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Perturbation of Burkholder's martingale transform and Monge–Ampère equation

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Abstract

Given a sequence of martingale differences, Burkholder found the sharp constant for the L^p-norm of the corresponding martingale transform. We are able to determine the sharp L^p-norm of a small "quadratic perturbation" of the martingale transform in L^p. By "quadratic perturbation" of the martingale transform, we mean the L^p norm of the square root of the squares of the martingale transform and the original martingale (with a small constant). The problem of perturbation of martingale transform appears naturally if one wants to estimate the linear combination of Riesz transforms (as, for example, in the case of Ahlfors–Beurling operator). Let $\{d_k\}_{k\geq 0}$ be a complex martingale difference in $L^p[0,1]$, where $1<p<\infty$, and $\{\varepsilon_k\}_{k\geq 0}$ a sequence in $\{\pm 1\}$. We obtain the following generalization of Burkholder's famous result. If $\tau\in[-\frac{1}{2},\frac{1}{2}]$ and $n\in\mathbb{Z}_+$, then

$$\left\|\sum_{k=0}^{n}\binom{\varepsilon_k}{\tau} d_k\right\|_{L^p([0,1],\mathbb{C}^2)} \leq \left((p^*-1)^2+\tau^2\right)^{\frac{1}{2}} \left\|\sum_{k=0}^{n} d_k\right\|_{L^p([0,1],\mathbb{C})},$$

where $((p^*-1)^2+\tau^2)^{\frac{1}{2}}$ is sharp and $p^*-1=\max\{p-1,\frac{1}{p-1}\}$. For $2\leq p<\infty$, the result is also true with the sharp constant for $\tau\in\mathbb{R}$.

Keywords: Martingale; Martingale transform; Perturbation; Riesz transform; Ahlfors–Beurling transform; Bellman function; Monge–Ampère equation

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1. Introduction

In a series of papers, [5–12], Burkholder was able to compute the L^p operator norm of the martingale transform, which we will denote as MT. This was quite a revolutionary result, not only because of the result itself but because of the method for approaching the problem. Burkholder's method in these early papers was the inspiration for the Bellman function technique, which has been a very useful tool in approaching modern and classical problems in harmonic analysis (this paper will demonstrate the Bellman function technique as well). But, the result itself has many applications. One particular application of his result is for obtaining sharp estimates for singular integrals. Consider the Ahlfors–Beurling operator, which we will denote as T. Lehto, [16], showed in 1965 that $\|T\|_p := \|T\|_{p\to p} \geq (p*-1) = \max\left\{p-1, \frac{1}{p-1}\right\}$. Iwaniec conjectured in 1982, [15], that $\|T\|_p = p^* - 1$. The only progress toward proving that conjecture has been using Burkholder's result, see [17,2,1] for the major results toward proving the conjecture. However, Burkholder's estimates have been useful for lower bound estimates as well. For example, Geiss et al. [14] were able to show that $\|\Re T\|_p, \|\Im T\|_p \geq p^* - 1$, by using Burkholder's estimates. The upper bound for these two operators was determined as $p^* - 1$ by Nazarov and Volberg, [17] and Bañuelos et al. [2], so we now have $\|\Re T\|_p = \|\Im T\|_p = p^* - 1$. Note that $\Re T$ the difference of the squares of the planar Riesz transforms, i.e. $T = R_1^2 - R_2^2$.

A recent result of Geiss et al. [14] points to the following observation, though not immediately. We can estimate linear combinations of squares of Riesz transforms if we know the corresponding estimate for a linear combination of the martingale transform and the identity operator. In other words, one can get at estimates of the norm of $(R_1^2 - R_2^2) + \tau \cdot I$, by knowing the estimates of the norm of $MT + \tau \cdot I \cdot \|MT + \tau \cdot I\|_p$ has only been computed for either $\tau = 0$ by Burkholder [8] or $\tau = \pm 1$ by Choi [13]. The problem is still open for all other τ-values and seems to be very difficult, though we have had some progress. But, if we consider "quadratic" rather than linear perturbations then things become more manageable (see [3,4]). This brings us to the focus of this paper, which is determining estimates for quadratic perturbations of the martingale transform, which will have connections to quadratic combinations of squares of Riesz transforms.

To prove our main result, we are going to take a slightly indirect approach. Burkholder (see [8]) defined the martingale transform, MT_ε, as

$$MT_\varepsilon\left(\sum_{k=1}^n d_k\right) := \sum_{k=1}^n \varepsilon_k d_k.$$

Then the main result can be stated as

$$\sup_{\vec{\varepsilon}} \left\| \begin{pmatrix} MT_{\vec{\varepsilon}} \\ \tau I \end{pmatrix} \right\|_{L^p(\mathbb{C})\to L^p(\mathbb{C}^2)} = \sup_{\vec{\varepsilon}} \frac{\left\| \sum_{k=1}^n \begin{pmatrix} \varepsilon_k \\ \tau \end{pmatrix} d_k \right\|_p}{\left\| \sum_{k=1}^n d_k \right\|_p} = ((p^*-1)^2 + \tau^2)^{\frac{1}{2}},$$

where I is the identity transformation and τ is "small". However, rather than working with this martingale transform in terms of the martingale differences, in a probabilistic setting, we will define another martingale transform in terms of the Haar expansion of $L^p[0, 1]$ functions and set up a Bellman function in that context. Burkholder showed, in [12], that these two different martingale transforms have the same L^p operator norm, for $\tau = 0$, so we expected a perturbation of these to act similarly and it turns out that they do. For convenience, we will work with the martingale transform in the Haar setting. Using the Bellman function technique will turn the problem of finding the sharp constant of the above estimate into solving a second order partial differential equation. The beauty of this approach is that it gets right to the heart of the problem with very little advanced techniques needed in the process. In fact, the only background material that is needed for the Bellman function technique approach, is some basic knowledge of partial differential equations and some elementary analysis.

Observe that for $2 \le p < \infty$, the estimate from above is just an application of Minkowski's inequality on $L^{\frac{p}{2}}$ and Burkholder's original result. But, this argument does not address sharpness, even though the constant obtained turns out to be the sharp constant for small τ. For $1 < p < 2$, Minkowski's inequality (in $l^{\frac{2}{p}}$) also plays a role, but to a lesser extent and cannot give the sharp constant, as we will see Proposition 30. It is, indeed, very strange that such sloppy estimation could give the estimate with sharp constant for $1 \le p < \infty$. We will now rigorously develop some background ideas needed to set up the Bellman function.

In our calculations, we follow the scheme of [19], but our "Dirichlet problem" for Monge–Ampère is different. For small τ the scheme works. For large τ and $1 < p < 2$ it definitely must be changed as [3] shows. The amazing feature is the "splitting" of the result into two quite different cases: $1 < p < 2$ and $2 \le p < \infty$, where in the former case we know the result only for small τ, but in the latter one τ is unrestricted.

1.1. Motivation of the Bellman function

Let I be an interval and $\alpha^{\pm} \in \mathbb{R}^+$ such that $\alpha^+ + \alpha^- = 1$. These $\alpha^{\pm}$ generate two subintervals $I^{\pm}$ such that $|I^{\pm}| = \alpha^{\pm}|I|$ and $I = I^- \cup I^+$. We can continue this decomposition indefinitely as follows. Any sequence $\{\alpha_{n,m} : 0 < \alpha_{n,m} < 1, 0 \le m < 2^n, 0 < n < \infty, \alpha_{n,2k} + \alpha_{n,2k+1} = 1\}$, generates the collection $\mathcal{I} := \{I_{n,m} : 0 \le m < 2^n, 0 < n < \infty\}$ of subintervals of I, where $I_{n,m} = I^-_{n,m} \cup I^+_{n,m} = I_{n+1,2m+1} \cup I_{n+1,2m+1}$ and $\alpha^- = \alpha_{n+1,2m}, \alpha^+ = \alpha_{n+1,2m+1}$. Note that $I_{0,0} = I$.

For any $J \in \mathcal{I}$ we define the Haar function $h_J := -\sqrt{\frac{\alpha^+}{\alpha^-|J|}}\chi_{J^-} + \sqrt{\frac{\alpha^-}{\alpha^+|J|}}\chi_{J^+}$. If max $\{|I_{n,m}| : 0 \le m < 2^n\} \to 0$ as $n \to \infty$ then $\{h_J\}_{J\in\mathcal{I}}$ is an orthonormal basis for $L^2_0(I) := \{f \in L^2(I) : \int_I f = 0\}$. However, if we add one extra function then Haar functions form an orthonormal basis in $L^2[0, 1]$. Fix $I_0 = [0, 1]$ and $\mathcal{I} = \mathcal{D}$ as the dyadic subintervals of I. Let $\mathcal{D}_n = \{I \in \mathcal{D} : |I| = 2^{-n}\}$. We use the notation $\langle f\rangle_J$ to represent the average integral of f over the interval $J \in \mathcal{D}$ and $\sigma(\mathcal{D}_n)$ to be the σ-algebra generated by $\mathcal{D}_n$. For any $f \in L^1(I_0)$ we have the identity

$$\sum_{I\in\mathcal{D}_n} \langle f\rangle_I \chi_I = \langle f\rangle_{(I_0)}\chi_{[0,1)} + \sum_{I\in\sigma(\mathcal{D}_n)} (f, h_I)h_I. \tag{1.1}$$

By Lebesgue differentiation, the left-hand side in (1.1) converges to f almost everywhere, as $n \to \infty$. So any $f \in L^p(I_0) \subset L^1(I_0)$ can be decomposed in terms of the Haar system as

$$f = \langle f\rangle_{(I_0)}\chi_{(I_0)} + \sum_{I\in\mathcal{D}} (f, h_I)h_I.$$

In terms of the expansion in the Haar system we define the martingale transform, g of f, as

$$g := \langle g\rangle_{(I_0)}\chi_{(I_0)} + \sum_{I\in\mathcal{D}} \varepsilon_I (f, h_I)h_I,$$

where $\varepsilon_I \in \{\pm 1\}$. Requiring that $|(g, h_J)| = |(f, h_J)|$, for all $J \in \mathcal{D}$, is equivalent to g being the martingale transform of f, for $f, g \in L^p(I_0)$.

Now we define the Bellman function as $\mathcal{B}(x_1, x_2, x_3) :=$

$$\sup_{f,g}\{\langle (g^2+\tau^2 f^2)^{\frac{p}{2}}\rangle_I : x_1 = \langle f\rangle_I, x_2 = \langle g\rangle_I, x_3 \\ = \langle |f|^p\rangle_I, |(f,h_J)| = |(g,h_J)|,\ \forall J\in\mathcal{D}\}$$

on the domain $\Omega = \{x \in \mathbb{R}^3 : x_3 \geq 0, |x_1|^p \leq x_3\}$. The Bellman function is defined in this way, since we would like to know the value of the supremum of $\left\|\begin{pmatrix} g \\ \tau f\end{pmatrix}\right\|_p$, where g is the martingale transform of f. Note that $|x_1|^p \leq x_3$ is just Hölder's inequality. Even though the Bellman function is only being defined for real-valued functions, we can "vectorize" it to work for complex-valued (and even Hilbert-valued) functions, as we will later demonstrate. Finding the Bellman function will make proving the following main result quite easy. We will call $\langle (g^2+\tau^2 f^2)^{\frac{p}{2}}\rangle_I^{\frac{1}{p}}$ the "quadratic perturbation" of the martingale transform's norm $\langle |g|^p\rangle_I^{\frac{1}{p}}$.

Theorem 1. *Let $\{d_k\}_{k\geq 1}$ be a complex martingale difference in $L^p[0,1]$, where $1 < p < \infty$, and $\{\varepsilon_k\}_{k\geq 1}$ a sequence in $\{\pm 1\}$. If $\tau \in [-\frac{1}{2}, \frac{1}{2}]$ and $n \in \mathbb{Z}_+$ then*

$$\left\|\sum_{k=1}^{n}\begin{pmatrix}\varepsilon_k \\ \tau\end{pmatrix} d_k\right\|_{L^p([0,1],\mathbb{C}^2)} \leq ((p^*-1)^2+\tau^2)^{\frac{1}{2}} \left\|\sum_{k=1}^{n} d_k\right\|_{L^p([0,1],\mathbb{C})},$$

where $((p^*-1)^2+\tau^2)^{\frac{p}{2}}$ is sharp. The result is also true with the sharp constant for $2 \leq p < \infty$ and $\tau \in \mathbb{R}$.

Note that when $\tau = 0$ we get Burkholder's famous result [8].

Now that we have the problem formalized, notice that $\mathcal{B}$ is independent of the initial choice of I_0 (which we will just denote I from now on) and $\{\alpha_{n,m}\}_{n,m}$, so we return to having them arbitrary. Finding $\mathcal{B}$ when $p = 2$ is easy, so we will do this first.

Proposition 2. *If $p = 2$ then $\mathcal{B}(x) = x_2^2 - x_1^2 + (1+\tau^2)x_3$.*

Proof. Since $f \in L^2(I)$ then $f = \langle f\rangle_I\chi_I + \sum_{J\in\mathcal{D}}(f, h_J)h_J$ implies

$$\begin{aligned}\langle |f|^2\rangle_I &= \frac{1}{|I|}\int_I |f|^2 \\ &= \langle f\rangle_I^2 + 2\langle f\rangle_I \sum_{J\in\mathcal{D}}(f,h_J)\frac{1}{|I|}\int_I h_J + \frac{1}{|I|}\int_I \sum_{J,K\in\mathcal{D}}(f,h_J)(f,h_K)h_J h_K \\ &= \langle f\rangle_I^2 + \frac{1}{|I|}\sum_{J\in\mathcal{D}}|(f,h_J)|^2.\end{aligned}$$

So $\|f\|_2^2 = |I|x_3 = |I|x_1^2 + \sum_{J\in\mathcal{D}} |(f, h_J)|^2$ and similarly

$$\|g\|_2^2 = |I|x_2^2 + \sum_{J\in\mathcal{D}} |(g, h_J)|^2 = |I|x_2^2 + \sum_{J\in\mathcal{D}} |(f, h_J)|^2.$$

Now we can compute $\mathcal{B}$ explicitly, $(p = 2)$

$$\begin{aligned}\langle (g^2 + \tau^2 f^2)^{\frac{p}{2}}\rangle_I &= \langle |g|^2\rangle_I + \tau^2\langle |f|^2\rangle_I = x_2^2 + \tau^2 x_1^2 + (1+\tau^2)\frac{1}{|I|}\sum_{J\in\mathcal{D}} |(f, h_J)|^2 \\ &= x_2^2 + \tau^2 x_1^2 + (1+\tau^2)(x_3 - x_1^2). \quad \square\end{aligned}$$

1.2. Outline of argument to prove main result

Computing the Bellman function, $\mathcal{B}$, for $p \neq 2$, is much more difficult, so more machinery is needed. In Section 1.3, we will derive properties of the Bellman function, the most notable of which is concavity under certain conditions. Finding a $\mathcal{B}$ to satisfy the concavity will amount to solving a partial differential equation, after adding an assumption. This PDE has a solution on characteristics that is well known, so we just need to find an explicit solution from this, using the Bellman function properties. How the characteristics behave in the domain of definition for the Bellman function will give us several cases to consider. In Section 2, we will get a Bellman function candidate for $1 < p < \infty$ by putting together several cases. Once we have what we think is the Bellman function, we need to show that it has the necessary smoothness and that Assumption 7 was not too restrictive to give us the Bellman function. This is covered in Section 3. Finally the main result is shown in Section 4. In Section 6, we show why several cases did not lead to a Bellman function candidate and why the value of τ was restricted for the Bellman function candidate.

1.3. Properties of the Bellman function

One of the properties we nearly always have (or impose) for any Bellman function, is concavity (or convexity). It is not true that $\mathcal{B}$ is globally concave, on all of Ω, but under certain conditions it is concave. The needed condition is that g is the martingale transform of f, or $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ in terms of the variables in Ω.

Definition 3. We say that the function B on Ω has restrictive concavity if for all $x^{\pm} \in \Omega$ such that $x = \alpha^+ x^+ + \alpha^- x^-, \alpha^+ + \alpha^- = 1$ and $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ then $\mathcal{B}(x) \geq \alpha^+\mathcal{B}(x^+) + \alpha^-\mathcal{B}(x^-)$.

