(3) \qquad a^{-1} a_1 a = 1_x^{-1} a_1 1_x.$$

Now let $A = \mathrm{Cost}_G\, e$, $B = Ob(G)$. Then A, B inherit group structures from that of G, and the initial point map $\partial: A \to B$ is a morphism of groups. Further we have an operation $(a, x) \mapsto a^x$ of B on the group A given by $a^x = 1_x^{-1} a 1_x$, $x \in B$, $a \in A$, and we clearly have

$$\partial(a^x) = x^{-1}(\partial a) x,$$

$$a^{-1} a_1 a = a^{\partial(a)}, \text{ by (3)},$$

for $a, a_1 \in A$, $x \in B$. Thus (A, B, ∂) is a *crossed* module [5, 8, 9, 13] (also called a *crossed group* in [6]).

We can express this more categorically. Let $\mathscr{G}$ be the category of $\mathscr{G}$-groupoids (i.e. group objects in the category of groupoids, with morphisms functors of groupoids preserving the group structure). Let $\mathscr{C}$ be the category of crossed modules where a *morphism* $(f_2, f_1)\colon (A, B, \eth) \to (A', B', \eth')$ of crossed modules is a pair $f_2\colon A \to A'$, $f_1\colon B \to B'$ of morphism of groups such that $f_1\eth = \eth' f_2$, and f_2 is an operator morphism over f_1. Then we have clearly defined a functor $\delta\colon \mathscr{G} \to \mathscr{C}$. The result of Verdier is then:

THEOREM 1. The functor $\delta\colon \mathscr{G} \to \mathscr{C}$ is an equivalence of categories.

We sketch the proof, partly following unpublished notes of Duskin [6].

Let $M = (A, B, \eth)$ be a crossed module. A $\mathscr{G}$-groupoid $\theta(M)$ is defined as follows. The group of objects of $\theta(M)$ is the group B. The group of arrows of $\theta(M)$ is the semi-direct product $A \mathbin{\tilde{\times}} B$ with the usual group structure

$$(a', b')(a, b) = (a'^{b}a, b'b);$$

the initial and final maps are defined respectively by $(a, b) \mapsto b\eth(a)$, $(a, b) \mapsto b$, while composition is given by

$$(a', b') \circ (a, b) = (aa', b).$$

It is then routine to check that $\theta(M)$ is a $\mathscr{G}$-groupoid. Clearly θ extends to a functor $\mathscr{C} \to \mathscr{G}$.

We easily obtain a natural equivalence $T\colon 1_{\mathscr{C}} \to \delta\theta$, where if $M = (A, B, \eth)$ is a crossed module, then T_M is the identity on B and on A is given by $a \mapsto (a, e)$.

To define the natural equivalence $S\colon \theta\delta \to 1_{\mathscr{G}}$, let G be a $\mathscr{G}$-groupoid. A map $S_G\colon \theta\delta(G) \to G$ is defined to be the identity on objects and on arrows is given by $(a, y) \mapsto 1_y a$. Clearly S_G is bijective on arrows so it only remains to check that S_G preserves composition and the group operation. This is routine and so is omitted.

We next show how the equivalence between $\mathscr{G}$-groupoids and crossed modules given by Theorem 1 extends to an equivalence between notions of homotopy in the two categories.

The standard notion of homotopy in $\mathscr{C}$ is as follows [5]. Let (f_2, f_1) and $(g_2, g_1)\colon M \to M'$ be morphisms of crossed modules $M = (A, B, \eth)$, $M' = (A', B', \eth')$. A *homotopy* $d\colon (f_2, f_1) \simeq (g_2, g_1)$ is a function $d\colon B \to A'$ such that

(i) $d(b'b) = d(b')^{g_1(b)} d(b)$, all $b, b' \in B$,
(ii) $\eth' d(b) = g_1(b)^{-1} f_1(b)$, all $b \in B$, and
(iii) $d\eth(a) = g_2(a)^{-1} f_2(a)$, all $a \in A$.

