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Weakly Integrable Semigroups on Locally Convex Spaces

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A class of weakly integrable semigroups on locally convex spaces is introduced and studied. The results are illustrated by examples of semigroups of unbounded operators on a Banach space, which include fractional powers of a closed operator and spectral local semigroups.

1. INTRODUCTION

The connection between Markov processes and semigroups of continuous linear operators is now well established. However, early in the development of the theory W. Feller observed that the use of strongly continuous semigroups is not always appropriate [3]. The purpose of this note is to introduce a class of semigroups of operators on locally convex spaces which is well adapted to the study of Markov processes and other areas of analysis where the assumption of strong continuity for the semigroups concerned fails to hold. It turns out that a surprising amount of information about general semigroups is easily deduced from very few conditions.

The approach suggested here is based on a number of principles common to many problems in analysis. Weak (Pettis-type) integration is the most appropriate tool for this situation, as opposed to the strong (Bochner) integrals used in the standard theory [4]. Moreover, even to utilize the weak integral it is often necessary to weaken the topology of the underlying vector space: a feature in common with the spectral theory of operators [10]. Another aspect of the present approach is that the "infinitesimal generator" of the semigroup is defined directly in terms of the resolvent instead of the conventional definition by differention. In the usual theory it makes no difference which definition one chooses; they agree.

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However, in the present context the resolvent approach has the advantage of substituting integral operators and equations for differential ones: a time-honoured and successful method in analysis.

Much of the existing work extending parts of the Hille–Yosida–Phillips theory of strongly continuous semigroups on Banach spaces to the locally convex space setting appeals to conditions which limit its applicability to, say, Markov processes. The approach followed here is closest to those of H. Komatsu [8] and S.-Y. Shaw [13] in certain respects.

The paper is organized as follows. Section 2 deals with the definition and basic properties of F-semigroups on a locally convex space E, where F is a subspace of the continuous dual E' of E. Section 3 treats some special cases, and examines the relationship between various "generators." In Section 4, the inductive limit of weakly integrable semigroups is defined, and applied in Section 5 to examples of semigroups of unbounded operators on a Banach space treated by R. J. Hughes [5]. One example is the semigroup of fractional powers of a closed operator, and another is the spectral local semigroup associated with a Klein–Landau system of operators [7].

An application to diffusion processes is given elsewhere [6]. A characterization of those operators which act as generators of a large class of weakly integrable semigroups will be given in a later paper.

2. Definition and Properties

Let E be a locally convex space with continuous dual E'. Denote the family of continuous linear operators on E by $L(E)$. The identity map on E is denoted by I.

A semigroup of continuous linear operators on E is a map $S: (0, \infty) \to L(E)$ such that $S(s+t) = S(s)\,S(t)$ for all $s, t > 0$.

Given an operator $A \in L(E)$, the adjoint $A': E' \to E'$ of A is defined by $\langle Ax, \xi \rangle = \langle x, A'\xi \rangle$, $x \in E$, $\xi \in E'$. The adjoint of A is continuous for the weak topology $\sigma(E', E)$ on E'. For a semigroup $S: (0, \infty) \to L(E)$, set $S'(t) = S(t)'$ for each $t > 0$. We say that a subspace X of E (resp. of E') is S-invariant (resp., S'-invariant) if for each $t > 0$ it is left invariant by the operator $S(t)$ (resp., by $S(t)'$).

Now suppose that F is a subspace of E' separating points in E.

Definition 2.1. A semigroup $S: (0, \infty) \to L(E)$ is said to be an F-*semigroup* if the following two conditions are satisfied.

(S1) There exists an S'-invariant subspace of F separating points in E.

(S2) There exists $\omega_S \geqslant 0$ such that for all $\lambda > \omega_S$ and $x \in E$, the map

$t \to e^{-\lambda t}\langle S(t)x, \xi\rangle$, $t > 0$ is integrable on $(0, \infty)$ for every $\xi \in F$, and there exists $R(\lambda)x \in E$ such that

$$\langle R(\lambda)x, \xi\rangle = \int_0^\infty e^{-\lambda t}\langle S(t)x, \xi\rangle\, dt$$

for all $\xi \in F$.

Condition (S1) is equivalent to the requirement that the set of all $\xi \in F$ such that $S(t)'\xi \in F$ for every $t > 0$ is $\sigma(F, E)$-dense in F.