Proposition 4. *The Bellman function $\mathcal{B}$ is restrictively concave in the x-variables.*

Proof. Let $\varepsilon > 0$ be given and $x^{\pm} \in \Omega$. By the definition of $\mathcal{B}$, there exists $f^{\pm}, g^{\pm}$ on $I^{\pm}$ such that $\langle f\rangle_{J^\pm} = x_1^{\pm}$, $\langle g\rangle_{J^\pm} = x_2^{\pm}$, $\langle |f^{\pm}|^p\rangle_{I^\pm} = x_3^{\pm}$ and

$$\mathcal{B}(x^{\pm}) - \langle [(g^{\pm})^2 + \tau^2 (f^{\pm})^2]^{\frac{p}{2}}\rangle_{I^\pm} \leq \varepsilon.$$

On $I = I^+ \cup I^-$ we define f and g as $f := f^+\chi_{I^+} + f^-\chi_{I^-}$, $g := g^+\chi_{I^+} + g^-\chi_{I^-}$. So,

$$|x_1^+ - x_1^-| = |\langle f\rangle_{I^+} - \langle f\rangle_{I^-}| = \left|\frac{1}{|I^+|}\int_{|I^-|} f - \frac{1}{|I^-|}\int_{I^-} f\right|$$

$$= \left| \frac{1}{\alpha^+|I|} \int_{|I^-|} f - \frac{1}{\alpha^-|I|} \int_{I^-} f \right| = \frac{1}{|I|} \left| \int f \left(\frac{1}{\alpha^+} \chi_{I^+} - \frac{1}{\alpha^-} \chi_{I^-} \right) \right|$$
$$= \sqrt{\frac{|I|}{\alpha^+\alpha^-}} \left| \int f h_I \right| =: \sqrt{\frac{|I|}{\alpha^+\alpha^-}} \, |(f, h_I)| .$$

Similarly, $|x_2^+ - x_2^-| = \sqrt{\frac{|I|}{\alpha^+\alpha^-}} \, |(g, h_I)|$. So our assumption $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ is equivalent to $|(f, h_I)| = |(g, h_I)|$. Since $x_1 = \langle f \rangle_I, x_2 = \langle g \rangle_I$ and $x_3 = \langle |f|^p \rangle_I$ then f and g are test functions and so

$$\begin{aligned} \mathcal{B}(x) &\geq \langle (g^2 + \tau^2 f^2)^{\frac{p}{2}} \rangle_I \\ &= \alpha^+ \langle [(g^+)^2 + \tau^2 (f^+)^2]^{\frac{p}{2}} \rangle_{I^+} + \alpha^- \langle [(g^-)^2 + \tau^2 (f^-)^2]^{\frac{p}{2}} \rangle_{I^-} \\ &\geq \alpha^+ \mathcal{B}(x^+) + \alpha^- \mathcal{B}(x^-) - \varepsilon. \quad \square \end{aligned}$$

At this point we do not quite have concavity of $\mathcal{B}$ on Ω since there is the restriction $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ needed. To make this condition more manageable, we will make a change of coordinates. Let $y_1 := \frac{x_2+x_1}{2}$, $y_2 := \frac{x_2-x_1}{2}$ and $y_3 := x_3$. We will also change notation for the Bellman function and corresponding domain in the new variable y. Let $\mathcal{M}(y_1, y_2, y_3) := \mathcal{B}(x_1, x_2, x_3) = \mathcal{B}(y_1 - y_2, y_1 + y_2, y_3)$. Then the domain of definition for $\mathcal{M}$ will be $\Xi := \{y \in \mathbb{R}^3 : y_3 \geq 0, |y_1 - y_2|^p \leq y_3\}$.

If we consider $x^\pm \in \Omega$ such that $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$, then the corresponding points $y^\pm \in \Xi$ satisfy either $y_1^+ = y_1^-$ or $y_2^+ = y_2^-$. This implies that fixing y_1 as $y_1^+ = y_1^-$ or y_2 as $y_2^+ = y_2^-$ will make $\mathcal{M}$ concave with respect to y_2, y_3 under fixed y_1 and with respect to y_1, y_3 under y_2 fixed.

Rather than using Proposition 4 to check the concavity of the Bellman function we can just check it in the following way, assuming $\mathcal{M}$ is C^2. Let $j \neq i \in \{1, 2\}$ and fix y_i as $y_i^+ = y_i^-$. Then $\mathcal{M}$ as a function of y_j, y_3 is concave if

$$\begin{pmatrix} \mathcal{M}_{y_j y_j} & \mathcal{M}_{y_j y_3} \\ \mathcal{M}_{y_3 y_j} & \mathcal{M}_{y_3 y_3} \end{pmatrix} \leq 0,$$

which is equivalent to

$$\mathcal{M}_{y_j y_j} \leq 0, \qquad \mathcal{M}_{y_3 y_3} \leq 0, \qquad D_j = \mathcal{M}_{y_j y_j} \mathcal{M}_{y_3 y_3} - \mathcal{M}_{y_3 y_j} \mathcal{M}_{y_j y_3} \geq 0.$$

Proposition 5 (*Restrictive Concavity in y-Variables*)**.** *Let $j \neq i \in \{1, 2\}$ and fix y_i as $y_i^+ = y_i^-$. If $\mathcal{M}_{y_j y_j} \leq 0, \mathcal{M}_{y_3 y_3} \leq 0$ and $D_j = \mathcal{M}_{y_j y_j} \mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_j y_3})^2 \geq 0$ for $j = 1$ and $j = 2$ then $\mathcal{M}$ is Restrictively concave.*

The Bellman function, as it turns out, has many other nice properties.

Proposition 6. *Suppose that $\mathcal{M}$ is $C^1(\mathbb{R}^3)$, then $\mathcal{M}$ has the following properties.*

(i) *Symmetry: $\mathcal{M}(y_1, y_2, y_3) = \mathcal{M}(y_2, y_1, y_3) = \mathcal{M}(-y_1, -y_2, y_3)$.*
(ii) *Dirichlet boundary data: $\mathcal{M}(y_1, y_2, (y_1 - y_2)^p) = ((y_1 + y_2)^2 + \tau^2 (y_1 - y_2)^2)^{\frac{p}{2}}$.*
(iii) *Neumann conditions: $\mathcal{M}_{y_1} = \mathcal{M}_{y_2}$ on $y_1 = y_2$ and $\mathcal{M}_{y_1} = -\mathcal{M}_{y_2}$ on $y_1 = -y_2$.*
(iv) *Homogeneity: $\mathcal{M}(r y_1, r y_2, r^p y_3) = r^p \mathcal{M}(y_1, y_2, y_3), \forall r > 0$.*
(v) *Homogeneity relation: $y_1 \mathcal{M}_{y_1} + y_2 \mathcal{M}_{y_2} + p y_3 \mathcal{M}_{y_3} = p \mathcal{M}$.*

Proof. (i) Note that we get $\mathcal{B}(x_1, x_2, x_3) = \mathcal{B}(-x_1, x_2, x_3) = \mathcal{B}(x_1, -x_2, x_3)$ by considering test functions $\widetilde{f} = -f$ and $\widetilde{g} = -g$. Change coordinates from x to y and the result follows.

(ii) On the boundary $\{x_3 = |x_1|^p\}$ of Ω we see that

$$\frac{1}{|I|}\int_I |f|^p = \langle |f|^p\rangle_I = x_3 = |x_1|^p = |\langle f\rangle_I|^p = \left|\frac{1}{|I|}\int_I f\right|^p$$

is only possible if $f \equiv$ const. (i.e. $f = x_1$). But, $|(f, h_J)| = |(g, h_J)|$ for all $J \in \mathcal{I}$, which implies that $g \equiv$ const. (i.e. $g = x_2$). Then $\mathcal{B}(x_1, x_2, |x_1|^p) = \langle (g^2 + \tau^2 f^2)^{\frac{p}{2}}\rangle_I = (x_2^2 + \tau^2 x_1^2)^{\frac{p}{2}}$. Changing coordinates gives the result.

(iii) This follows from (i).

(iv) Consider the test functions $\widetilde{f} = rf, \widetilde{g} = rg$.

(v) Differentiate (iv) with respect to r and evaluate it at $r = 1$. □

Now that we have all of the properties of the Bellman function we will turn our attention to actually finding it. Proposition 5 gives us two partial differential inequalities to solve, $D_1 \geq 0$, $D_2 \geq 0$, that the Bellman function must satisfy. Since the Bellman function is the supremum of the left-hand side of our estimate under the condition that g is the martingale transform of f, and must also satisfy the estimates in Proposition 5, then it seems reasonable that the Bellman function (being the optimal such function) may satisfy the following, for either $j = 1$ or $j = 2$:

$$D_j = \mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_3 y_j})^2 = 0.$$

The PDE that we now have is the well known Monge–Ampère equation which has a solution. Let us make it clear that we have added an assumption.

Assumption 7. $D_j = \mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_3 y_j})^2 = 0$, for either $j = 1$ or $j = 2$.

Adding this assumption comes with a price. Any function that we construct, satisfying all properties of the Bellman function, must somehow be shown to be the Bellman function. We will refer to any function satisfying some, or all Bellman function properties as a Bellman function candidate.

Proposition 8. *For $j = 1$ or 2, $\mathcal{M}_{y_j y_j}\mathcal{M}_{y_3 y_3} - (\mathcal{M}_{y_3 y_j})^2 = 0$ has the solution $M(y) = y_j t_j + y_3 t_3 + t_0$ on the characteristics $y_j dt_j + y_3 dt_3 + dt_0 = 0$, which are straight lines in the $y_j \times y_3$ plane. Furthermore, t_0, t_j, t_3 are constant on characteristics with the property $M_{y_j} = t_j$, $M_{y_3} = t_3$.*

This is a result of Pogorelov; see [18,20]. Now that we have a solution M to the Monge–Ampère, we need to get rid of t_0, t_j, t_3 so that we have an explicit form of M, without the characteristics. We note that a solution to the Monge–Ampère is not necessarily the Bellman function. It must satisfy the restrictive concavity of Proposition 5, be C^1-smooth, and satisfy the properties of Proposition 6. The restrictive concavity property is one of the key deciding factors of whether or not we have a Bellman function in many cases. Even if the Monge–Ampère solution satisfies all of those conditions, it must still be shown to be equal to the Bellman function, because we added an additional assumption (Assumption 7) to get the Monge–Ampère solution as a starting point. This will be considered rigorously in Section 3, after we obtain a solution to the Monge–Ampère equation, with the appropriate Bellman function properties. So from this point on we will use M and B to denote solutions to the Monge–Ampère equation, i.e. Bellman function candidates, and $\mathcal{M}$ and $\mathcal{B}$ to denote the true Bellman function.

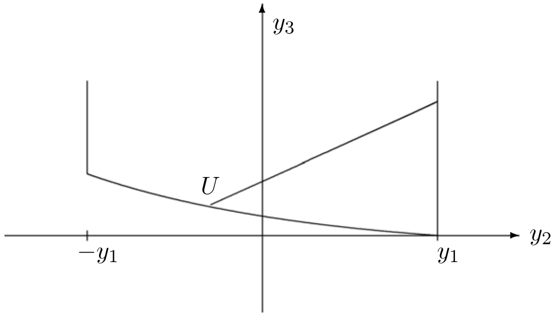


Fig. 1. Sample characteristic of solution from Case (1_2).

2. Computing the Bellman function candidate from the Monge–Ampère solution

Due to the symmetry property of $\mathcal{M}$, from Proposition 6, we only need to consider a portion of the domain Ξ, which we will denote as, $\Xi_+ := \{y : -y_1 \leq y_2 \leq y_1, y_3 \geq 0, (y_1 - y_2)^p \leq y_3\}$. Since the characteristics are straight lines, then one end of each line must be on the boundary $\{y : (y_1 - y_2)^p = y_3\}$. Let U denote the point at which the characteristic touches the boundary. Furthermore, the characteristics can only behave in one of the following four ways, since they are straight lines in the plane:

(1) The characteristic goes from U to $\{y : y_1 = y_2\}$.
(2) The characteristic goes from U to infinity, running parallel to the y_3-axis.
(3) The characteristic goes from U to $\{y : y_1 = -y_2\}$.
(4) The characteristic goes from U to $\{y : (y_1 - y_2)^p = y_3\}$.

To find a Bellman function candidate we must first fix a variable (y_1 or y_2) and a case for the characteristics. Then we use the Bellman function properties to get rid of the characteristics. If the Monge–Ampère solution satisfies restrictive concavity, then it is a Bellman function candidate. However, checking the restrictive concavity is quite difficult in many of the cases, since it amounts to doing second derivative estimates for an implicitly defined function. Let us now find our Bellman function candidate.

Remark 9. Since we will have either y_1 or y_2 fixed in each case, then there will be eight cases in all. Let (1_j), (2_j), (3_j), (4_j) denote the case when $M_{y_j y_j} M_{y_3 y_3} - (M_{y_3 y_j})^2 = 0$ and y_i is fixed, where $i \neq j$. Also, we will denote $G(z_1, z_2) := (z_1 + z_2)^{p-1}[z_1 - (p-1)z_2]$ and $\omega := \left(\frac{\mathcal{M}(y)}{y_3}\right)^{\frac{1}{p}}$ from this point on.

2.1. Bellman candidate for $2 < p < \infty$

The solution to the Monge–Ampère equation when $2 < p < \infty$, is only partially valid on the domain in two cases, due to restrictive concavity. Case (1_2), will give us an implicit solution that is valid on part of Ξ_+ and Case (2_2) will give us an explicit solution for the remaining part of Ξ_+. First, we deal with Case (1_2). (See Fig. 1)

2.1.1. Case (1_2)

Since we are considering Case (1_2), then $y_1 \geq 0$ is fixed until the point that we have the implicit solution independent of the characteristics satisfying all of the Bellman function properties.

Proposition 10. *For* $1 < p < \infty$ *and* $\frac{p-2}{p}y_1 < y_2 < y_1$, M *is given implicitly by the relation* $G(y_1+y_2, y_1-y_2) = y_3 G(\sqrt{\omega^2-\tau^2}, 1)$, *where* $G(z_1, z_2) := (z_1+z_2)^{p-1}[z_1-(p-1)z_2]$ *on* $z_1+z_2 \geq 0$ *and* $\omega := \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}$.

This is proven through a series of lemmas.

Lemma 11. $M(y) = t_2y_2 + t_3y_3 + t_0$ *on the characteristic* $y_2dt_2 + y_3dt_3 + dt_0 = 0$ *can be simplified to* $M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$, *where* u *is the unique solution to the equation* $\frac{y_2+(\frac{2}{p}-1)y_1}{y_3} = \frac{u+\left(\frac{2}{p}-1\right)y_1}{(y_1-u)^p}$ *and* $\frac{p-2}{p}y_1 < y_2 < y_1$.

Proof. A characteristic in Case (1_2) is from $U = (y_1, u, (y_1-u)^p)$ to $W = (y_1, y_1, w)$. Throughout the proof we will use the properties of the Bellman function from Proposition 6. Using the Neumann property and the property from Proposition 8 we get $M_{y_1} = M_{y_2} = t_2$ at W. By homogeneity at W we get

$$py_2t_2 + pwt_3 + pt_0 = pM(W) = y_1M_{y_1} + y_2M_{y_2} + py_3M_{y_3} = 2y_1t_2 + pwt_3.$$

Then $t_0 = (\frac{2}{p}-1)y_1t_2$ and $dt_0 = (\frac{2}{p}-1)y_1dt_2$, since y_1 is fixed. So $M(y) = [y_2+(\frac{2}{p}-1)y_1]t_2 + y_3t_3$ on $[y_2+(\frac{2}{p}-1)y_1]dt_2 + y_3dt_3 = 0$. By substitution we get, $M(y) = y_3[t_3 - t_2\frac{dt_3}{dt_2}]$ on characteristics. But, $t_2, t_3, \frac{dt_3}{dt_2}$ are constant on characteristics, which gives that $\frac{M(y)}{y_3} \equiv$ const. as well. We can calculate the value of the constant by using the Dirichlet boundary data for M at U. Therefore, $M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$, where u is the solution to the equation

$$\frac{y_2 + \left(\frac{2}{p}-1\right)y_1}{y_3} = \frac{u + \left(\frac{2}{p}-1\right)y_1}{(y_1-u)^p}. \tag{2.1}$$

Now fix $u = -(\frac{2}{p}-1)y_1$. Then we see that $y_2 = -(\frac{2}{p}-1)y_1 = u$ is also fixed by (2.1). This means that the characteristics are limited to part of the domain, as shown in Fig. 2, since they start at U and end at $W \in \{y_1 = y_2\}$. All that remains is verifying Eq. (2.1) has exactly one solution $u = u(y_1, y_2, y_3)$ in the sector $\frac{p-2}{p}y_1 < y_2 < y_1$. Indeed, the function

$$f(u) := y_3\left[u + \left(\frac{2}{p}-1\right)y_1\right] - (y_1-u)^p\left[y_2 + \left(\frac{2}{p}-1\right)y_1\right]$$

is monotone increasing for $u < y_1$, $f(-(\frac{2}{p}-1)y_1) = -\left(\frac{2}{p}y_1\right)^p\left[y_2 + (\frac{2}{p}-1)y_1\right] < 0$ and $f(y_1) = \frac{2}{p}y_1y_3 > 0$. Therefore, we do get a unique solution, u, in the sector. □

Lemma 12. $M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$ *can be rewritten as* $G(y_1+y_2, y_1-y_2) = y_3G(\sqrt{\omega^2-\tau^2}, 1)$ *for* $\frac{p-2}{p}y_1 < y_2 < y_1$.