Given such a homotopy d, let $\theta(d)\colon B \to A' \mathbin{\tilde{\times}} B'$ be the function into the semi-direct product $A' \mathbin{\tilde{\times}} B'$ such that $\theta(d)$ has components d and g_1. Then condition (i) on d is equivalent to $\theta(d)$ being a morphism of groups.

Now B is the set of objects of $\theta(M)$, $A' \tilde{\times} B'$ is the set of arrows of $\theta(M')$, while (f_2, f_1), (g_2, g_1) determine morphisms $\theta(f_2, f_1), \theta(g_2, g_1): \theta(M) \to \theta(M')$ of $\mathscr{G}$-groupoids, $i=0, 1$. It is now straightforward to verify that $\theta(d)$ is a homotopy or natural equivalence between these morphisms of groupoids.

We therefore define a *$\mathscr{G}$-homotopy* $V: h \simeq k$ of morphisms $h, k: G \to H$ of $\mathscr{G}$-groupoids to be a homotopy (or natural equivalence) in the usual sense with the additional property that $V: 0b(G) \to \mathrm{Arr}(H)$ is a morphism of groups. This notion of homotopy gives $\mathscr{G}$ the structure of a 2-category. Similarly $\mathscr{C}$ has, with its notion of homotopy, the structure of a 2-category.

THEOREM 2. The 2-categories of $\mathscr{G}$-groupoids and of crossed modules are equivalent 2-categories.

We omit further details.

2. THE FUNDAMENTAL GROUPOID OF A TOPOLOGICAL GROUP

Let $M=(A, B, \partial)$ be a crossed module. Then M determines an obstruction class or k-invariant $k(M) \in H^3(Q, A)$ where $Q = \mathrm{Coker}\, \partial$, $A = \mathrm{Ker}\, \partial$ [8, 9], which for free B classifies the crossed module up to homotopy equivalences which are the identity on A and on Q.

Thus a $\mathscr{G}$-groupoid G also determines a k-invariant $k(G) \in H^3(\pi_0 G, G\{e\})$, namely the k-invariant of its associated crossed module. (Here $G\{e\}$ is the abelian group $G(e, e)$.)

A particular example of a $\mathscr{G}$-groupoid is the fundamental groupoid πX of a topological group X, with group structure on πX induced by that of X (using the rule $\pi(X \times X) = \pi X \times \pi X$ [1]).

The object of this section is to prove:

THEOREM 3. If X is a topological group, then the k-invariant of the $\mathscr{G}$-groupoid πX can be identified with the first Postnikov invariant of B_{SX}, the classifying space of the singular complex of X.

In this Theorem, the singular complex SX is a simplicial group, and its classifying space $K = B_{SX}$ is the CW-complex which is the realisation $|L|$ of the simplicial set $L = \overline{W}SX$ [10]. We follow [9] in describing the Postnikov invariant of K in terms of its cell-structure as the k-invariant of the crossed module $M_K = (\pi_2(K, K^1), \pi_1(K^1), \partial')$, where K^1 is the 1-skeleton of K and ∂' is the homotopy boundary. Now the k-invariant of πX is that of the crossed module $M_X = (\mathrm{Cost}_{\pi X}\, e, X, \partial)$. We prove Theorem 3 by constructing a morphism $f: M_K \to M_X$ such that f induces isomorphisms $f_0: \mathrm{Coker}\, \partial' \to \mathrm{Coker}\, \partial$, $f_3: \mathrm{Ker}\, \partial' \to \mathrm{Ker}\, \partial$. The construction of k-invariants [9] then implies that $f_{0*}\, k(M_K) = f_3{}^* k(M_X)$ and this is what is required for Theorem 3.

Since $K=|L|$ where $L=\overline{W}SX$, we may use the descriptions of the i-simplices of L given in [10]. Thus L_0 is a point, and L_1 consists of the points of X. Hence $\pi_1(K^1)$ is the free group on generators $[u]$ for $u \in X$ with relation $[e]=1$, and a morphism $f_1: \pi_1(K^1) \to X$ is obtained by extending the identity map on generators.