The linear map $R(\lambda)\colon E \to E$ is well defined for every $\lambda > \omega_S$ because F separates points in E. It is convenient to write $R(\lambda) = \int_0^\infty e^{-\lambda t}S(t)\, dt$, $\lambda > \omega_S$, with the understanding that the integral exists in the sense of (S2). It is called the *resolvent* of S.

In the case that E is a Banach space and $F = E'$, condition (S2) is similar to the familiar notion of the Pettis integrability of the E-valued function $t \mapsto e^{-\lambda t}S(t)x$, $t > 0$ for each $x \in E$ and $\lambda > \omega_S$ [1].

Recent progress [14] towards a better understanding of Pettis integrals can be brought to bear on the verification of condition (S2). For the moment we note that whenever $e^{-\omega_S t}S(t)$, $t > 0$, is a bounded family of continuous linear operators on a Banach space E, and the function $t \mapsto \langle S(t)x, \xi\rangle$, $t > 0$ is Lebesgue measurable for every $x \in E$ and $\xi \in E'$, then it is consistent with ZFC to assume that (S2) holds with $F = E'$ [2]; at least for bounded semigroups on a Banach space, it appears that the conditions of Definition 2.1 are truly "minimal."

Let S be an F-semigroup on E with $R(\lambda) = \int_0^\infty e^{-\lambda t}S(t)\, dt$, $\lambda > \omega_S$, its resolvent. Set $F_S = \{\xi \in F\colon S(t)'\xi \in F \text{ for all } t > 0\}$.

LEMMA 2.2. *$R(\lambda) - R(\nu) = (\nu - \lambda)R(\lambda)R(\nu)$ for all $\lambda, \nu > \omega_S$.*

Proof. Let $x \in E$, $\xi \in F_S$ and $\lambda > \nu > \omega_S$. Then

$$\begin{aligned}
\langle R(\lambda)R(\nu)x, \xi\rangle &= \left\langle \int_0^\infty e^{-\lambda t}S(t)R(\nu)x\,dt, \xi\right\rangle \\
&= \int_0^\infty e^{-\lambda t}\left\langle S(t)R(\nu)x, \xi\right\rangle dt \\
&= \int_0^\infty e^{-\lambda t}\left\langle R(\nu)x, S(t)'\xi\right\rangle dt \\
&= \int_0^\infty e^{-\lambda t}\left\langle \int_0^\infty e^{-\nu s}S(s)x\,ds, S(t)'\xi\right\rangle dt.
\end{aligned}$$

Because $\xi \in F_S$, we have for each $t > 0$

$$\left\langle \int_0^\infty e^{-\nu s} S(s)\, x ds, S(t)'\xi \right\rangle = \int_0^\infty e^{-\nu s} \left\langle S(s)\, x, S(t)'\xi \right\rangle ds$$

$$= \int_0^\infty e^{-\nu s} \left\langle S(t)\, S(s)\, x, \xi \right\rangle ds$$

$$= \int_0^\infty e^{-\nu s} \left\langle S(t+s)\, x, \xi \right\rangle ds.$$

Take a change of variables $u = t + s$, $s > 0$. Then

$$\int_0^\infty e^{-\nu s} \left\langle S(t+s)\, x, \xi \right\rangle ds = e^{\nu t} \int_t^\infty e^{-\nu u} \left\langle S(u)\, x, \xi \right\rangle du,$$

so we have

$$\langle R(\lambda)\, R(\nu)\, x, \xi \rangle = \int_0^\infty e^{-(\lambda-\nu)t} \left[\int_t^\infty e^{-\nu u} \langle S(u)\, x, \xi \rangle \, du \right] dt.$$

Now integrate by parts to get

$$\langle R(\lambda)\, R(\nu)\, x, \xi \rangle = (\lambda - \nu)^{-1} \left[\int_0^\infty e^{-\nu u} \langle S(u)\, x, \xi \rangle \, du - \int_0^\infty e^{-\lambda t} \langle S(t)\, x, \xi \rangle \, dt \right]$$

$$= (\lambda - \nu)^{-1} \langle [R(\nu) - R(\lambda)]\, x, \xi \rangle.$$

To see that $\langle R(\lambda)\, R(\nu)\, x, \xi \rangle = \langle R(\nu)\, R(\lambda)\, x, \xi \rangle$, observe that the function $(t, s) \to \langle S(t+s)\, x, \xi \rangle$, $t, s > 0$ is measurable and

$$\int_0^\infty e^{-\lambda t} \left[\int_0^\infty e^{-\nu s} |\langle S(t+s)\, x, \xi \rangle| \, ds \right] dt$$

$$= \int_0^\infty e^{-(\lambda-\nu)t} \int_t^\infty e^{-\nu u} |\langle S(u)\, x, \xi \rangle| \, du\, dt$$

$$\leqslant \int_0^\infty e^{-(\lambda-\nu)t} \int_0^\infty e^{-\nu u} |\langle S(u)\, x, \xi \rangle| \, du\, dt.$$