Proof. $\omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}} = \frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u} \geq |\tau|\frac{|y_1-u|}{y_1-u} = |\tau|.$

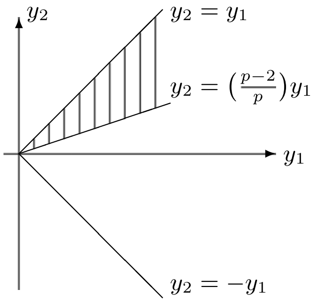


Fig. 2. Sector for characteristics in Case (1_2), when $p > 2$.

Since $y_1 \pm u \geq 0$ and $\omega^2 - \tau^2 \geq 0$, then $u = \frac{\sqrt{\omega^2-\tau^2}-1}{\sqrt{\omega^2-\tau^2}+1} y_1$ by inversion. Substituting this into $\frac{y_2+\left(\frac{2}{p}-1\right)y_1}{y_3} = \frac{u+\left(\frac{2}{p}-1\right)y_1}{(y_1-u)^p}$ gives

$$2^{p-1} y_1^{p-1}[py_2 - (p-2)y_1] = y_3(\sqrt{\omega^2 - \tau^2} + 1)^{p-1}[\sqrt{\omega^2 - \tau^2} - (p-1)]$$

or $(x_1+x_2)^{p-1}[x_2-(p-1)x_1] = \left[\sqrt{B^{\frac{2}{p}} - \left(\tau x_3^{\frac{1}{p}}\right)^2} + x_3^{\frac{1}{p}}\right]^{p-1} \left[\sqrt{B^{\frac{2}{p}} - \left(\tau x_3^{\frac{1}{p}}\right)^2} - (p-1) x_3^{\frac{1}{p}}\right]$. Thus, $G(x_2, x_1) = G\left(\sqrt{B^{\frac{2}{p}} - \left(\tau x_3^{\frac{1}{p}}\right)^2}, x_3^{\frac{1}{p}}\right)$ or $G(y_1 + y_2, y_1 - y_2) = y_3 G(\sqrt{\omega^2 - \tau^2}, 1)$. $\square$

This proves Proposition 10. We have constructed a partial Bellman function candidate from the Monge–Ampère solution in Case (1_2), so y_1 no longer needs to be fixed. All of the properties of the Bellman function were used to derive this partial Bellman candidate, but the restrictive concavity from Proposition 5 still needs to be verified. To verify restrictive concavity, we need that $M_{y_2y_2} \leq 0, M_{y_3y_3} \leq 0, D_2 \geq 0$ and $M_{y_1y_1} \leq 0, D_1 \geq 0$. By assumption $D_2 = 0$, we need not worry about that estimate. The remaining estimates will be verified in a series of lemmas. The first lemma is an idea taken from Burkholder [6] to make the calculations for computing mixed partials shorter. In the lemma, we compute the partials of arbitrary functions which we will choose specifically later, although it is not hard to see what the appropriate choices should be.

Lemma 13. *Let* $H = H(y_1, y_2)$, $\Phi(\omega) = \frac{H(y_1,y_2)}{y_3}$, $R_1 = R_1(\omega) := \frac{1}{\Phi'}$ *and* $R_2 = R_2(\omega) := R_1' = -\frac{\Phi''}{\Phi'^2}$. *Then*

$$M_{y_3y_3} = \frac{p\omega^{p-2}R_1H^2}{y_3^3}[\omega R_2 + (p-1)R_1]$$

$$M_{y_3y_i} = -\frac{p\omega^{p-2}R_1HH'}{y_3^2}[\omega R_2 + (p-1)R_1]$$

$$M_{y_i y_i} = \frac{p\omega^{p-2}R_1}{y_3}\Big([\omega R_2 + (p-1)R_1](H')^2 + \omega y_3 H''\Big)$$
$$D_i = M_{y_3 y_3} M_{y_i y_i} - M^2_{y_3 y_i} = \frac{p^2\omega^{2p-3}R_1^2 H^2 H''}{y_3^3}[\omega R_2 + (p-1)R_1].$$

Proof. First of all we calculate the partial derivatives of ω:

$$\Phi'\omega_{y_3} = -\frac{H}{y_3^2} \Longrightarrow \omega_{y_3} = -\frac{R_1 H}{y_3^2},$$
$$\Phi'\omega_{y_i} = \frac{H_{y_i}}{y_3} \Longrightarrow \omega_{y_i} = \frac{R_1 H_{y_i}}{y_3} = \frac{R_1 H'}{y_3}, \quad i = 1, 2.$$

Here and further we shall use notation H' for any partial derivative H_{y_i}, $i = 1, 2$. This cannot cause any confusion since only one $i \in \{1, 2\}$ participate in the calculation of D_i.

$$\omega_{y_3 y_3} = -\frac{R_2\omega_{y_3} H}{y_3^2} + 2\frac{R_1 H}{y_3^3} = \frac{R_1 H}{y_3^4}(R_2 H + 2y_3),$$
$$\omega_{y_3 y_i} = -\frac{R_2\omega_{y_i} H}{y_3^2} - \frac{R_1 H'}{y_3^2} = -\frac{R_1 H'}{y_3^3}(R_2 H + y_3),$$
$$\omega_{y_i y_i} = \frac{R_2\omega_{y_i} H}{y_3} + \frac{R_1 H'}{y_3} = \frac{R_1}{y_3^2}(R_2(H')^2 + y_3 H'').$$

Now we pass to the calculation of derivatives of $M = y_3\omega^p$:

$$M_{y_3} = py_3\omega^{p-1}\omega_{y_3} + \omega^p,$$
$$M_{y_i} = py_3\omega^{p-1}\omega_{y_i};$$
$$\begin{aligned} M_{y_3 y_3} &= py_3\omega^{p-1}\omega_{y_3 y_3} + 2p\omega^{p-1}\omega_{y_3} + p(p-1)y_3\omega^{p-2}\omega^2_{y_3} \\ &= \frac{p\omega^{p-2}R_1 H^2}{y_3^3}[\omega R_2 + (p-1)R_1], \end{aligned} \tag{2.2}$$
$$\begin{aligned} M_{y_3 y_i} &= py_3\omega^{p-1}\omega_{y_3 y_i} + p\omega^{p-1}\omega_{y_i} + p(p-1)y_3\omega^{p-2}\omega_{y_3}\omega_{y_i} \\ &= -\frac{p\omega^{p-2}R_1 H H'}{y_3^2}[\omega R_2 + (p-1)R_1], \end{aligned}$$
$$\begin{aligned} M_{y_i y_i} &= py_3\omega^{p-1}\omega_{y_i y_i} + p(p-1)y_3\omega^{p-2}\omega^2_{y_i} \\ &= \frac{p\omega^{p-2}R_1}{y_3}\Big([\omega R_2 + (p-1)R_1](H')^2 + \omega y_3 H''\Big). \end{aligned} \tag{2.3}$$

This yields

$$D_i = M_{y_3 y_3} M_{y_i y_i} - M^2_{y_3 y_i} = \frac{p^2\omega^{2p-3}R_1^2 H^2 H''}{y_3^3}[\omega R_2 + (p-1)R_1]. \quad \square \tag{2.4}$$

Lemma 14. *If* $\alpha_i, \beta_i \in \{\pm 1\}$ *and* $H(y_1, y_2) = G(\alpha_1 y_1 + \alpha_2 y_2, \beta_1 y_1 + \beta_2 y_2)$ *then*

$$H'' = \begin{cases} 4G_{z_1 z_2}, & \alpha_j = \beta_j \\ 0, & \alpha_j = -\beta_j. \end{cases}$$

Consequently, in Case (1_2), $\operatorname{sign} H'' = -\operatorname{sign}(p-2)$.

Proof.

$$\begin{aligned} H'' &= \frac{\partial^2}{\partial y_i^2} G(\alpha_1 y_1 + \alpha_2 y_2, \beta_1 y_1 + \beta_2 y_2) \\ &= \alpha_i^2 G_{z_1 z_1} + 2\alpha_i \beta_i G_{z_1 z_2} + \beta_i^2 G_{z_2 z_2} \\ &= G_{z_1 z_1} + G_{z_2 z_2} \pm 2G_{z_1 z_2}, \end{aligned}$$

where the "+" sign has to be taken if the coefficients in front of y_i are equal and the "−" sign in the opposite case.

The derivatives of G are simple:

$$\begin{aligned} G_{z_1} &= p(z_1 + z_2)^{p-2}\big[z_1 - (p-2)z_2\big], \\ G_{z_2} &= -p(p-1)z_2(z_1 + z_2)^{p-2}; \\ G_{z_1 z_2} &= p(p-1)(z_1 + z_2)^{p-3}\big[z_1 - (p-3)z_2\big], \\ G_{z_1 z_2} &= -p(p-1)(p-2)z_2(z_1 + z_2)^{p-3}, \\ G_{z_2 z_2} &= -p(p-1)(z_1 + z_2)^{p-3}\big[z_1 + (p-1)z_2\big]. \end{aligned}$$

Note that $G_{z_1 z_1} + G_{z_2 z_2} = 2G_{z_1 z_2}$, and therefore,

$$H'' = \begin{cases} 4G_{z_1 z_2}, & \alpha_j = \beta_j \\ 0, & \alpha_j = -\beta_j. \end{cases}$$

Now in Case (1_2), we must choose $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1$ and $\beta_2 = -1$ for H to match how the implicit solution was defined in terms of G in Proposition 10. Then $G_{z_1 z_2} = -p(p-1)(p-2)(y_1 - y_2)(2y_1)^{p-3}$. □

Remark 15. Let $\beta := \sqrt{\omega^2 - \tau^2}$ from this point on. In Case (1_2), $\beta > p-1$ in the sector $\frac{p-2}{p} y_1 \langle y_2 < y_1$. Equivalently, $B\rangle(\tau^2 + (p-1)^2)^{\frac{p}{2}}$ in $\frac{p-2}{p} y_1 < y_2 < y_1$.

This is an easy application of Proposition 10:

$$\begin{aligned} (\beta+1)^{p-1}[\beta - p + 1] = G(\beta, 1) &= \frac{1}{y_3} G(y_1 + y_2, y_1 - y_2) \\ &= (2y_1)^{p-1}[-(p-2)y_1 + p y_2] > 0. \end{aligned}$$

Before we can compute the signs of $M_{y_1 y_1}$, $M_{y_2 y_2}$, $M_{y_3 y_3}$ and D_1 we need a technical lemma.

Lemma 16. *If* $1 < p < \infty$ *and* $\tau \in \mathbb{R}$, *then*

$$g(\beta) := -p(p-2)\omega\beta^{-3}(\beta+1)^{p-3}[(\tau^2 + p - 1)\beta^2 - \tau^2(p-3)\beta + \tau^2]$$

satisfies $\operatorname{sign} g(\beta) = -\operatorname{sign}(p-2)$ *in Case* (1_2).

Proof. The only terms controlling the sign in g are $(p-2)$ and the quadratic part, which we will denote $q(\beta)$. So we need to simply figure out the sign of q. The discriminant of q is $\tau^2(p-1)[\tau^2(p-5)-4]$.

If $p \le 5$ then the discriminant of q is negative and so $q(\beta) > 0$. If $p > 5$, and $\tau^2(p-5)-4 < 0$ then $q(\beta) > 0$ once again.

The only case left to consider is if $p > 5$ and $\tau^2(p-5)-4 \ge 0$. The zeros of q are given by $\beta = \frac{\tau^2(p-3)\pm|\tau|\sqrt{p-1}\sqrt{\tau^2(p-5)-4}}{2(\tau^2+p-1)}$. Let β_1, β_2 be the zeros such that $\beta_2 \ge \beta_1$. We claim that $\max\{p-1, \beta_2\} = p-1$. Indeed, $p-1-\beta_2 > 0$

$$\begin{aligned}
&\Longleftrightarrow (p+1)\tau^2 + 2(p-1)^2 > |\tau|\sqrt{p-1}\sqrt{\tau^2(p-5)-4} \\
&\Longleftrightarrow 4(p-1)^4 + 4\tau^2(p+1)(p-1)^2 + \tau^4(p+1)^2 > \tau^2(p-1)(\tau^2(p-5)-4) \\
&\Longleftrightarrow (p-1)^4 + \tau^2 p^2(p-1) + \tau^4(2p-1) > 0,
\end{aligned}$$

which is obviously true for all $\tau \in \mathbb{R}$. Now that we have proven the claim, recall that $\beta > p-1$, as shown in Remark 15. Therefore, $\beta > \beta_2$, so $q(\beta) > 0$ in this case. □

Lemma 17. *$D_1 > 0$ in Case (1_2) for all $\tau \in \mathbb{R}$.*

Proof. We use the partial derivatives of G computed in the proof of Lemma 14 to make the computations of Φ' and Φ'' easier.

$$\begin{aligned}
\Phi(\omega) &= G(\beta, 1) \\
\Phi'(\omega) &= p\omega[\beta+1]^{p-2}[1-(p-2)\beta^{-1}] \qquad (2.5) \\
\Phi''(\omega) &= p(\beta+1)^{p-2}[1-(p-2)\beta^{-1}] \\
&\quad + p(p-2)\omega^2\beta^{-1}[\beta+1]^{p-3}[1-(p-2)\beta^{-1}] \\
&\quad + p(p-2)\omega^2\beta^{-3}[\beta-1]^{p-2}
\end{aligned}$$

$$\begin{aligned}
\Lambda &= (p-1)\Phi' - \omega\Phi'' \\
&= p(p-2)\omega\beta^{-1}(\beta+1)^{p-2}[1-(p-2)\beta^{-1}] \\
&\quad - p(p-2)\omega^3\beta^{-3}(\beta+1)^{p-3}[\beta(\beta-p+2)+\beta+1] \\
&= p(p-2)\omega\beta^{-2}(\beta+1)^{p-3}[\beta-p+2]\{\beta(\beta+1)-\omega^2\} \\
&\quad - p(p-2)\omega^3\beta^{-3}(\beta+1)^{p-2} \\
&= p(p-2)\omega\beta^{-3}[\beta(\beta-p+2)(\beta-\tau^2)-\omega^2(\beta+1)] \\
&= -p(p-2)\omega\beta^{-3}(\beta+1)^{p-3}[(\tau^2+p-1)\beta^2 - \tau^2(p-3)\beta + \tau^2]. \qquad (2.6)
\end{aligned}$$

So we can see that $\operatorname{sign}\Lambda = \operatorname{sign} g(\beta) = -\operatorname{sign}(p-2)$, by Lemma 16. Therefore, $\operatorname{sign} D_1 = \operatorname{sign} H'' \operatorname{sign}\Lambda = [-\operatorname{sign}(p-2)]^2$ by (2.4) and Lemma 14. □

Since $D_1 > 0$, then all that remains to be checked, for the restrictive concavity of M, is that $M_{y_iy_i}$ (for $i = 1, 2$) and $M_{y_3y_3}$ have the appropriate signs. But, it turns out that only for $2 < p < \infty$, will these have the appropriate signs.

Lemma 18. $\operatorname{sign} M_{y_1y_1} = \operatorname{sign} M_{y_2y_2} = \operatorname{sign} M_{y_3y_3} = -\operatorname{sign}(p-2)$ *in Case* (1_2) *for all* $\tau \in \mathbb{R}$. *Therefore, M is a partial Bellman function candidate for $2 < p < \infty$ but not for $1 < p < 2$, since it does not satisfy the required restrictive concavity.*

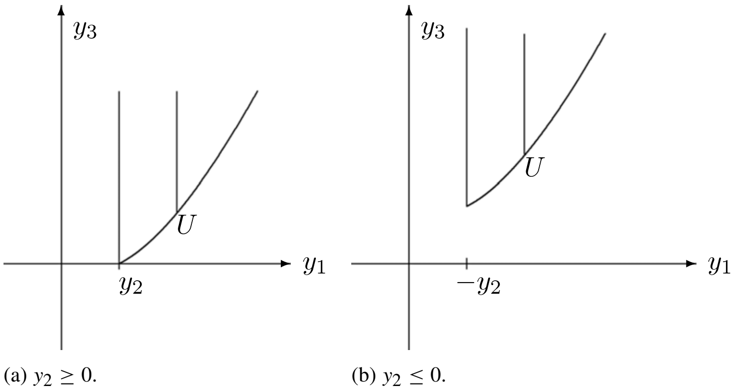


Fig. 3. Sample characteristic of the Monge–Ampère solution in Case (2_1).

Proof. By (2.2),

$$M_{y_3y_3} = \frac{p\omega^{p-2}R_1^2H^2}{y_3^3}\left[\frac{\Lambda}{\Phi'}\right].$$

Remark 15 gives $\Phi' > 0$. From Lemma 16, $\operatorname{sign} M_{y_3y_3} = \operatorname{sign}\Lambda = \operatorname{sign} g(\beta) = -\operatorname{sign}(p-2)$. By (2.3), for $i = 1$ or 2,

$$\begin{aligned} M_{y_iy_i} &= \frac{p\omega^{p-2}R_1}{y_3}\left[(\omega R_2 + (p-1)R_1)(H')^2 + \omega y_3 H''\right] \\ &= \frac{p\omega^{p-2}}{y_3(\Phi')^3}\left[\Lambda(H')^2 + \omega y_3 H''(\Phi')^2\right], \end{aligned}$$

giving $\operatorname{sign} M_{y_2y_2} = -\operatorname{sign}(p-2)$. □

The previous two lemmas established that the partial Bellman function candidate, from Case (1_2) satisfies the restrictive concavity property, for $2 < p < \infty$. The candidate was constructed using the remaining Bellman function properties, so it is in fact a partial candidate. Now we will turn our attention to Case (2). As it turns out, Case (2_2) also gives a partial Bellman function candidate, which, as luck would have it, is the missing half of the partial Bellman candidate just constructed.