According to [9] M_K is isomorphic to $M_K'=(\varrho_2/d_3\varrho_3, \pi_1(K^1), \bar{d}_2)$ where $\bar{d}_2$ is induced by d_2 in the operator sequence [13]

$$\to \varrho_3 \xrightarrow{d_3} \varrho_2 \xrightarrow{d_2} \pi_1(K^1),$$

and $\varrho_n=\pi_n(K^n, K^{n-1})$ is for $n=2$ the free crossed (ϱ_1, d_2)-module on the 2-cells of K and for $n=3$ is the free $\pi_1(K^2)$-module on the 3-cells of K [13]. Thus to define $f_2': \varrho_2/d_3\varrho_3 \to \mathrm{Cost}_{\pi X}\, e$ we need only define $\bar{f}_2: \varrho_2 \to \mathrm{Cost}_{\pi X}\, e$ by specifying $\bar{f}_2(\delta)$ for each 2-cell δ of K in such a way that (i) $\partial\bar{f}_2(\delta)=$ $=f_1 d_2(\delta)$ and (ii) $\bar{f}_2 d_3 \varrho_3=0$.

The elements of L_2 are pairs (λ, u) such that $u \in X$ and λ is a path in X from $\lambda(0)$ to $\lambda(1)$. Then $\partial_0(\lambda, u)=u$, $\partial_1(\lambda, u)=\lambda(1)u$ [1]), $\partial_2(\lambda, u)=\lambda(0)$ and so $d_2(\lambda, u)=[\lambda(0)][u][\lambda(1)u]^{-1}$. So we define $\bar{f}_2(\lambda, u)$ to be e if (λ, u) is degenerate and otherwise to be $[\lambda](1_{\lambda(1)})^{-1}$. It follows that

$$\begin{aligned} f_1 d_2(\lambda, u) &= \lambda(0)\, u\, u^{-1}\, \lambda(1)^{-1} \\ &= \partial \bar{f}_2(\lambda, u). \end{aligned}$$

This proves (i).

For the proof of (ii) it is enough to show that $\bar{f}_2 d_3(\varkappa)=0$ for each non-degenerate 3-simplex $\varkappa$ of L, since these form a set of free generators of the $\pi_1(K^2)$-module ϱ_3, and d_3 is an operator morphism. For such a $\varkappa$ the homotopy addition lemma gives

$$d_3(\varkappa)=[\partial_3\varkappa][\partial_1\varkappa][\partial_2\varkappa]^{-1}([\partial_0\varkappa]^{a^{-1}})^{-1}$$

where a is the element of $\pi_1(K^1)$ determined by $\partial_3\partial_2(\varkappa)$ and $[\partial_i\varkappa]$ denotes the generator of ϱ_2 corresponding to $\partial_i\varkappa$. Now $\varkappa$ is a triple (σ, λ, u) where $u \in X$, λ is a path in X and σ is a singular 2-simplex in X. Let $\partial_0\sigma=\beta$, $\partial_1\sigma=\gamma$, $\partial_2\sigma=\alpha$ (so that, in πX, $[\gamma]=[\alpha]\circ[\beta]$), and let $\alpha(0)=x=\gamma(0)$, $\alpha(1)=\beta(0)=y$, $\beta(1)=\gamma(1)=z$, $\lambda(1)=s$, $\lambda(0)=t$. By the formulae [1]) for ∂_i in $\overline{W}G$, and since $\bar{f}_2$ is an operator morphism

$$\bar{f}_2 d_3(\varkappa)=\bar{f}_2(\alpha, t)\bar{f}_2(\beta\lambda, u)\bar{f}_2(\gamma, su)^{-1}\bar{f}_2((\lambda, u)^{x^{-1}})^{-1}$$

$$=[\alpha]1_y^{-1}[\beta\lambda](1_z 1_s)^{-1}([\gamma]1_z^{-1})^{-1}(1_x[\lambda]1_s^{-1}1_x^{-1})^{-1} \tag{1}$$

$$=[\alpha]1_y^{-1}[\beta]([\lambda]1_s^{-1})\{[\beta]^{-1}1_y\}\{[\alpha]^{-1}1_x\}(1_s[\lambda]^{-1})1_x^{-1} \tag{2}$$

$$=e \tag{3}$$

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[1]) The formula for ∂_{i+1} in $\overline{W}G$ on p. 87 of [10] should interchange $\partial_0 g_{n-i}$ and g_{n-i+1}.