By the Fubini–Tonelli theorem,

$$\int_0^\infty e^{-\lambda t} \left[\int_0^\infty e^{-\nu s} \langle S(t+s)\, x, \xi \rangle \, ds \right] dt$$

$$= \int_0^\infty e^{-\nu s} \left[\int_0^\infty e^{-\lambda t} \langle S(t+s)\, x, \xi \rangle \, dt \right] ds,$$

which means that $\langle R(\lambda)\, R(\nu)\, x, \xi \rangle = \langle R(\nu)\, R(\lambda)\, x, \xi \rangle$.

Since F_S separates points in E, the desired relation holds.

The resolvent relation of Lemma 2.2 implies that the ranges of the operators $R(\lambda)$, $\lambda > \omega_S$ are all the same, and if one of them is injective then they all are. In this case there exists a linear map $G_S: \mathscr{D}(G_S) \to E$ such that $R(\lambda) = (\lambda I - G_S)^{-1}$, $\mathscr{D}(G_S) = R(\lambda) E$ for all $\lambda > \omega_S$. The operator G_S, when it exists, is called the *E-generator* of S.

LEMMA 2.3. *For each* $t > 0$, $\lambda > \omega_S$, $S(t) R(\lambda) = R(\lambda) S(t)$.

Proof. Let $x \in E$ and $\xi \in F_S$. Then

$$\langle R(\lambda) S(t) x, \xi \rangle = \int_0^\infty e^{-\lambda s} \langle S(s) S(t) x, \xi \rangle \, ds = \int_0^\infty e^{-\lambda s} \langle S(t) S(s) x, \xi \rangle \, ds$$

$$= \int_0^\infty e^{-\lambda s} \langle S(s) x, S(t)' \xi \rangle \, ds$$

$$= \langle R(\lambda) x, S(t)' \xi \rangle = \langle S(t) R(\lambda) x, \xi \rangle.$$

Since F_S separates points in E, $R(\lambda) S(t) = S(t) R(\lambda)$.

COROLLARY 2.4. *Suppose that that E-generator* G_S *of* S *exists; that is, each operator* $R(\lambda)$, $\lambda > \omega_S$ *is injective. Then for all* $t > 0$,

$$S(t) \mathscr{D}(G_S) \subset \mathscr{D}(G_S), \; G_S S(t) x = S(t) G_S x \qquad \text{for all } x \in \mathscr{D}(G_S).$$

Proof. $\mathscr{D}(G_S) = R(\lambda) E$ for every $\lambda > \omega_S$, so Lemma 2.3 immediately implies that $S(t) \mathscr{D}(G_S) \subset \mathscr{D}(G_S)$.

If $x \in \mathscr{D}(G_S)$, then for some $y \in E$ and $\lambda > \omega_S$, $x = R(\lambda) y$. But

$$G_S S(t) x = G_S S(t) R(\lambda) y = G_S R(\lambda) S(t) y = \lambda R(\lambda) S(t) y - S(t) y$$
$$= S(t)[\lambda R(\lambda) - I] y = S(t) G_S R(\lambda) y = S(t) G_S x.$$

PROPOSITION 2.5. *Let* $A: \mathscr{D}(A) \to E$ *be a linear map with domain* $\mathscr{D}(A) \subset E$.

If for some $\lambda > \omega_S$, $R(\lambda)(\lambda I - A) x = x$ *for all* $x \in \mathscr{D}(A)$, *then*

$$\langle S(t) x, \xi \rangle = \langle x, \xi \rangle + \int_0^t \langle S(s) Ax, \xi \rangle \, ds \tag{1}$$

for all $x \in \mathscr{D}(A)$, $\xi \in F$ *and* $t > 0$.

Conversely, if for every $x \in D(A)$ *the set of all* $\xi \in F$ *for which* (1) *holds for almost* $t > 0$ *separates points in* E, *then* $R(\lambda)(\lambda I - A) x = x$ *for all* $x \in \mathscr{D}(A)$.