2.1.2. *Case* (2_2) *for* $2 < p < \infty$

We can obtain a Bellman candidate from Case (2) without having to separately fix y_1 or y_2. Let us compute the solution in this case. (Figs. 3 and 4)

Lemma 19. *In Case* (2) *we obtain*

$$M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1y_2 + y_2^2]^{\frac{p}{2}} + c[y_3 - (y_1 - y_2)^p] \tag{2.7}$$

as a Bellman function candidate, where $c > 0$ *is some constant and* $\gamma = \frac{1-\tau^2}{1+\tau^2}$.

Proof. In Case (2), on the characteristic $y_i dt_i + y_3 dt_3 + dt_0 = 0$, y_1 and y_2 are fixed. Furthermore, on the characteristic, t_0, t_i, t_3 are fixed, so we have

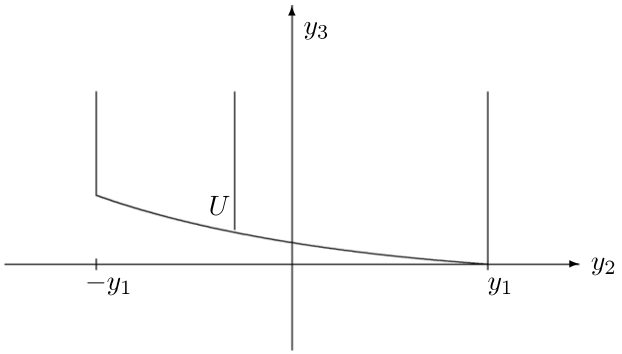


Fig. 4. Sample characteristic of the Monge–Ampère solution for Case (2_2).

$$\begin{aligned} M(y) &= y_i t_i + y_3 t_3 + t_0 \\ &= (y_i t_i + t_0) + y_3 t_3 \\ &= c_1(y_1, y_2) + c_2(y_1, y_2) y_3. \end{aligned}$$

Then $M_{y_3 y_3} = 0$ and $M_{y_3 y_i} = \partial_{y_i} c_2$. Recall that $D_i \geq 0$ by Proposition 5, so $\partial_{y_i} c_2(y_1, y_2) = 0$. This implies that c_2 is a constant. Using the boundary data from Proposition 6 gives $((y_1+y_2)^2 + \tau^2(y_1 - y_2)^2)^{\frac{p}{2}} = M(y_1, y_2, (y_1 - y_2)^p) = c_1(y_1, y_2) + c_2(y_1 - y_2)^p$. Solving for $c_1(y_1, y_2)$ gives the result. To see that $c_2 > 0$, just notice that as $y_3 \to \infty$, $M(y) \to \infty$. $\square$

It is not possible to determine if this Bellman function candidate satisfies restrictive concavity, unless we know the value of the constant c in Lemma 19. This constant can be computed by using the fact that (2.7) must agree with the partial candidate in Case (1_2) at $y_2 = \frac{p-2}{p} y_1$, if (2.7) is in fact a candidate itself.

Lemma 20. *In Case* (2_2), *the value of the constant in Lemma* 19 *is* $c = ((p-1)^2 + \tau^2)^{\frac{p}{2}}$ *for* $2 < p < \infty$.

Proof. If $M(y) = (1+\tau^2)^{\frac{p}{2}} [y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + c[y_3 - (y_1 - y_2)^p]$ (where $\gamma = \frac{1-\tau^2}{1+\tau^2}$) is to be a candidate, or partial candidate, then it must agree at $y_2 = \frac{p-2}{p} y_1$, with the solution M given implicitly by the relation $G(y_1 + y_2, y_1 - y_2) = y_3 G(\sqrt{\omega^2 - \tau^2}, 1)$, from Proposition 10. At $y_2 = \frac{p-2}{p} y_1$,

$$\begin{aligned} (\sqrt{\omega^2 - \tau^2} + 1)^{p-1} [\sqrt{\omega^2 - \tau^2} - p + 1] &= G(\sqrt{\omega^2 - \tau^2}, 1) \\ &= \frac{1}{y_3} (2y_1)^{p-1} [-(p-2) y_1 + (p-2) y_1] \\ &= 0. \end{aligned}$$

Since $\sqrt{\omega^2 - \tau^2} + 1 \neq 0$ then $\sqrt{\omega^2 - \tau^2} = p - 1$, which implies $\omega = ((p-1)^2 + \tau^2)^{\frac{1}{2}}$. So,

$$\begin{aligned} ((p-1)^2 + \tau^2)^{\frac{p}{2}} y_3 &= \omega^p y_3 \\ &= M\left(y_1, \frac{p-2}{p} y_1, y_3\right) \end{aligned}$$

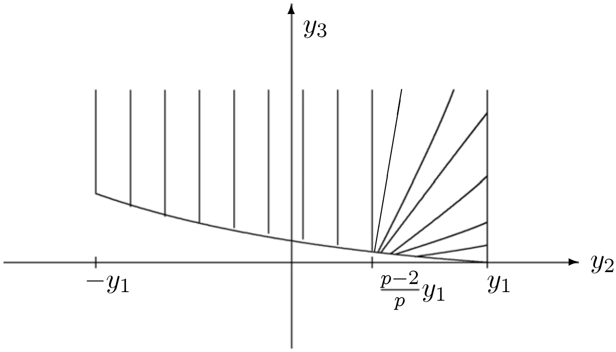


Fig. 5. Characteristics of Bellman candidate for $2 < p < \infty$.

$$= \left[\left(2\frac{p-1}{p}y_1\right)^2 + \tau^2\left(\frac{2}{p}y_1\right)^2\right]^{\frac{p}{2}} + c\left[y_3 - \left(\frac{p}{2}y_1\right)^p\right].$$

Now just solve for c. $\square$

2.1.3. Gluing together partial candidates from cases (1_2) *and* (2_2)

It turns out that the Bellman function candidate obtained from Case (2_2) is only valid on part of the domain Ξ_+, since it does not remain concave throughout (for example at or near (y_1, y_1, y_3)). As luck would have it, the partial candidate has the necessary restrictive concavity on the part of the domain where the candidate from Case (1_2) left off, i.e. in $-y_1 < y_2 < \frac{p-2}{p}y_1$. This means that we can glue together the partial candidate from Cases (1_2) and (2_2) to get a candidate on Ξ_+ for $2 < p < \infty$. The characteristics for this solution can be seen in Fig. 5.

Proposition 21. *For* $2 < p < \infty, \gamma = \frac{1-\tau^2}{1+\tau^2}$ *and* $\tau \in \mathbb{R}$, *the solution to the Monge–Ampère equation is given by*

$$M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + ((p-1)^2 + \tau^2)^{\frac{p}{2}}[y_3 - (y_1 - y_2)^p]$$

when $-y_1 < y_2 \le \frac{p-2}{p}y_1$ *and is given implicitly by*

$G(y_1 + y_2, y_1 - y_2) = y_3 G(\sqrt{\omega^2 - \tau^2}, 1)$ *when* $\frac{p-2}{p}y_1 \le y_2 < y_1$, *where* $G(z_1, z_2) = (z_1 + z_2)^{p-1}[z_1 - (p-1)z_2]$ *and* $\omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}$. *This solution satisfies all properties of the Bellman function.*

We already know that the implicit part of the solution has the correct restrictive concavity property of the Bellman function, as shown in Section 2.1.1. However, the restrictive concavity still needs to be verified for the explicit part. Since the explicit part of the solution satisfies $M_{y_3 y_i} = M_{y_3 y_3} = 0$, then $D_i = 0$, for $i = 1, 2$. So all that remains to be verified for the restrictive concavity of the explicit part is checking the sign of $M_{y_i y_i}$, for $i = 1, 2$. Observe that the explicit part can be written as

$$M(y) = [(y_1 + y_2)^2 + \tau^2(y_1 - y_2)^2]^{\frac{p}{2}} + C_{p,\tau}[y_3 - (y_1 - y_2)^p]. \tag{2.8}$$

It is easy to check that $M_{y_2 y_2} \le M_{y_1 y_1}$ on $-y_1 < y_2 \le \frac{p-2}{p}y_1$ for $2 < p < \infty$. So we only need to find the largest range of τ's such that $M_{y_1 y_1} \le 0$.

Lemma 22. *In Case* (2_2), $M_{y_1y_1} \leq 0$ *on* $-y_1 < y_2 \leq \frac{p-2}{p} y_1$ *for all* $\tau \in \mathbb{R}$.

Proof. Changing coordinates back to x will make the estimates much easier. So we would like to show that, on $0 \leq x_2 \leq (p-1)x_1$, we have,

$$M_{y_1y_1} \leq 0, \tag{2.9}$$

where $C_{p,\tau} = ((p-1)^2 + \tau^2)^{\frac{p}{2}}$ and

$$\frac{1}{p} M_{y_1y_1} = (p-2)(x_2^2 + \tau^2 x_1^2)^{\frac{p-4}{2}} (x_2 + \tau^2 x_1)^2 + (1+\tau^2) \times (x_2^2 + \tau^2 x_1^2)^{\frac{p-2}{2}} - (p-1) C_{p,\tau} x_1^{p-2}.$$

First, consider $4 \leq p < \infty$. If $p \neq 4$, then showing (2.9) is equivalent to

$$(p-2)(p-1+\tau^2)^2 + (1+\tau^2)((p-1)^2 + \tau^2) - (p-1)((p-1)^2+\tau^2)^2 \leq 0,$$

which can be verified using direct calculations, for all τ. Let $s = \frac{x_2}{x_1}$, then (2.9) simplifies to showing,

$$F(s) = (p-2)(s+\tau^2)^2 + (1+\tau^2)(s^2+\tau^2) - C_{p,\tau}(p-1)(s^2+\tau^2)^{\frac{4-p}{2}} \leq 0,$$

where $0 \leq s \leq p-1$. For $p = 4$, F is a quadratic function that is increasing on $(\frac{-2\tau^2}{\tau^2+3}, p-1)$. Since $F(3) \leq 0$, then $F(s) \leq 0$ on $(0, 3)$.

Now we will consider $2 \leq p < 4$. Note that $F(s) = 0$ at $p = 2$, so we can assume that $p \neq 2$. Breaking up the domain of F will make things easier. For $s \in (1, p-1)$, we have the following estimate, $(s+\tau^2)^2 \leq (s^2+\tau^2)^2$. Let $t = s^2 + \tau^2$, then

$$\frac{1}{t} F(s) \leq (p-2)t + 1 + \tau^2 - C_{p,\tau}(p-1)t^{\frac{2-p}{2}} := g_1(t).$$

Observe that g_1 is increasing on $1 + \tau^2 \leq t \leq (p-1)^2 + \tau^2$ and $g_1((p-1)^2 + \tau^2) \leq 0$. Therefore, $F(s) \leq 0$ on $(1, p-1)$.

For $s \in (0, \frac{1-\tau^2}{2})$, we have the estimate $(s+\tau^2)^2 \leq s^2 + \tau^2$. Let $t = s^2 + \tau^2$, then

$$\frac{1}{t} F(s) \leq p - 1 + \tau^2 - C_{p,\tau}(p-1)t^{\frac{2-p}{2}} := g_2(t).$$

Since g_2 is increasing on $(\tau^2, \frac{(1-\tau)^2}{4} + \tau^2)$ and $g_2(\frac{(1-\tau)^2}{4} + \tau^2) \leq 0$, then $F(s) \leq 0$ on $(1, p-1)$ and on $(0, \frac{1-\tau^2}{2})$.

All that remains is to show that $F(s) \leq 0$ on $(\frac{1-\tau^2}{2}, 1)$. If we estimate in the crudest possible way, on this interval, then we obtain:

$$\frac{1}{p-1} F(s) \leq (1+\tau^2)^2 - ((p-1)^2 + \tau^2)^{\frac{p}{2}} \left(\frac{(1-\tau^2)^2}{4} + \tau^2 \right)^{\frac{4-p}{2}} \leq 0,$$

for all $|\tau| \leq 1$ and $3 \leq p \leq 4$, by direct calculations. So we need to estimate a little more carefully. On $(\frac{1-\tau^2}{2}, 1)$, let $t = s^2 + \tau^2$. So t must be in the range $\frac{(1-\tau)^2}{4} - \tau^2 \leq t \leq 1 + \tau^2$.

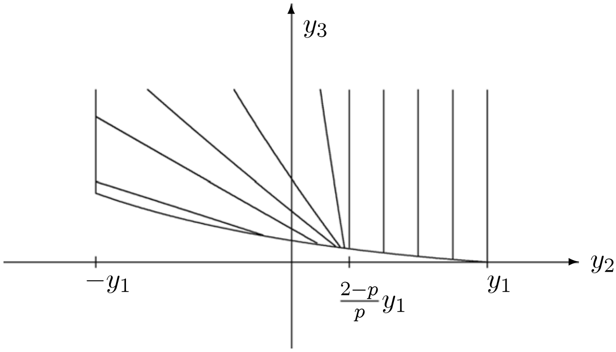


Fig. 6. Characteristics of Bellman candidate for $1 < p < 2$ and $|\tau| \le \frac{1}{2}$.

Then,

$$\begin{aligned} F(s) &= (p-2)(\sqrt{t-\tau^2}+\tau^2)^2 + (1+\tau^2)t - C_{p,\tau}(p-1)t^{\frac{4-p}{2}} \\ &= (p-2)(t-\tau^2+2\tau^2\sqrt{t-\tau^2}+\tau^4) + (1+\tau^2)t - C_{p,\tau}(p-1)t^{\frac{4-p}{2}} \\ &\le (p-2)(t+\tau^2+\tau^4) + (1+\tau^2)t - C_{p,\tau}(p-1)t^{\frac{4-p}{2}} =: g_3(t). \end{aligned}$$

One can see that g_3 is decreasing for $2 \le p < 3.95$ and $g_3\left(\frac{(1-\tau^2)^2}{4}+\tau^2\right) \le 0$, for all $|\tau| \le 1$, by direct calculations. Thus, $F(s) \le 0$ on $(\frac{1-\tau^2}{2}, 1)$. $\square$

We have now verified that the explicit part of the Bellman function candidate, from Case (2_2), has the appropriate restrictive concavity. So we have proven Proposition 21, by Lemmas 17, 18 and 22. Now that we have a Bellman candidate for $2 < p < \infty$, we will turn our attention to p-values in the dual range $1 < p < 2$.

2.2. The Bellman function candidate for $1 < p < 2$

In order to get a Bellman function candidate for $1 < p < 2$ we just need to glue together candidates from Cases (2_2) and (3_2) in almost the same way as we did for $2 < p < \infty$ in Section 2.1. Refer to Addendum 1 (Section 5) for full details.

Proposition 23. *Let* $1 < p < 2$ *and* $\gamma = \frac{1-\tau^2}{1+\tau^2}$. *If* $|\tau| \le \frac{1}{2}$, *then a solution to the Monge–Ampère equation is given by*

> $M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1 y_2 + y_2^2]^{\frac{p}{2}} + \left(\frac{1}{(p-1)^2}+\tau^2\right)^{\frac{p}{2}}[y_3 - (y_1-y_2)^p]$ *when* $\frac{2-p}{p}y_1 \le y_2 < y_1$ *and is given implicitly by*
>
> $G(y_1-y_2, y_1+y_2) = y_3 G(1, \sqrt{\omega^2-\tau^2})$ *when* $-y_1 < y_2 \le \frac{2-p}{p}y_1$, *where* $\omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}$.
> *This solution satisfies all of the properties of the Bellman function.*

Most of the remaining cases do not yield a Bellman function candidate. If we fix y_2 then the Monge–Ampère solution from Cases (1) and (3) does not satisfy the restrictive concavity needed to be a Bellman function candidate. Case (2) yields the same partial solution if we first fix y_1 or y_2, since restrictive concavity is only valid on part of the domain. So, all that remains is Case (4).

However, we do not know whether or not Case (4) gives a Bellman function candidate. For $\tau = 0$, it was shown in [19] that Case (4) does not produce a Bellman function candidate, since some simple extremal functions give a contradiction to linearity of the Monge–Ampère solution on characteristics. However, for $\tau \neq 0$ these extremal functions only work as a counterexample for some p-values and some signs of the martingale transform. Case (4) could give a solution throughout Ξ_+ or could yield a partial solution that would work well with the characteristics from Case (2_1). Since Case (4) does not provide a Bellman candidate for $\tau = 0$, we expect the same for small τ. The picture probably changes most drastically for large τ. But it does not matter, since we will now show that our Bellman candidate is actually the Bellman function (which we would have to check anyways because of the added assumption). The details for the remaining cases that do not yield a Bellman function candidate are in Addendum 2 (Section 6).