where to deduce (2) from (1) we use the substitution $[\gamma]=[\alpha]1_\nu^{-1}[\beta]$, and to deduce (3) from (2) we note that each term in round brackets lies in $\mathrm{Cost}_{\pi X}\, e$ and so by § 1 commutes with elements of $\mathrm{St}_{\pi X}\, e$, and in particular with each term in curly brackets.

This completes the proof of (ii) and so gives the construction of $f'\colon M'_K \to M_X$. Now $\pi_1(K)=\mathrm{Coker}\,(d_2\colon \varrho_2 \to \varrho_1)$ is the free group on generators $[u]$, $u \in X$, with relations $[e]=1$, $[u\cdot\lambda(0)]=[u][\lambda(1)]$, for $u \in X$, λ a path in X. It follows easily that $f'_0\colon \mathrm{Coker}\, d_2 \to \mathrm{Coker}\, \partial$ is an isomorphism $\pi_1(K) \to \pi_0(X)$. We also need to prove that $f'_3\colon \mathrm{Ker}\, \bar{d}_2 \to \mathrm{Ker}\, \partial$ is an isomorphism, where $\mathrm{Ker}\, \bar{d}_2 = \pi_2(K)$ and $\mathrm{Ker}\, \partial = \pi_1(X, e)$. But $\pi_2(K)= =\pi_2(L)$ and L is a Kan complex ([7] Proposition 10.4); hence elements of $\pi_2(L)$ are represented by elements of L_2 with faces at the base point, i.e. by pairs (λ, e) where λ is a loop at e. Then f'_3 is given by $[(\lambda, e)] \to [\lambda]$, and this is the standard map $\pi_2(L) \to \pi_1(SX)$ which is known to be an isomorphism.

This completes the proof of Theorem 3.

There should be a better proof of Theorem 3.

What would seem to be required is either a description of the first k-invariant of a topological space K in terms of some crossed module defined using fibrations, or else a description of the first k-invariant of a simplicial set L in terms of a crossed module defined directly by the simplicial structure of L. Neither such description is known to us.

Our final result gives an application of Theorem 3.

PROPOSITION 4. Let X be a topological group which is a split extension of a discrete group by a path-connected group. Then the first k-invariant of B_{SX} is zero.

PROOF. According to [9] p. 43 the k-invariant of the crossed module $M_X = (\mathrm{Cost}_{\pi X}\, e, X, \partial)$ is determined from the exact sequence

$$e \to \pi_1(X, e) \to \mathrm{Cost}_{\pi X}\, e \xrightarrow{\partial} X \xrightarrow{\nu} F \to e$$

(where in our case $F=\pi_0 X$ is discrete) by considering first the deviation from being a morphism of a section s of ν. However $\nu\colon X \to F$ has a section which is a morphism, since X is a split extension. Therefore the k-invariant of M_X is zero. The result follows from Theorem 3.

COROLLARY 5. The first k-invariant of the classifying space B_X is zero if X is any quotient of $0(n)$ by a normal subgroup.

PROOF. If X is connected the result is clear. Otherwise it is well-known that the determinant map $0(n) \to Z_2$ has a section s which is a morphism, and so a section for $X \to \pi_0 X$ is the composite $Z_2 \xrightarrow{s} 0(n) \xrightarrow{p} X$ where p is the projection.

There are examples of topological groups X such that M_X has non-trivial k-invariant – such an X is $G(Y)$ where Y is a connected finite simplicial complex with non-trivial first Postnikow invariant, and $G(Y)$ is Milnor's group model of the loop-space of Y [11].

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