Proof. First, let A be an operator such that for some $\lambda > \omega_S$, $R(\lambda)(\lambda I - A)\, x = x$ for all $x \in \mathscr{D}(A)$. By Lemma 2.2, the equation is valid for all values of $\lambda > \omega_S$.

Let $x \in \mathscr{D}(A)$, $\xi \in F$ and $\lambda > \omega_S$. Then

$$\langle R(\lambda)(\lambda I - A)\, x, \xi \rangle = \int_0^\infty e^{-\lambda t} \langle S(t)(\lambda I - A)\, x, \xi \rangle \, dt = \langle x, \xi \rangle,$$

$$\lambda \int_0^\infty e^{-\lambda t} \langle S(t)\, x, \xi \rangle \, dt - \int_0^\infty e^{-\lambda t} \langle S(t)\, Ax, \xi \rangle \, dt = \langle x, \xi \rangle, \tag{2}$$

$$\int_0^\infty e^{-\lambda t} \langle S(t)\, x, \xi \rangle \, dt - \int_0^\infty e^{-\lambda t} \int_0^t \langle S(s)\, Ax, \xi \rangle \, ds\, dt = \int_0^\infty e^{-\lambda t} \langle x, \xi \rangle \, dt. \tag{3}$$

Here we have integrated by parts and noted that

$$\begin{aligned} \left| e^{-\lambda t} \int_0^t \langle S(s)\, Ax, \xi \rangle \, ds \right| &\leqslant e^{-\lambda t} \int_0^t |\langle S()s\, Ax, \xi \rangle| \, ds \\ &\leqslant e^{-(\lambda - \mu)t} \int_0^t e^{-\mu s} |\langle S(s)\, Ax, \xi \rangle| \, ds \\ &\leqslant e^{-(\lambda - \mu)t} \int_0^\infty e^{-\mu s} |\langle S(s)\, Ax, \xi \rangle| \, ds \end{aligned}$$

for all $\omega_S < \mu < \lambda$ and all $t > 0$, so that

$$\lim_{t \to \infty} e^{-\lambda t} \int_0^t \langle S(s)\, Ax, \xi \rangle \, ds = 0.$$

Now (3) is true for all $\lambda > \omega_S$, so the uniqueness property of Laplace transforms implies that

$$\langle S(t)\, x, \xi \rangle = \langle x, \xi \rangle + \int_0^t \langle S(s)\, Ax, \xi \rangle \, ds$$

for almost all $t > 0$.

The equality (1) holds for all values of $t > 0$ because $\mathscr{D}(A) \subset R(\lambda)\, E$, $\lambda > \omega_S$, and for every $y \in E$

$$\begin{aligned} \langle S(t)\, R(\lambda)\, y, \xi \rangle = \langle R(\lambda)\, S(t)\, y, \xi \rangle &= \int_0^\infty e^{-\lambda s} \langle S(s+t)\, y, \xi \rangle \, ds \\ &= e^{\lambda t} \int_t^\infty e^{-\lambda u} \langle S(u)\, y, \xi \rangle \, du \end{aligned}$$

so $\langle S(\cdot)\, x, \xi \rangle$ is continuous on $(0, \infty)$.

Assume that the set of all $\xi \in F$ for which (1) holds for almost all $t > 0$ separates points in E. Then for $\omega_S < \mu < \lambda$

$$\int_0^\infty e^{-\lambda t} \left| \int_0^t \langle S(s)\, Ax, \xi \rangle \, ds \right| dt \leqslant \int_0^\infty e^{-(\lambda-\mu)t} dt \int_0^\infty e^{-\mu s} |\langle S(s)\, Ax, \xi \rangle| \, ds$$
$$< \infty.$$

Thus on integrating (1) with respect to $e^{-\lambda t} dt$ we obtain (3). Integration by parts now gives (2), which is just a reformulation of the desired equation.

COROLLARY 2.6. *Suppose that the E-generator G_S of S exists. Then the function $\langle G_S S(\cdot)\, x, \xi \rangle$ is integrable on $[0, t]$ and*

$$\langle S(t)\, x, \xi \rangle = \langle x, \xi \rangle + \int_0^t \langle G_S S(s)\, x, \xi \rangle \, ds$$

for all $x \in \mathscr{D}(G_S)$, $\xi \in F$ and $t > 0$.

An operator $A: \mathscr{D}(A) \to E$, $\mathscr{D}(A) \subset E$ with the property that $R(\lambda)(\lambda I - A)\, x = x$ for all $x \in \mathscr{D}(A)$ and $\lambda > \omega_S$ is called a *subgenerator* of S.