3. The Monge–Ampère solution is the Bellman function

We will now show that the Monge–Ampère solution obtained in Propositions 21 and 23 is actually the Bellman function. To this end, let us revert back to the x-variables. We will denote the Bellman function candidate as B_τ and use $\mathcal{B}_\tau$ to denote the true Bellman function. Extending the function G on part of Ω_+ to U_τ on all of Ω, appropriately, makes it possible to define the solution in terms of a single relation.

Definition 24. Let $v(x, y) := v_{p,\tau}(x, y) = (\tau^2|x|^2+|y|^2)^{\frac{p}{2}} - ((p^*-1)^2+\tau^2)^{\frac{p}{2}}|x|^p$, $u(x, y) := u_{p,\tau}(x, y) = p(1 - \frac{1}{p*})^{p-1}\left(1 + \frac{\tau^2}{(p^*-1)^2}\right)^{\frac{p-2}{2}}(|x| + |y|)^{p-1}[|y| - (p^* - 1)|x|]$ and

$$U(x, y) := U_{p,\tau}(x, y) = \begin{cases} v(x, y) & : |y| \geq (p^* - 1)|x| \\ u(x, y) & : |y| \leq (p^* - 1)|x|. \end{cases}$$

for $1 < p < 2$. For $2 < p < \infty$ we interchange the two pieces in U.

Proposition 25. *For $1 < p < 2$ and $|\tau| \leq \frac{1}{2}$ or $2 < p < \infty$ and $\tau \in \mathbb{R}$ the Bellman function candidate is the unique positive solution given by*

$$U(x_1, x_2) = U\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right).$$

Furthermore, U is C^1-smooth on Ω.

Proof. First consider $2 \leq p < \infty$. It is clear that

$$U(x_1, x_2) = U\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right), \tag{3.1}$$

by comparing the solution obtained in Proposition 21 and using the symmetry property in Proposition 6. The constant $\alpha_{p,\tau} = p(1 - \frac{1}{p^*})^{p-1}\left(1 + \frac{\tau^2}{(p^*-1)^2}\right)^{\frac{p-2}{2}}$ was determined so that $U_x = U_y$ at $|y| = (p^* - 1)|x|$. The partial derivatives are given by,

$$u_x = \alpha_{p,\tau}(p - 1)x'(|x| + |y|)^{p-2}(|y| - (p^* - 1)|x|) - \alpha_{p,\tau}(p^* - 1)x'(|x| + |y|)^{p-1},$$
$$v_x = p\tau^2 x(\tau^2|x|^2 + |y|^2)^{\frac{p-2}{2}} - px'((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}|x|^{p-1},$$

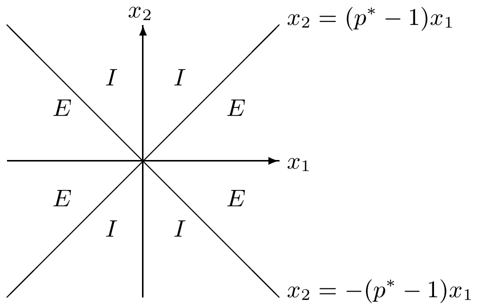


Fig. 7. Location of Implicit (I) and Explicit (E) part of B_τ for $2 \le p < \infty$.

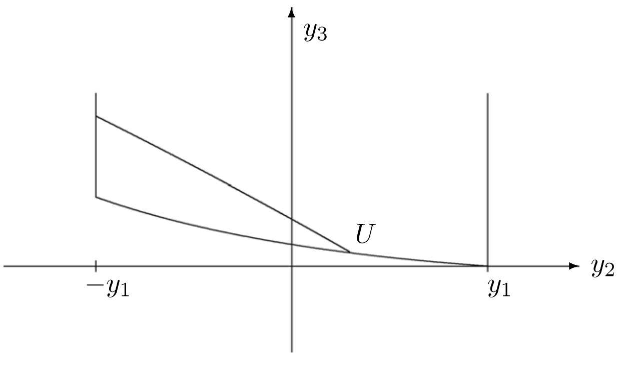


Fig. 8. Sample characteristic of the Monge–Ampère solution in Case (3_2).

$$u_y = \alpha_{p,\tau}(p-1)y'(|x|+|y|)^{p-2}(|y|-(p^*-1)|x|) + \alpha_{p,\tau}y'(|x|+|y|)^{p-1},$$
$$v_y = py(\tau^2|x|^2+|y|^2)^{\frac{p-2}{2}},$$

where $x' = \frac{x}{|x|}$ and $y' = \frac{y}{|y|}$. U is C^1-smooth, except possibly at gluing and symmetry lines. It is easy to verify that u_x is continuous at $\{x = 0\}$, U_x and U_y are continuous at $\{|y| = (p^*-1)|x|\}$ and v_y is continuous at $\{y = 0\}$. This proves that U is C^1-smooth on Ω.

Observe that $U_y > 0$ for $y \neq 0$ and $U_x < 0$ for $x \neq 0$. This is enough to show that B_τ is the unique positive solution to (3.1). Indeed, if $x \in \Omega$ such that $|x_1| = x_3^{\frac{1}{p}}$, then $\sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}} = |x_2|$ by the Dirichlet boundary conditions. This gives us (3.1) uniquely at $B_\tau(x)$. Fix x_1, such that $|x_1| < x_3^{\frac{1}{p}}$, then $U\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right) < U\left(x_1, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right)$. Since x_1 is fixed, then $\sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}} > |x_2|$, so $U\left(x_1, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right)$ strictly decreases to $U(x_1, x_2)$, as $\sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}$ decreases to $|x_2|$, giving us a unique $B_\tau(x)$ for which (3.1) holds.

Now consider $1 < p < 2$. U is C^1-smooth on Ω, since v_x is continuous at $\{x = 0\}$, u_y is continuous at $\{y = 0\}$ and U_x and U_y are continuous at $\{|y| = (p^*-1)|x|\}$. This is easily verified since the partial derivatives are computed above (just switch the two pieces of each function). Observe that for $x \neq 0$ and $y \neq 0$, $U_x < 0$ and for $y \neq 0$, $U_y > 0$. Then the argument above showing $U(x_1, x_2) = U\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right)$ uniquely determines B_τ also holds for this range of p-values as well, except maybe at $x_1 = x_2 = 0$. Suppose $U(0,0) = U\left(x_3^{\frac{1}{p}}, \sqrt{B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right)$, then $B_\tau(x) = ((p^*-1)^2 + \tau^2)^{\frac{p}{2}} x_3$. So $B_\tau(x)$ is uniquely determined by the fixed x-value. □

Corollary 26. *B_τ is continuous in Ω (Figs. 7 and 8).*

Proof. In this proof, only we will revert back to the notation $U_{p,\tau}$, rather than U, to make clear the distinction when $\tau = 0$ or $\tau \neq 0$. We only consider $2 < p < \infty$ as the dual range is handled identically. By Proposition 25, we have that B_τ is the unique positive solution to (3.1). Since this is true for all $\tau \in \mathbb{R}$, then $B_0 = \left(B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}\right)^{\frac{p}{2}}$ on $|x_2| \geq (p^*-1)|x_1|$, since $U_{p,\tau} = \left(1 + \frac{\tau^2}{(p^*-1)^2}\right)^{\frac{p-2}{2}} U_{p,0}$. Equivalently, we have

$$B_\tau = \left(B_0^{\frac{2}{p}} + \tau^2 x_3^{\frac{2}{p}}\right)^{\frac{p}{2}}. \tag{3.2}$$

Since B_0 was shown to be continuous in [19, p. 26] then B_τ is also continuous on $|x_2| \geq (p^*-1)|x_1|$, using the relation. This takes care of the implicit part of B_τ. The explicit part of B_τ is clearly continuous on $|x_2| \leq (p^*-1)|x_1|$. □

Lemma 27. *Let $1 < p < \infty$. Then, $B_\tau|_L$ is C^1-smooth on Ω, where L is any line in Ω.*

Proof. Since $B_\tau|_L$ is C^2-smooth on Ω_+, all that remains to be checked is the smoothness at the gluing and symmetry lines, i.e. at $\{x_1 = 0\}, \{x_2 = 0\}$ and $\{|x_2| = (p^*-1)|x_1|\}$. Let $L = L(t), t \in \mathbb{R}$ be any line in Ω passing through any of the planes in question, such that $L(0)$ is on the plane. Now plug $L(t)$ into (3.1) and differentiate with respect to t. Let $t \to 0^+$ and $t \to 0^-$ and equate the two relations. This gives

$$\left.\frac{d}{dt} B_\tau(L(t))\right|_{t=0^-} = \left.\frac{d}{dt} B_\tau(L(t))\right|_{t=0^+}. \quad \square$$

Proposition 28 (*Restrictive Concavity*)**.** *Let $1 < p < 2$ and $|\tau| \leq \frac{1}{2}$ or $2 \leq p < \infty$ and $\tau \in \mathbb{R}$. Suppose $x^\pm \in \Omega$ such that $x = \alpha^+ x^+ + \alpha^- x^-$, $\alpha^+ + \alpha^- = 1$. If $|x_1^+ - x_1^-| = |x_2^+ - x_2^-|$ then $B_\tau(x) \geq \alpha^+ B_\tau(x^+) + \alpha^- B_\tau(x^-)$.*

Proof. Recall that Propositions 21 and 23, together with the symmetry property of B_τ, establish this result everywhere, except at $\{x_1 = 0\}, \{x_2 = 0\}$ and $\{|x_2| = (p^*-1)|x_1|\}$. Let $f(t) = B_\tau|_{L(t)}$, where L is any line in Ω, such that $L(0) \in \{x_1 = 0\}, \{x_2 = 0\}$ or $\{|x_2| = (p^*-1)|x_1|\}$. Since $f'' < 0$ for $t < 0$ and $t > 0$ and f is C^1-smooth (by Lemma 27), then f is concave. □

Proposition 29. *Let $1 < p < \infty$. If a function $\widetilde{B}$ has restrictive concavity and $\widetilde{B}_\tau(x_1, x_2, |x_1|^p) \geq (\tau^2 x_1^2 + x_2^2)^{\frac{p}{2}}$, then $\widetilde{B}_\tau \geq \mathcal{B}_\tau$. In particular, $B_\tau \geq \mathcal{B}_\tau$.*

Proof. This was proven in [19] for B_0 (Lemma 2 on page 29). The same proof will apply here to B_τ. $\square$

Proposition 30. *For $1 < p < \infty$, $B_\tau \leq \mathcal{B}_\tau$.*

Proof. For $1 < p < 2$ there is a direct proof, which will be discussed first. By (3.2) we know that $B_0 = \left(B_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}\right)^{\frac{p}{2}}$ on $\{|x_2| \leq (p^* - 1)|x_1|\}$. Consider, $\widetilde{\mathcal{B}}_0 = \left(\mathcal{B}_\tau^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}\right)^{\frac{p}{2}}$. It suffices to show that $B_0 \leq \widetilde{\mathcal{B}}_0$. But, $B_0 = \mathcal{B}_0$ (as Burkholder showed), so without suprema we can reduce to simply showing

$$\langle |g|^p\rangle_I^{\frac{2}{p}} + \tau^2 \langle |f|^p\rangle_I^{\frac{2}{p}} \leq \langle (\tau^2|f|^2 + |g|^2)^{\frac{p}{2}}\rangle_I.$$

Apply Minkowski: $\left\| \int_I (A, C)\right\|_{l^{\frac{2}{p}}} \leq \int_I \|(A, C)\|_{l^{\frac{2}{p}}}$. Choosing $A = |g|^p$ and $C = |\tau f|^p$ proves the result. So we have shown that $B_\tau \leq \mathcal{B}_\tau$ on $\{|x_2| \leq (p^* - 1)|x_1|\}$.

Now we would like to show that $B_\tau \leq \mathcal{B}_\tau$ on $\{|x_2| \geq (p^* - 1)|x_1|\}$. Let $H_1(x_1, x_2, x_3) = B_\tau(x_1, x_2, x_3) - B_\tau(0, 0, 1)x_3$. Lemma 34, in the next section, proves that $H_1(x_1, x_2, \cdot)$ is an increasing function starting at $H_1(x_1, x_2, |x_1|^p) = v_\tau(x_1, x_2)$ and increasing to $\widetilde{U}_{p,\tau}(x, y) := \sup_{t \geq |x|^p}\{B_\tau(x, y, t) - B_\tau(0, 0, 1)t\}$. The same proof works for $H_2(x_1, x_2, x_3) = \mathcal{B}_\tau(x_1, x_2, x_3) - \mathcal{B}_\tau(0, 0, 1)x_3$. So

$$H_2(x_1, x_2, x_3) \geq v_\tau(x_1, x_2) = B_\tau(x_1, x_2, x_3) - B_\tau(0, 0, 1)x_3.$$

Since $B_\tau(0, 0, 1) \leq \mathcal{B}_\tau(0, 0, 1)$, then $B_\tau \leq \mathcal{B}_\tau$ on $\{|x_2| \geq (p^* - 1)|x_1|\}$.

Now we consider $2 < p < \infty$. Let $\varepsilon > 0$ be arbitrarily small and consider the following extremal functions

$$f(x) = \begin{cases} -c & : 1 < x < \varepsilon \\ \gamma f\left(\dfrac{t - \varepsilon}{1 - 2\varepsilon}\right) & : \varepsilon < x < 1 - \varepsilon \\ c & : 1 - \varepsilon < x < 1, \end{cases}$$

$$g(x) = \begin{cases} d_- & : 1 < x < \varepsilon \\ \gamma g\left(\dfrac{t - \varepsilon}{1 - 2\varepsilon}\right) & : \varepsilon < x < 1 - \varepsilon \\ d_+ & : 1 - \varepsilon < x < 1, \end{cases}$$

where $c, d_\pm$ and γ are defined so that f and g are a pair of test functions at $(0, x_2, x_3)$. We can use f and g to show, just as in [19, Lemma 3, p. 30], that

$$B_\tau(0, x_2, x_3) \leq \mathcal{B}_\tau(0, x_2, x_3). \tag{3.3}$$

Now we need to take care of the estimate when $x_1 \neq 0$. Making a change of coordinates from x to y we only need to consider $y \in \Xi_+$, by the symmetry property of the Bellman function and Bellman function candidate. So far we have that $M_\tau(y_1, y_1, y_3) \leq \mathcal{M}_\tau(y_1, y_1, y_3)$ by (3.3). The Dirichlet boundary conditions give that $M(y_1, y_2, (y_1 - y_2)^p) = \mathcal{M}(y_1, y_2, (y_1 - y_2)^p)$. On any characteristic in $\{\frac{p-2}{p} y_1 \leq y_2 \leq y_1\}$ (see Fig. 5) M_τ is linear (since it is the Monge–Ampère solution) and $\mathcal{M}_\tau$ is concave (by Proposition 4). Therefore, $M_\tau(y_1, y_2, y_3) \leq \mathcal{M}_\tau(y_1, y_2, y_3)$ on

$\{\frac{p-2}{p}y_1 \le y_2 \le y_1\}$. For the remaining part of Ξ_+, we can use the same proof as for $1 < p < 2$ to get $M_\tau(y_1, y_2, y_3) \le \mathcal{M}_\tau(y_1, y_2, y_3)$ on $\{-y_1 \le y_2 \le \frac{p-2}{p}y_1\}$. $\square$

Now that we have proven $B = \mathcal{B}$, we will mention another surprising fact.

Definition 31. We define $\mathcal{B}^l = \mathcal{B}^l(x_1, x_2, x_3)$ as the least restrictively concave majorant of $(x_2^2 + \tau^2 x_1^2)^{\frac{p}{2}}$ in Ω.

Proposition 32. *For $1 < p < 2$ and $\tau \le \frac{1}{2}$ or $2 \le p < \infty$ and $\tau \in \mathbb{R}$ we have $B = \mathcal{B} = \mathcal{B}^l$.*

This is proven in [3].

4. Proving the main result

Now that we have the Bellman function, the main result can be proven without too much difficulty. But first, we will find another relationship between U and v. Quite surprisingly, U is the least zigzag-biconcave majorant of v.

Definition 33. A function of (x, y) that is biconcave in $(x + y, x - y)$ we call zigzag-biconcave.

Lemma 34. *Let $1 < p < \infty$ and $\widetilde{U}(x, y) = \sup_{t \ge |x|^p}\{B_\tau(x, y, t) - B_\tau(0, 0, 1)t\}$. Fix (x, y). The function $H(x, y, t) = B_\tau(x, y, t) - B_\tau(0, 0, 1)t$ is increasing in t from $H(x, y, |x|^p) = v(x, y) := (\tau^2|x|^2 + |y|^2)^{\frac{p}{2}} - ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}|x|^p$ to $\widetilde{U}_{p,\tau}(x, y)$.*

Proof. Recall that B_τ is continuous in Ω and for (x, y) fixed, $B_\tau(x, y, \cdot)$ is concave. Then $H(x, y, \cdot)$ is also concave. Since $\widetilde{U}_{p,\tau}(x, y) = \sup_{t \ge |x|^p}\{B_\tau(x, y, t) - B_\tau(0, 0, 1)t\}$, then it either increases to $\widetilde{U}(x, y)$, or there exists t_0 such that $H(x, y, t_0) = \widetilde{U}(x, y)$ and H is decreasing for $t > t_0$. If H is decreasing for $t > t_0$, then $H \longrightarrow -\infty$ as $t \longrightarrow \infty$ by concavity. Then there exists $\varepsilon > 0$ and $t' > t_0$ such that $H(x, y, t') < \varepsilon t'$. So we have, $\limsup_{t\to\infty} \frac{H(x,y,t)}{t} < -\varepsilon$. But,

$$\lim_{t\to\infty} \frac{H(x, y, t)}{t} = \lim_{t\to\infty}\left[B_\tau\left(\frac{x}{t^{\frac{1}{p}}}, \frac{y}{t^{\frac{1}{p}}}, 1\right) - B_\tau(0, 0, 1)\right] = 0,$$

by continuity of B_τ at $(0, 0, 1)$. This gives us a contradiction. Therefore, $H(x, y, t) \ge -\varepsilon t$, for all t and all $\varepsilon > 0$, i.e. H is non-negative concave function on $[|x|^p, \infty)$. So $H(x, y, \cdot)$ is increasing and $H(x, y, |x|^p) = v_{p,\tau}(x, y)$ by the Dirichlet boundary conditions of B_τ in Proposition 6. $\square$

Proposition 35. *For $1 < p < 2$ and $|\tau| \le \frac{1}{2}$ or $2 \le p < \infty$ and $\tau \in \mathbb{R}$, $U_{p,\tau}(x, y) = \widetilde{U}_{p,\tau}(x, y)$.*

Proof. Suppose $2 \le p < \infty$ and $|y| \ge (p - 1)|x|$. Then

$$\begin{aligned}
\widetilde{U}_0(x, y) &= \lim_{t\to\infty}(B_0(x, y, t) - B_0(0, 0, 1)t) \\
&= \lim_{t\to\infty} \frac{B_0\left(\frac{x}{t^{\frac{1}{p}}}, \frac{y}{t^{\frac{1}{p}}}, 1\right) - B_0(0, 0, 1)}{1/t} \\
&= \frac{d}{du}B_0(u^{\frac{1}{p}}x, u^{\frac{1}{p}}y, 1)\bigg|_{u=0}.
\end{aligned}$$

Now we repeat the same steps and obtain

$$\begin{aligned}
\widetilde{U}_\tau(x,y) &= \lim_{t\to\infty}(B_\tau(x,y,t) - B_\tau(0,0,1)t)\\
&= \frac{d}{du}\left[\left(B_0^{\frac{2}{p}}(u^{\frac{1}{p}}x, u^{\frac{1}{p}}y, 1) + \tau^2\right)^{\frac{p}{2}}\right]\Bigg|_{u=0}\\
&= \Bigg[\left(B_0^{\frac{2}{p}}(u^{\frac{1}{p}}x, u^{\frac{1}{p}}y, 1) + \tau^2\right)^{\frac{p-2}{2}} B_0^{\frac{2-p}{p}}\\
&\quad \times (u^{\frac{1}{p}}x, u^{\frac{1}{p}}y, 1)\frac{d}{du}B_0(u^{\frac{1}{p}}x, u^{\frac{1}{p}}y, 1)\Bigg]\Bigg|_{u=0}\\
&= \left(1 + \frac{\tau^2}{(p-1)^2}\right)^{\frac{p-2}{2}} \widetilde{U}_0(x,y)\\
&= \left(1 + \frac{\tau^2}{(p-1)^2}\right)^{\frac{p-2}{2}} U_0(x,y),
\end{aligned}$$

where the last equality is by [8]. Therefore, $\widetilde{U}_\tau(x,y) = U_\tau(x,y)$.

Now suppose $|y| \le (p-1)|x|$. Looking at the explicit form of B_τ in the region, note that $B_\tau(x,y,\cdot)$ is linear. So $\widetilde{U}_\tau(x,y) = \sup_{t\ge|x|^p}\{B_\tau(x,y,t) - B_\tau(0,0,1)t\} = \sup_{t\ge|x|^p}\{B_\tau(x,y,0)\} = v_\tau(x,y) = U_\tau(x,y)$.

We can apply the same proof to show that $\widetilde{U}_\tau(x,y) = U_\tau(x,y)$ for $1 < p < 2$. $\square$

Proposition 36. *U is the least zigzag-biconcave majorant of v.*

Refer to [3] for the proof.

We now have enough machinery to easily prove the main result, in terms of the Haar expansion of an $\mathbb{R}$-valued L^p function.

Theorem 37. *Let $1 < p < 2, |\tau| \le \frac{1}{2}$ or $2 \le p < \infty, \tau \in \mathbb{R}$. Let $f, g : [0,1] \to \mathbb{R}$. If $|\langle g\rangle_{[0,1]}| \le (p^*-1)|\langle f\rangle_{[0,1]}|$ and $|(f,h_J)| = |(g,h_J)|$ for all $J \in \mathcal{D}$, then $\langle(\tau^2|f|^2 + |g|^2)^{\frac{p}{2}}\rangle_{[0,1]} \le ((p^*-1)^2+\tau^2)^{\frac{p}{2}}\langle|f|^p\rangle_{[0,1]}$, where $((p^*-1)^2+\tau^2)$ is the sharp constant and $p^*-1 = \max\left\{p-1, \frac{1}{p-1}\right\}$.*

Proof. Suppose that $2 \le p < \infty$ and $\tau \in \mathbb{R}$. The proof relies on the fact that the $B = \mathcal{B}$ (Propositions 29 and 30) and $U(x,y) = \sup_{t\ge|x|^p}\{B(x,y,t) - B(0,0,1)t\}$ (Proposition 35).

Since $|y| \le (p^*-1)|x|$ on Ω, then

$$U(x,y) = v(x,y) = (|y|^2 + \tau^2|x|^2)^{\frac{p}{2}} - ((p^*-1)^2+\tau^2)^{\frac{p}{2}}|x|^p \le 0.$$

Then,

$$\sup_{\substack{t>|x|^p\\ |y|\le(p^*-1)|x|}}\{B(x,y,t) - B(0,0,1)t\} \le 0.$$

But, $U(0,0) = 0$, therefore

$$\sup_{\substack{t>|x|^p \\ |y|\le(p^*-1)|x|}} \frac{B(x,y,t)}{t} = B(0,0,1) = ((p^*-1)^2+\tau^2)^{\frac{p}{2}}. \tag{4.1}$$

Observing the relationship $B = \mathcal{B}$ gives the desired result.

For $1 < p < 2$, $|\tau| \le \frac{1}{2}$ and $|y| \le (p^*-1)|x|$,

$$U(x,y) = p\left(1-\frac{1}{p^*}\right)\left(1+\frac{\tau^2}{(p^*-1)^2}\right)^{\frac{p-2}{2}} (|x|+|y|)^{p-1}[|y|-(p^*-1)|x|] \le 0,$$

so we have (4.1) by the same reasoning as for $2 \le p < \infty$. $\square$

Remark 38. Note that Minkowski's inequality together with Burkholder's original result gives the same upper estimate for $2 \le p < \infty$.

Indeed, if $f \in L^p[0,1]$ and g is the corresponding martingale transform then Minkowski's inequality gives,

$$\begin{aligned} \|g^2+\tau^2 f^2\|_{L^{\frac{p}{2}}}^{\frac{p}{2}} &\le (\|g^2\|_{L^{\frac{p}{2}}} + \|\tau^2 f^2\|_{L^{\frac{p}{2}}})^{\frac{p}{2}} = (\|g\|_{L^p}^2 + \|\tau f\|_{L^p}^2)^{\frac{p}{2}} \\ &\le \|f\|_{L^p}^p((p^*-1)^2+\tau^2)^{\frac{p}{2}}. \end{aligned}$$

This is very surprising in the sense that the "trivial" constant $((p^*-1)^2+\tau^2)^{\frac{p}{2}}$ is actually the sharp constant.

Now we will prove the main result for Hilbert-valued martingales. The same ideas can be used to extend the previous result to Hilbert-valued L^p-functions as well. Let $\mathbb{H}$ be a separable Hilbert space with $\|\cdot\|_{\mathbb{H}}$ as the induced norm.

Theorem 39. *Let $1 < p < \infty$, $(W, \mathcal{F}, \mathbb{P})$ be a probability space and $\{f_k\}_{k\in\mathbb{Z}^+}, \{g_k\}_{k\in\mathbb{Z}^+} : W \to \mathbb{H}$ be two $\mathbb{H}$-valued martingales with the same filtration $\{\mathcal{F}_k\}_{k\in\mathbb{Z}^+}$. Denote $d_k = f_k - f_{k-1}, d_0 = f_0, e_k = g_k - g_{k-1}, e_0 = g_0$ as the associated martingale differences. If $\|e_k(\omega)\|_{\mathbb{H}} \le \|d_k(\omega)\|_{\mathbb{H}}$, for all $\omega \in W$ and all $k \ge 0$ and $\tau \in [-\frac{1}{2}, \frac{1}{2}]$ then*

$$\left\|\left(\sum_{k=0}^n e_k, \tau\sum_{k=0}^n d_k\right)\right\|_{L^p(W,\mathbb{H}^2)} \le ((p^*-1)^2+\tau^2)^{\frac{p}{2}} \left\|\sum_{k=0}^n d_k\right\|_{L^p(W,\mathbb{H})},$$

where $((p^*-1)^2+\tau^2)^{\frac{p}{2}}$ is the best possible constant and $p^*-1 = \max\{p-1, \frac{1}{p-1}\}$. For $2 \le p < \infty$, the result is also true, with the best possible constant, if $\tau \in \mathbb{R}$.

In the theorem, "best possible" constant means that if $C_{p,\tau} < ((p^*-1)^2+\tau^2)^{\frac{1}{2}}$, then for some probability space $(W, \mathcal{G}, P)$ and a filtration $\mathcal{F}$, there exist $\mathbb{H}$-valued martingales $\{f\}_k$ and $\{g\}_k$, such that

$$\|(g_k, \tau f_k)\|_{L^p([0,1],\mathbb{H}^2)} > C_{p,\tau}\, \|f_k\|_{L^p([0,1],\mathbb{H})}\,.$$

Proof. We will prove the result for $2 \le p < \infty$, since the result for $1 < p < 2$ is similar. Replace $|\cdot|$ with $\|\cdot\|_{\mathbb{H}}$, in $U_{p,\tau}$. Let $f_n = \sum_{k=0}^n d_k$ and $g_n = \sum_{k=0}^n e_k$. Recall that $U := U_{p,\tau}$ is the least zigzag-biconcave majorant of $v := v_{p,\tau}$. As in [9, pp. 77–79],

$$U_{p,\tau}(x+h, y+k) \le U_{p,\tau}(x,y) + \Re(\partial_x U_{p,\tau}, h) + \Re(\partial_y U_{p,\tau}, k), \tag{4.2}$$

for all $x, y, h, k \in \mathbb{H}$, such that $|k| \le |h|$ and $\|x+ht\|_{\mathbb{H}}\|x+kt\|_{\mathbb{H}} > 0$. The result in (4.2) follows from the zigzag-biconcavity and implies that $\mathbb{E}[U(f_k, g_k)]$ is a supermartingale. Lemma 34 gives that $v(f_n, g_n) \le U(f_n, g_n)$. Therefore,

$$\mathbb{E}[v(f_n, g_n)] \le \mathbb{E}[U(f_n, g_n)] \le \mathbb{E}[U(f_{n-1}, g_{n-1})] \le \cdots \le \mathbb{E}[U(d_0, e_0)].$$

But, $\mathbb{E}[U(d_0, e_0)] \le 0$ in both pieces of U_τ since $2 - p^* \le 0$ and $\|e_0\|_{\mathbb{H}} \le \|d_0\|_{\mathbb{H}}$. Thus, $\mathbb{E}[v_\tau(f_n, g_n)] \le 0$. The constant, in the estimate, is best possible, since it was attained in Theorem 37. □

Remark 40. For $1 < p < 2$ and $|\tau| > \frac{1}{2}$, the "trivial" constant $((p^* - 1)^+\tau^2)^{\frac{p}{2}}$ in the main result is no longer sharp because of a "phase transition". In [3], there is an L^p-function, f, constructed so that together with it's martingale transform, g, we have $\langle(\tau^2|f|^2 + |g|^2)^{\frac{p}{2}}\rangle_{[0,1]} > ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}}\langle|f|^p\rangle_{[0,1]}$ for large τ.

5. Addendum 1

Throughout this section, the arguments may seem brief in comparison to Section 2.1. The reason for this is because we cover the exact same argument as in Section 2.1, only with slightly different cases. So if any arguments are unclear, then returning to Section 2.1 should help to clear up any difficulties. We will first consider Case (3_2) to get a partial Bellman function candidate.

5.1. *Considering Case* (3_2)

Proposition 41. *For $1 < p < \infty$ and $-y_1 < y_2 < \frac{2-p}{p}y_1$, M is given implicitly by the relation $G(y_1 - y_2, y_1 + y_2) = y_3 G(1, \sqrt{\omega^2 - \tau^2})$.*

This is proven through a series of lemmas.

Lemma 42. *$M(y) = t_2y_2 + t_3y_3 + t_0$ on the characteristic $y_2dt_2 + y_3dt_3 + dt_0 = 0$ can be simplified to $M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$, where u is the unique solution to the equation $\frac{y_2+(1-\frac{2}{p})y_1}{y_3} = \frac{u+(1-\frac{2}{p})y_1}{(y_1-u)^p}$ and $-y_1 < y_2 < \frac{2-p}{p}y_1$.*

Proof. Any characteristic, in Case(3_2), goes from $U = (y_1, u, (y_1 - u)^p)$ to $W = (y_1, -y_1, w)$. Recall the properties of the Bellman function we derived in Proposition 6, as we will be using them throughout the proof. Using the Neumann property and the property from Proposition 8, we get $M_{y_1} = -M_{y_2} = -t_2$ at W. By homogeneity at W we get

$$-py_1t_2 + pwt_3 + pt_0 = pM(W) = y_1M_{y_1} + y_2M_{y_2} + py_3M_{y_3} = -2y_1t_2 + pwt_3.$$

Now we follow the same idea as in Lemma 11, to get $M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$, where $u = u(y_1, y_2, y_3)$ is the solution to the equation

$$\frac{y_2 + \left(1 - \frac{2}{p}\right)y_1}{y_3} = \frac{u + \left(1 - \frac{2}{p}\right)y_1}{(y_1 - u)^p}. \tag{5.1}$$

Fix $u = -(1 - \frac{2}{p})y_1$, then we see that $y_2 = -(\frac{2}{p} - 1)y_1 = u$ is also fixed by (5.1). This means that the characteristics must lie in the sector, shown in Fig. 9, since they go from U to

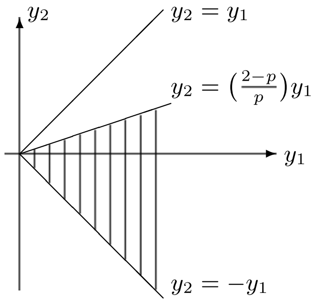


Fig. 9. Range of characteristics in Case (3_2) for $1 < p < 2$.

$W \in \{y_2 = -y_1\}$. The same argument as in Lemma 11 can be used to verify that Eq. (5.1) has a unique solution in the sector $-y_1 < y_2 < \frac{2-p}{p} y_1$. $\square$

Lemma 43. *$M(y) = \left(\frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u}\right)^p y_3$ can be rewritten as $G(y_1 - y_2, y_1 + y_2) = y_3 G(1, \sqrt{\omega^2 - \tau^2})$ for $-y_1 < y_2 < \frac{2-p}{p} y_1$.*

Proof. $\omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}} = \frac{\sqrt{(y_1+u)^2+\tau^2(y_1-u)^2}}{y_1-u} \geq |\tau|$.

Since $y_1 \pm u \geq 0$ and $\omega^2 - \tau^2 \geq 0$, then $u = \frac{\sqrt{\omega^2-\tau^2}-1}{\sqrt{\omega^2-\tau^2}+1} y_1$ by inversion. Substituting u into $\frac{y_2+(1-\frac{2}{p})y_1}{y_3} = \frac{u+(1-\frac{2}{p})y_1}{(y_1-u)^p}$ gives

$$2^{p-1} y_1^{p-1}[p y_2 + (p-2) y_1] = y_3(\sqrt{\omega^2 - \tau^2} + 1)^{p-1}[\sqrt{\omega^2 - \tau^2} - (p-1)]$$

or $(x_1 + x_2)^{p-1}[(p-1)x_2 - x_1] = \left[\sqrt{B^{\frac{2}{p}} - \left(\tau x_3^{\frac{1}{p}}\right)^2} + x_3^{\frac{1}{p}}\right]^{p-1} \left[(p-1)\sqrt{B^{\frac{2}{p}} - \left(\tau x_3^{\frac{1}{p}}\right)^2} - x_3^{\frac{1}{p}}\right]$. Thus, $G(x_1, x_2) = G\left(x_3^{\frac{1}{p}}, \sqrt{B^{\frac{2}{p}} - \left(\tau x_3^{\frac{1}{p}}\right)^2}\right)$ or $G(y_1 - y_2, y_1 + y_2) = y_3 G(1, \sqrt{\omega^2 - \tau^2})$. $\square$

As before, we must verify that this partial Bellman function candidate has the restrictive concavity property, so y_1 is no longer fixed. To check restrictive concavity, we must show that $M_{y_1 y_1} \leq 0$, $M_{y_2 y_2} \leq 0$, $M_{y_3 y_3} \leq 0$ and $D_1 \geq 0$ (note that $D_2 = 0$ by assumption). These estimates are verified in the following series of lemmas.