3. SOME SPECIAL CASES

To obtain comparisons with standard semigroup theory, additional continuity conditions need to be imposed upon F-semigroups.

First of all we note that for any subgenerator $A: \mathscr{D}(A) \to E$ of S, if $x \in \mathscr{D}(A)$ then $S(t)\, x \to x$ in $\sigma(E, F)$ as $t \to 0^+$ and the E-valued function $S(\cdot)\, x$ is $\sigma(E, F)$-continuous on $(0, \infty)$.

PROPOSITION 3.1. *Let $A: \mathscr{D}(A) \to E$ be a subgenerator of S. Let ρ be a locally convex topology on E with a fundamental system of $\sigma(E, F)$-closed convex neighborhoods of zero.*

If for all $y \in E$, the E-valued function $S(\cdot)\, y$ is ρ-bounded in a neighborhood of $t = 0$, then $S(t)\, x \to x$ in ρ as $t \to 0^+$ for every $x \in \mathscr{D}(A)$.

Furthermore, if $\mathscr{D}(A)$ is ρ-dense in E and S is ρ-equicontinuous in a neighborhood of $t = 0$, then $S(t)\, x \to x$ in ρ as $t \to 0^+$ for all $x \in E$.

Proof. Let U be an arbitrary $\sigma(E, F)$-closed, disked neighborhood of zero of ρ. Let $U^0 = \{\xi \in F: |\langle x, \xi \rangle| \leqslant 1$ for all $x \in U\}$ be the polar of U in F. Then $U^{00} = \{x \in E: |\langle x, \xi \rangle| \leqslant 1$ for all $\xi \in U^0\} = U$ [10, IV, 1.5].

The claims now follow from Proposition 2.5 and the estimate

$$\sup_{\xi \in U^0} |\langle S(t)\, x - x, \xi \rangle| \leqslant \sup_{\xi \in U^0} \int_0^t |\langle S(s)\, Ax, \xi \rangle| \, ds, \qquad t > 0. \tag{4}$$

Proposition 3.2. *Suppose that there exists a $\sigma(F, E)$-dense subspace M of F such that $S(t)\,x \to x$ in $\sigma(E, M)$ as $t \to 0^+$ for each $x \in E$.*

Then for all $\lambda > \omega_S$, $R(\lambda) = \int_0^\infty e^{-\lambda t} S(t)\,dt$ is injective and $R(\lambda)\,E$ is $\sigma(E, M)$-dense in E.

Proof. For each $x \in E$, $\xi \in M$ and $\lambda > \omega_S$

$$\langle \lambda\, R(\lambda)\, x - x, \xi \rangle = \int_0^\infty \lambda e^{-\lambda t} \langle S(t)\, x - x, \xi \rangle\, dt.$$

Given $\varepsilon > 0$, choose $\delta > 0$ such that $|\langle S(t)\, x - x, \xi \rangle| < \varepsilon$ for all $t \in (0, \delta)$. Then

$$|\langle \lambda\, R(\lambda)\, x - x, \xi \rangle| \leqslant \varepsilon \int_0^\delta \lambda e^{-\lambda t} dt + \int_\delta^\infty \lambda e^{-\lambda t} |\langle S(t)\, x - x, \xi \rangle|\, dt.$$

Now for λ sufficiently large, the function $\lambda \mapsto \lambda e^{-\lambda t}$ is decreasing for each $t \geqslant \delta$ and $\lim_{\lambda \to \infty} \lambda e^{-\lambda t} = 0$, so by monotone convergence $\lim_{\lambda \to \infty} |\langle \lambda\, R(\lambda)\, x - x, \xi \rangle| < \varepsilon$, proving that $\lambda R(\lambda)\, x \to x$ in $\sigma(E, M)$. The conclusions follow immediately.

Under the above assumptions the E-generator G_S exists and its domain $\mathscr{D}(G_S)$ is $\sigma(E, M)$-dense in E.

Suppose that ρ is some locally convex topology on E with a fundamental system of $\sigma(E, F)$-closed convex neighbourhoods of zero. Then estimates of the form (4) yield conclusions similar to those of Proposition 3.2 for the topology ρ whenever $S(t)\, x \to x$ in ρ as $t \to 0^+$ for each $x \in E$; namely, $\mathscr{D}(G_S)$ is ρ-dense in E and $\lambda\, R(\lambda)\, x \to x$ in ρ as $\lambda \to \infty$ for every $x \in E$.