Lemma 44. *In Case* (3_2) *we choose $H(y_1, y_2) = G(y_1 - y_2, y_1 + y_2)$ because of how the implicit solution is defined and obtain* $\operatorname{sign} H'' = -\operatorname{sign}(p-2)$.

Proof. We already computed

$$H'' = \begin{cases} 4G_{z_1 z_2}, & \alpha_j = \beta_j \\ 0, & \alpha_j = -\beta_j \end{cases}$$

in Lemma 14. Since, $\alpha_1 = 1, \alpha_2 = -1, \beta_1 = 1$ and $\beta_2 = 1$ then $G_{z_1 z_2} = -p(p-1)(p-2)(y_1 + y_2)(2y_1)^{p-3}$. $\square$

Remark 45. In Case (3_2), $\beta > \frac{1}{p-1}$ in the sector $-y_1 < y_2 < \frac{2-p}{p} y_1$, where $\beta := \sqrt{\omega^2 - \tau^2}$. Equivalently, $B(x_1, x_2, x_3) \geq ((p^* - 1)^2 + \tau^2)^{\frac{p}{2}} x_3$ in $-y_1 < y_2 < \frac{2-p}{p} y_1$.

This is trivial since

$$
\begin{aligned}
(\beta + 1)^{p-1}[1 - (p-1)\beta] = G(1, \beta) &= \frac{1}{y_3} G(y_1 - y_2, y_1 + y_2) \\
&= (2y_1)^{p-1}[(p-2)y_1 + p y_2] < 0.
\end{aligned}
$$

Now we have enough information to check the sign of D_1. We will start limiting the values of τ, since it will be essential for having the restrictive concavity of the partial Bellman candidate from Case (2_2) (see Remark 50).

Lemma 46. *$D_1 > 0$ in Case (3_2) for all $|\tau| \leq 1$.*

Proof. We use the partial derivatives of G computed in the proof of Lemma 14 to make the computations of Φ' and Φ'' easier.

$$
\begin{aligned}
&\Phi(\omega) = G(1, \beta) \\
&\Phi'(\omega) = -p(p-1)\omega[\beta + 1]^{p-2} \qquad (5.2) \\
&\Phi''(\omega) = -\frac{p(p-1)(1+\beta)^{p-3}}{\beta}\left[\beta(1+\beta) + (p-2)\omega^2\right] \\
&\Lambda = (p-1)\Phi' - \omega\Phi'' \\
&\quad = -p(p-1)^2\omega(\beta+1)^{p-2} + \frac{p(p-1)\omega(1+\beta)^{p-3}}{\beta}\left[\beta(1+\beta) + (p-2)\omega^2\right] \\
&\quad = \frac{p(p-1)\omega(1+\beta)^{p-3}}{\beta}\left[-(p-1)(1+\beta)\beta + \beta(1+\beta) + (p-2)\omega^2\right] \\
&\quad = -\frac{p(p-1)(p-2)\omega(1+\beta)^{p-3}[\beta - \tau^2]}{\beta}. \qquad (5.3)
\end{aligned}
$$

Now we need to determine the sign of $\beta - \tau^2$ is for $1 < p < \infty$. By Remark 45, $\beta > \frac{1}{p-1} \geq \tau^2$ for $|\tau| \leq 1$ and $1 < p < 2$. But, what about $p > 2$? Using the form of the solution, M, in Lemma 42 we obtain $(y_1 + u)^2 + \tau^4(1 - \tau^2)(y_1 - u)^2 > 0$

$$
\begin{aligned}
&\iff \frac{(y_1 + u)^2 + \tau^2(y_1 - u)^2}{(y_1 - u)^2} \geq \tau^2(1 + \tau^2) \\
&\iff \omega = \left(\frac{M(y)}{y_3}\right)^{\frac{2}{p}} > \tau^2(1 + \tau^2) \\
&\iff \beta - \tau^2 > 0,
\end{aligned}
$$

where u is the unique solution to $\frac{y_2 + (1 - \frac{2}{p})y_1}{y_3} = \frac{u + (1 - \frac{2}{p})y_1}{y_3}$ and $|\tau| \leq 1$. Thus, $\operatorname{sign} D_1 = \operatorname{sign} H'' \operatorname{sign} \Lambda = [-\operatorname{sign}(p-2)]^2$ by (2.4) and Lemma 44. $\square$

The following lemma restricts the p-values for which our solution is a Bellman function candidate to $1 < p < 2$.

Lemma 47. sign $M_{y_1y_1} = \operatorname{sign} M_{y_2y_2} = \operatorname{sign} M_{y_3y_3} = \operatorname{sign}(p-2)$ *in Case* (3_2) *for all* $|\tau| \leq 1$. *Consequently, M is a Bellman function candidate for* $1 < p < 2$ *but not for* $2 < p < \infty$, *since it would not satisfy the restrictive concavity needed.*

Proof. By (2.2), (5.2), (5.3),

$$M_{y_3y_3} = \frac{p\omega^{p-2}R_1^2H^2}{y_3^3}\left[\frac{\Lambda}{\Phi'}\right],$$

giving sign $M_{y_3y_3} = (-1)[-\operatorname{sign}(p-2)]$. By (2.3), for $i = 1, 2$,

$$\begin{aligned} M_{y_iy_i} &= \frac{p\omega^{p-2}R_1}{y_3}\left[(\omega R_2 + (p-1)R_1)(H')^2 + \omega y_3 H''\right] \\ &= \frac{p\omega^{p-2}}{y_3(\Phi')^3}\left[\Lambda(H')^2 + \omega y_3 H''(\Phi')^2\right], \end{aligned}$$

giving sign $M_{y_iy_i} = (-1)[-\operatorname{sign}(p-2)]$, since $\Phi' < 0$. □

Now that we have a partial Bellman function candidate for $1 < p < 2$, from Case (3_2), satisfying all of the properties of the Bellman function, including restrictive concavity, we can turn our attention to Case (2_2). From Case (2_2) we will get a Bellman candidate on all Ξ_+, or part of it, depending on the τ- and p-values. The partial Bellman candidate, from Case (2_2), turns out to be the missing half for Case (3_2). We already have the solution for Case (2) from Lemma 19, but the value of the constant is needed before we can progress further.

5.2. *Case (2) for* $1 < p < 2$

Lemma 48. *If* $1 < p < 2$ *then in Case* (2_2), *the value of the constant in Lemma 19 is* $c = \left(\frac{1}{(p-1)^2} + \tau^2\right)^{\frac{p}{2}}$.

Proof. If $M(y) = (1+\tau^2)^{\frac{p}{2}}[y_1^2 + 2\gamma y_1y_2 + y_2^2]^{\frac{p}{2}} + c[y_3 - (y_1-y_2)^p]$ (where $\gamma = \frac{1-\tau^2}{1+\tau^2}$) is to be a candidate or partial candidate, then it must agree, at $y_2 = \frac{2-p}{p}y_1$, with the solution M given implicitly by the relation $G(y_1-y_2, y_1+y_2) = y_3G(1, \sqrt{\omega^2-\tau^2})$, from Proposition 41. At $y_2 = \frac{2-p}{p}y_1$,

$$\begin{aligned} (\sqrt{\omega^2-\tau^2}+1)^{p-1}[1-(p-1)\sqrt{\omega^2-\tau^2}] &= G(1, \sqrt{\omega^2-\tau^2}) \\ &= \frac{1}{y_3}(2y_1)^{p-1}[(2-p)y_1 + (p-2)y_1] \\ &= 0. \end{aligned}$$

Since $\sqrt{\omega^2-\tau^2}+1 \neq 0$ then $\sqrt{\omega^2-\tau^2} = \frac{1}{p-1}$, which implies $\omega = \left(\frac{1}{(p-1)^2}+\tau^2\right)^{\frac{1}{2}}$. So,

$$\begin{aligned} \left(\frac{1}{(p-1)^2}+\tau^2\right)^{\frac{p}{2}} y_3 &= \omega^p y_3 \\ &= M\left(y_1, \frac{2-p}{p}y_1, y_3\right) \end{aligned}$$

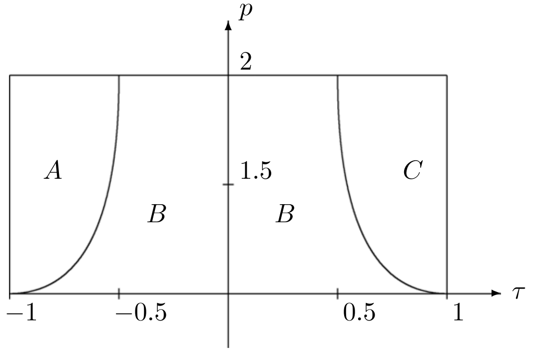


Fig. 10. Splitting $[-1,1]\times(1,2)$ in the $(\tau\times p)$-plane into three regions A, B and C. The curves separating regions A, B and C are where $M_{y_2y_2}=0$ in Case (2_2).

$$= \left[\left(\frac{2}{p}y_1\right)^2 + \tau^2\left(\frac{2(p-1)}{p}y_1\right)^2\right]^{\frac{2}{p}} + c\left[y_3 - \left(\frac{2(p-1)}{p}y_1\right)^p\right].$$

Now just solve for c. $\square$

In the following lemma, the value of τ has to be restricted to $|\tau|\le 1$, so that restrictive concavity is satisfied for our Bellman candidate. Actually, the τ-values play an even bigger role. Depending on the value of $(\tau,p)\in[-1,1]\times(1,2)$, there is either one or two Bellman function candidates. For $(\tau,p)\in B$, from Fig. 10, there is a partial Bellman candidate arising from Case (2_2). So we can glue this together with the other partial candidate obtained in Case (3_2). This gives a Bellman candidate, as before, having characteristics as in Fig. 6. For $(\tau,p)\in A\cup C$ the candidate obtained from Case (2_2) maintains restrictive concavity throughout Ξ_+ and is therefore requires no gluing. To avoid the difficulty of determining which candidate to choose and how to determine the optimal constant from Case (2), we restrict (τ,p) to region B, or require that $|\tau|\le\frac{1}{2}$. Recall that the partial Bellman candidate, M, obtained from Case (2_2), for $1<p<2$, satisfies $M_{y_iy_3}=M_{y_3y_3}=0$ and hence $D_i=0$, for $i=1,2$. So all that still needs to be checked for restrictive concavity is the sign of $M_{y_1y_1}$ and $M_{y_2y_2}$. Since $M_{y_1y_1}\le M_{y_2y_2}$, then we just need to show that $M_{y_2y_2}\le 0$ on $\frac{2-p}{p}y_1\le y_2\le y_1$ in Ξ_+. This is considered in the following lemmas.

Lemma 49. *In Case* (2_2), $M_{y_2y_2}(y_1,\frac{2-p}{p}y_1,y_3)\le 0$ *for* $|\tau|\le 1$ *and* $1<p<2$.

Proof. The solution M that we get from (2_2), when $1<p<2$, is obtained from Lemmas 19 and 48. Let $\gamma=\frac{1-\tau^2}{1+\tau^2}$, $f_1(y)=y_1^2+y_2^2+2\gamma y_1y_2$, $f_2(y)=(p-2)(y_2+\gamma y_1)^2+f_1(y)$ and $f_3(y)=y_1-y_2$. Then

$$M_{y_2y_2}=p(1+\tau^2)^{\frac{p}{2}}f_1^{\frac{p-4}{2}}f_2-p(p-1)\left(\frac{1}{(p-1)^2}+\tau^2\right)^{\frac{p}{2}}f_3^{p-2}.$$

By direct calculations one can verify, $M_{y_2y_2}(y_1,\frac{2-p}{p}y_1,y_3)\le 0$ when $|\tau|\le 1$. $\square$

Remark 50. Note that Lemma 49 is false for p near 2 when $|\tau|$ larger than 1, so we cannot take a larger value and still maintain the restrictive concavity.

Lemma 51. *In Case* (2_2), $M_{y_2y_2}(y_1, cy_1, y_3) \le 0$, *for all* $c \in \left[\frac{2-p}{p}, 1\right]$.

Proof. Using $M_{y_2y_2}$ from Lemma 49, we see that $M_{y_2y_2} \le 0$ is equivalent to

$$(1+\tau^2)^{\frac{p}{2}} f_3^{2-p} \frac{f_2}{f_1^{\frac{4-p}{2}}} - p(p-1)\left(\frac{1}{(p-1)^2}+\tau^2\right)^{\frac{p}{2}} \le 0.$$

Observe that the function $f_2/(f_1^{\frac{4-p}{2}})$ is strictly positive, has a horizontal asymptote at the y_2-axis, increases on $(-\infty, -\gamma)$, and decreases on $(-\gamma, \infty)$. As y_2 increases from $\frac{2-p}{p}$ to 1, f_3^{2-p} and $\frac{f_2}{f_1^{\frac{2}{4-p}}}$ both decrease. Since $M_{y_2y_2}(y_1, \frac{2-p}{p}y_1, y_3) \le 0$, as shown in Lemma 49, the result follows. □

Lemma 52. *The Monge–Ampère solution in Case* (2_2) *yields the following results for* $1 < p < 2$.$M_{y_2y_2}(y_1, y_1, y_3) < 0$ *for* $|\tau| \le 1$ *and* $M_{y_2y_2}(y_1, -y_1, y_3) > 0$ *for* $|\tau| \le \frac{1}{2}$.

Proof. Let f_1, f_2 and f_3 be as in Lemma 49 and

$$g = (1+\tau^2)^{\frac{p}{2}} f_3^{2-p} f_2 - (p-1)\left(\frac{1}{(p-1)^2}+\tau^2\right)^{\frac{p}{2}} f_1^{\frac{4-p}{2}}.$$

Note that $M_{y_2y_2}$ and g have the same signs. It is clear that $g(y_1, y_1, y_3) < 0$, proving the first inequality. One can now verify that $g(y_1, -y_1, y_3) > 0$ for $|\tau| \le \frac{1}{2}$ which proves the second inequality. □

Remark 53. One can see in the graph of $\frac{1}{y_1^{p-2}} g(y_1, y_1, y_3)$ that $g(y_1, y_1, y_3) < 0$, in regions A and C, (see Fig. 10). This tells us that the Bellman candidate from Case (2_2) will maintain restrictive concavity throughout the domain in for $(\tau, p) \in A \cup C$. Furthermore, there will be an improvement in the constant $((p^*-1)^2+\tau^2)^{\frac{p}{2}}$ that can still be used to maintain restrictive concavity in $A \cup C$.

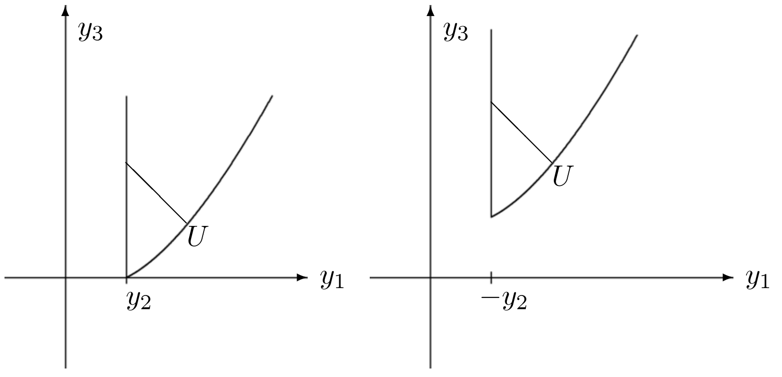
By Lemmas 51 and 52, we obtain a partial Bellman candidate from Case (2_2), when $1 < p < 2$ and $|\tau| \le \frac{1}{2}$. As before, we will glue this partial candidate from Case (2_2) to the partial candidate in Case (3_2) to obtain the Bellman candidate for $1 < p < 2$.

6. Addendum 2

Now that we have particular cases in which the Monge–Ampère solution gives a Bellman function candidate, we would like to discuss the remaining cases. It can be shown that all remaining cases do not yield a Bellman function candidate, except for Case (4) which is still not determined.

6.1. *Case* (1_2) *for* $1 < p < 2$ *and Case* (3_2) *for* $2 < p < \infty$ *do not lead to a Bellman candidate*

It was shown in Lemmas 18 and 47 that the Monge–Ampère solution obtained in each case does not have the appropriate restrictive concavity property to be a Bellman function candidate. We mention this here again simply for clarity.



(a) Case (1_1). (b) Case (3_1).

Fig. 11. Sample characteristic for the Monge–Ampère solution in Cases (1_1) and (3_1).

6.2. *Case* (1_1) *does not give a Bellman candidate*

We can consider Cases (1_1) and (3_1) simultaneously, for part of the calculation, since the same argument will work in both cases. In both cases, y_2 is fixed and the Monge–Ampère solution is given by $M(y) = t_1 y_1 + t_3 y_3 + t_0$ on the characteristics $dt_1 y_1 + dt_3 y_3 + dt_0 = 0$. As shown in Fig. 11, $y_2 \geq 0$ in case (1_2) and $y_2 \leq 0$ in Case (3_2), since if not then the characteristics go outside of the domain Ξ_+.

Lemma 54. *In Cases* (1_1) *and* (3_1)*, the solution to the Monge–Ampère can be written as,*

$$M(y) = \left(\frac{\sqrt{(u+y_2)^2 + \tau^2 (u-y_2)^2}}{u-y_2} \right)^p y_3$$

where $u = u(y_1, y_2, y_3)$ *is the solution to the equation* $\frac{y_1 + \left(\frac{2}{p}-1\right)|y_2|}{y_3} = \frac{u + \left(\frac{2}{p}-1\right)|y_2|}{(u-y_2)^p}$.

Proof. Any characteristic, in Cases (1_1) and (3_1), go from $U = (u, y_2, (u-y_2)^p)$ to $W = (|y_2|, y_2, w)$. Throughout the proof, we will use the properties of the Bellman function derived in Proposition 6. Using the Neumann property and the property from Proposition 8 we get $y_2 M_{y_2} = y_1 M_{y_1} = |y_2| t_1$ at W. By homogeneity at W we get

$$p|y_2|t_1 + pwt_3 + pt_0 = pM(W) = y_1 M_{y_1} + y_2 M_{y_2} + p y_3 M_{y_3} = 2t_1 |y_2| + pwt_3.$$

Following the same argument as in Lemma 11, gives $M(y) = \left(\frac{\sqrt{(u+y_2)^2+\tau^2(u-y_2)^2}}{u-y_2} \right)^p y_3$, where $u = u(y_1, y_2, y_3)$ is the solution to the equation

$$\frac{y_1 + \left(\frac{2}{p} - 1\right)|y_2|}{y_3} = \frac{u + \left(\frac{2}{p} - 1\right)|y_2|}{(u-y_2)^p}. \tag{6.1}$$

Since the solution, M, does not satisfy the restrictive concavity property necessary to be the Bellman function (as we will soon show), we are not concerned about the existence of the solution u in Eq. (6.1). $\square$

Lemma 55. *If* $\omega = \left(\frac{M(y)}{y_3}\right)^{\frac{1}{p}}$, *then in Cases* (1_1) *and* (3_1), *the solution* u *to Eq.* (6.1) *can be expressed as* $u = \frac{\sqrt{\omega^2-\tau^2}+1}{\sqrt{\omega^2-\tau^2}-1} y_2$ *and Eq.* (6.1) *can be rewritten as*

$$2^p|y_2|^{p-1}[py_1 + (2-p)|y_2|] = y_3|\beta - 1|^{p-1}[p(\beta+1) + (2-p)|\beta - 1|], \tag{6.2}$$

where $\beta = \sqrt{\omega^2 - \tau^2}$. *Furthermore,* $\operatorname{sign} y_2 = \operatorname{sign}(\beta - 1)$.

Proof. Let us show that $u = \frac{\sqrt{\omega^2-\tau^2}+1}{\sqrt{\omega^2-\tau^2}-1} y_2$ first. This follows from inverting

$$\omega = \frac{\sqrt{(u+y_2)^2 + \tau^2(u-y_2)^2}}{u - y_2},$$

and using the properties $\omega \geq |\tau|$ and $u \pm y_2 \geq 0$. Now that we have $u = \frac{\sqrt{\omega^2-\tau^2}+1}{\sqrt{\omega^2-\tau^2}-1} y_2$, we can use it to get the next result. Note that $u \geq 0$ and $\sqrt{\omega^2 - \tau^2} \geq 0$, which implies that $\operatorname{sign} y_2 = \operatorname{sign}(\sqrt{\omega^2 - \tau^2} - 1)$. To get (6.2), simply plug $u = \frac{\sqrt{\omega^2-\tau^2}+1}{\sqrt{\omega^2-\tau^2}-1} y_2$ in Eq. (6.1). $\square$

We can no longer discuss Cases (1_1) and (3_1) together, so for the remainder of the subsection the focus will be on Case (1_1) only.

Lemma 56. *In Case* (1_1), *the solution* M *from Lemma* 54 *can be rewritten in the implicit form* $G(y_2 + y_1, y_2 - y_1) = y_3 G(\sqrt{\omega^2 - \tau^2}, -1)$, *where* $G(z_1, z_2) = (z_1 + z_2)^{p-1}[z_1 - (p-1)z_2]$.

Proof. Recall that for Case (1_1) we have $y_2 > 0$.

$$y_2 = \frac{1}{2}(x_2 - x_1) > 0 \Longrightarrow x_2 > x_1$$

$$\operatorname{sign}(\sqrt{\omega^2 - \tau^2} - 1) = \operatorname{sign} y_2 > 0 \Longrightarrow \sqrt{\omega^2 - \tau^2} > 1 \Longrightarrow \omega > \sqrt{\tau^2 + 1}.$$

So, $\mathcal{B}(x) = M(y) > y_3(\tau^2 + 1)^{\frac{p}{2}}$. Now (6.2) can be rewritten as

$$(x_2 - x_1)^{p-1}[(p-1)x_1 + x_2] = \left[\sqrt{\mathcal{B}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}} - x_3^{\frac{1}{p}}\right]^{p-1} \times \left[\sqrt{\mathcal{B}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}} + (p-1)x_3^{\frac{1}{p}}\right].$$

Therefore,

$$G(x_2, -x_1) = G\left(\sqrt{\mathcal{B}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}, -x_3^{\frac{1}{p}}\right)$$

or by factoring out $x_3^{\frac{1}{p}}$ on the right side we get

$$G(y_2 + y_1, y_2 - y_1) = y_3 G(\sqrt{\omega^2 - \tau^2}, -1). \quad \square$$

Recall that the Monge–Ampère solution must satisfy the restrictive concavity conditions in Proposition 5 to be a Bellman function candidate. We will show that the Monge–Ampère solution obtained in Case (1_1) has $D_1 < 0$ and therefore cannot be a Bellman candidate.

Lemma 57. *In Case* (1_1) *we choose* $H(y_1, y_2) = G(y_1 + y_2, -y_1 + y_2)$ *because of how the implicit solution is defined and obtain* $\operatorname{sign} H'' = \operatorname{sign}(p-2)$.

Proof. We already computed

$$H'' = \begin{cases} 4G_{z_1 z_2}, & \alpha_j = \beta_j \\ 0, & \alpha_j = -\beta_j \end{cases}$$

in Lemma 14.

Since, $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = -1$ and $\beta_2 = 1$ then $G_{z_1 z_2} = p(p-1)(p-2)(y_1 - y_2)(2y_2)^{p-3}$. $\square$

Lemma 58. *If* $p \neq 2$ *then* $D_2 < 0$ *in Case* (1_1) *for all* τ.

Proof. We use the partial derivatives of G from the proof of Lemma 14 to make the computations of Φ' and Φ'' easier. Let $\alpha_p = \frac{p(p-1)\omega(\beta-1)^{p-3}}{\beta^3}$ and $\beta = \sqrt{\omega^2 - \tau^2}$.

$$\begin{aligned}
&\Phi(\omega) = G(\beta, 1) \\
&\Phi'(\omega) = p[\beta - 1]^{p-2}[\omega + (p-2)\omega\beta^{-1}] \\
&\Phi''(\omega) = G_{z_1 z_2}(\beta, -1)\beta^{-2}\omega^2 + G_{z_1}(\beta, -1)[-\omega\beta^{-3} + \beta^{-1}] \\
&\qquad = p(p-1)[\beta - 1]^{p-3}[\beta + p - 3]\frac{\omega^2}{\beta^2} - p\frac{\tau^2}{\beta^3}[\beta - 1]^{p-2}[\beta + p - 2]
\end{aligned}$$

$$\begin{aligned}
\Lambda &= (p-1)\Phi' - \omega\Phi'' \\
&= \alpha_p[(\beta - 1)\beta^3(\beta^{-1}(p-2) + 1) - \omega^2\beta(\beta + p - 3) + \tau^2(\beta - 1)(\beta + p - 2)] \\
&= \alpha_p[(\beta^2 + \tau^2)(\beta - 1)(\beta + p - 2) - \omega^2\beta(\beta + p - 3)] \\
&= \alpha_p\omega^2[\beta^2 + \beta(p-2) - \beta - (p-2) - \beta^2 - \beta(p-3)] \\
&= -\frac{p(p-1)(p-2)\omega^3(\sqrt{\omega^2 - \tau^2} - 1)^{p-3}}{(\sqrt{\omega^2 - \tau^2})^3}.
\end{aligned}$$

From Lemma 55, $\operatorname{sign}(\beta - 1) = \operatorname{sign} y_2 > 0$ and $\omega^2 > \tau^2 > 0$. Therefore, by Lemma 57 and (2.4) $\operatorname{sign} D_2 = \operatorname{sign} H'' \operatorname{sign} \Lambda = -(\operatorname{sign}(p-2))^2 < 0$. $\square$

Since $D_2 < 0$ in Case (1_1) then we get the following result.

Proposition 59. *Case* (1_1) *does not give a Bellman function candidate.*

6.3. Case (3_1) *does not provide a Bellman function candidate*

Much of the work needed to show that the Monge–Ampère solution cannot be the Bellman function, in Case (3_1), has already been started in Section 6.2. Let us finish the argument.

Lemma 60. *In Case* (3_1), *the solution* M *from Lemma 54 can be rewritten in the implicit form* $G(y_2 - y_1, -y_1 - y_2) = y_3 G(1, -\sqrt{\omega^2 - \tau^2})$, *where* $G(z_1, z_2) = (z_1 + z_2)^{p-1}[z_1 - (p-1)z_2]$.

Proof. Recall that in Case (3_2) we have that $y_2 < 0$.

$$\begin{aligned}
&y_2 = \frac{1}{2}(x_2 - x_1) < 0 \Longrightarrow x_2 < x_1 \\
&\operatorname{sign}(\sqrt{\omega^2 - \tau^2} - 1) = \operatorname{sign} y_2 < 0 \Longrightarrow \sqrt{\omega^2 - \tau^2} < 1 \Longrightarrow \omega < \sqrt{\tau^2 + 1}.
\end{aligned}$$

So, $\mathcal{B}(x) = M(y) < y_3(\tau^2+1)^{\frac{p}{2}}$. Now (6.2) can be rewritten as

$$(x_1-x_2)^{p-1}[x_1+(p-1)x_2] = \left[x_3^{\frac{1}{p}} - \sqrt{\mathcal{B}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right] \times \left[(p-1)\sqrt{\mathcal{B}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}} + x_3^{\frac{1}{p}}\right].$$

Therefore,

$$G(x_1,-x_2) = G\left(x_3^{\frac{1}{p}}, -\sqrt{\mathcal{B}^{\frac{2}{p}} - \tau^2 x_3^{\frac{2}{p}}}\right),$$

or by factoring out $x_3^{\frac{1}{p}}$ on the right side we get

$$G(y_1-y_2,-y_1-y_2) = y_3 G(1,-\sqrt{\omega^2-\tau^2}). \quad \square$$

Since y_2 is fixed then $D_2 \geq 0$ must be true in order that the Monge–Ampère solution from Case (3_1) is the Bellman function (see Proposition 5). However, the contrary is true: $D_2 < 0$.

Lemma 61. *In Case* (3_1) *we choose* $H(y_1,y_2) = G(y_1-y_2,-y_1+y_2)$ *because of how the implicit solution is defined and obtain* $\operatorname{sign} H'' = \operatorname{sign}(p-2)$.

Proof. We already computed

$$H'' = \begin{cases} 4G_{z_1z_2}, & \alpha_j = \beta_j \\ 0, & \alpha_j = -\beta_j \end{cases}$$

in Lemma 14.

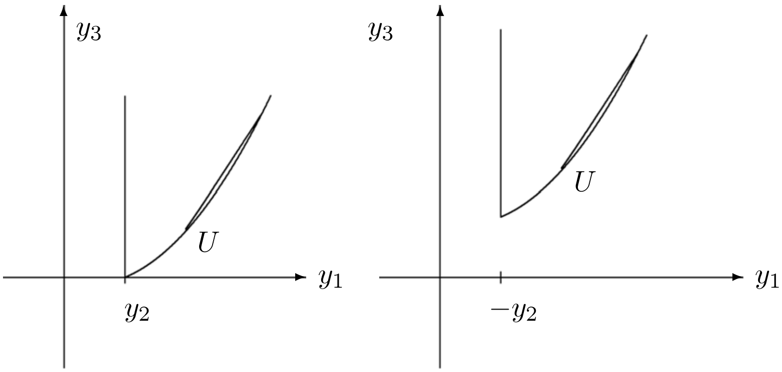
Since, $\alpha_1 = 1, \alpha_2 = -1, \beta_1 = -1$ and $\beta_2 = 1$ then $G_{z_1z_2} = p(p-1)(p-2)(y_1-y_2)(2y_2)^{p-3}$. $\square$

Lemma 62. *If* $p \neq 2$ *then* $D_2 < 0$ *in Case* (3_1) *for all* τ.

Proof. We use the partial derivatives of G computed in the proof of Lemma 14 to make the following computations of Φ' and Φ'' easier. Let $\beta = \sqrt{\omega^2-\tau^2}$.

$$\begin{aligned}
&\Phi(\omega) = G(1,-\beta)\\
&\Phi'(\omega) = -p(p-1)\omega(1-\beta)^{p-2}\\
&\Phi''(\omega) = -p(p-1)[(1-\beta)^{p-2} - (p-2)\omega^2\beta^{-1}]\\
&\Lambda = (p-1)\Phi' - \omega\Phi''\\
&\quad = p(p-1)\omega(1-\beta)^{p-3}[-(p-1)(1-\beta) + (1-\beta) - (p-2)\omega^2\beta^{-1}]\\
&\quad = -p(p-1)\omega(1-\beta)^{p-3}(p-2)[1-\beta+\omega^2\beta^{-1}]\\
&\quad = -p(p-1)(p-2)\omega(1-\beta)^{p-3}\left(1+\frac{\tau^2}{\beta}\right).
\end{aligned}$$

From Lemma 55, $1-\beta > 0$ and $\omega^2 > \tau^2 > 0$. Therefore, by Lemma 61 and (2.4) $\operatorname{sign} D_2 = \operatorname{sign} H'' \operatorname{sign} \Lambda = -(\operatorname{sign}(p-2))^2 < 0$. $\square$



(a) $y_2 \geq 0$. (b) $y_2 \leq 0$.

Fig. 12. Characteristic of the solution in Case (4_1).

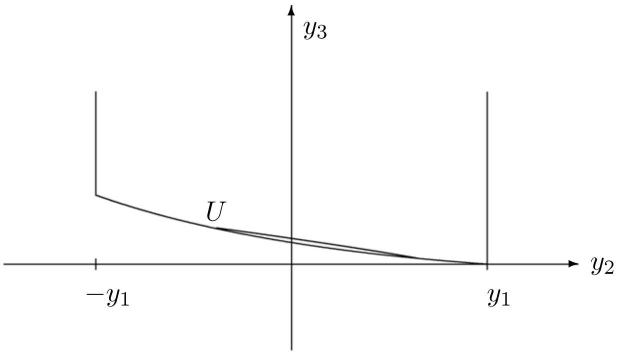


Fig. 13. Characteristic for the solution from Case (4_2).

Having shown that $D_2 < 0$ in Case (3_1) implies that the Monge–Ampère solution in that case cannot be the Bellman function.

Proposition 63. *Case* (3_1) *does not give a Bellman function candidate.*

6.4. *Case* (2_1) *gives a partial Bellman function candidate*

Case (2) was considered without having to fix either y_1 or y_2 first, so there is nothing new to do here. Refer to Sections 5.2 and 2.1.3 for more details.

6.5. *Case (4) may or may not yield a Bellman function candidate*

For $\tau = 0$, it was shown in [21] that Case (4) does not produce a Bellman function candidate, since some simple extremal functions give a contradiction to linearity of the Monge–Ampère solution on characteristics. However, for $\tau \neq 0$ it is much more difficult to show this. Those same extremal functions do contradict linearity for some p-values and some signs of the Martingale transform. For the sign of the Martingale transform where we do not have a contradiction, a new set of test of extremal functions would have to be found. Since the Bellman function has already been constructed from other cases, this case has not been investigated any further than just described. So, for p and τ values not mentioned in the main result, Case (4) could give a Bellman candidate throughout Ξ_+ or we could get a partial Bellman candidate that may work well with the characteristics from Case (2_1) Figs. 12 and 13.

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