Now let $\mathscr{D}(D_S)$ be the set of all $x \in E$ for which $(S(t)\, x - x)/t$ converges in $\sigma(E, F)$ as $t \to 0^+$ and define $D_S x$ as its limit.

Proposition 3.3. *Suppose that for each $t > 0$, $S(t)$ is $\sigma(E, F)$-continuous. Then $S(t)\, \mathscr{D}(D_S) \subset \mathscr{D}(D_S)$ and $S(t)\, D_S x = D_S S(t) x$ for all $x \in \mathscr{D}(D_S)$ and $t > 0$.*

If, furthermore, for each $x \in E$, $S(t)\, x \to x$ in $\sigma(E, F)$ as $t \to 0^+$, then the E-generator G_S of S exists and $G_S = D_S$.

Proof. The first part follows from the definition of D_S and the $\sigma(E, F)$-continuity of each operator $S(t)$, $t > 0$.

The E-generator G_S *of* S exists by Proposition 3.2, and according to Proposition 2.5

$$\langle S(t)\, x, \xi \rangle = \langle x, \xi \rangle + \int_0^t \langle S(s)\, G_S x, \xi \rangle\, ds$$

for all $x \in \mathscr{D}(G_S)$, $\xi \in F$ and $t > 0$. The convergence of $\langle S(s) G_S x, \xi \rangle$ to $\langle G_S x, \xi \rangle$ as $s \to 0^+$ implies that

$$\lim_{t \to 0^+} \langle S(t) x - x, \xi \rangle / t = \langle G_S x, \xi \rangle.$$

Therefore $\mathscr{D}(G_S) \subset \mathscr{D}(D_S)$ and $G_S x = D_S x$ for all $x \in \mathscr{D}(G_S)$. Now we show that $\mathscr{D}(D_S) \subset \mathscr{D}(G_S)$.

If $x \in \mathscr{D}(D_S)$, $\xi \in F$, then the function $\langle S(\cdot) x, \xi \rangle$ has a right derivative equal to $\langle S(t) D_S x, \xi \rangle$ at each point $t > 0$. Since $\langle S(\cdot) D_S x, \xi \rangle$ is continuous on $(0, \infty)$, we have

$$\langle S(t) x, \xi \rangle = \langle x, \xi \rangle + \int_0^t \langle S(s) D_S x, \xi \rangle \, ds, \qquad t > 0;$$

the right derivatives of both sides of the equation are equal. Another appeal to Proposition 2.5 yields the inclusion $\mathscr{D}(D_S) \subset \mathscr{D}(G_S)$.

Similar results can also be deduced for the topology ρ. As before $D_S x$ is defined as the limit $\lim_{t \to 0^+} (S(t) x - x)/t$ in ρ for every $x \in E$ for which it exists. If $S(t)$ is ρ-continuous for each $t > 0$, then of course $S(t)$ commutes with D_S. Combined with the estimate (4), the proof above shows that $G_S = D_S$ whenever, in addition, for each $x \in E$, $S(t) x \to x$ in ρ as $t \to 0^+$.

The last condition is implied by the assumptions of Proposition 3.1. Virtually all "weak" = "strong" properties of semigroups are proved in this manner by appealing to equicontinuity about $t = 0$.

The term "E-generator" was introduced because it may happen that the operators $R(\lambda)$, $\lambda > \omega_S$ are not injective on E, but there exists an invariant subspace H of E on which they are. Naturally one would then speak of an "H-generator" if the restriction of $R(\lambda)$ to H determined $R(\lambda)$ by, say, continuity. However, no continuity assumptions have been made for $R(\lambda)$ in general.

An important example of this phenomonon is provided by diffusion processes on $\mathbb{R}^d$. There E is the space of bounded Borel measurable functions on $\mathbb{R}^d$ and F is the space of (signed) Borel measures on $\mathbb{R}^d$. The resolvent $R(\lambda)$, $\lambda > \omega_S$ will *not* be injective on E (e.g., $R(\lambda)$ maps the characteristic function of a single point to zero), but it is injective when restricted to the continuous functions.

The resolvent $R(\lambda)$, $\lambda > \omega_S$ in general only partially determines S in the sense that if S_1 is another F-semigroup with resolvent $R_1(\lambda)$, $\lambda > \omega_1$ and $R_1(\lambda) = R(\lambda)$ for all $\lambda > \max(\omega_S, \omega_1)$, then for each $x \in E$ and $\xi \in F$, $\langle S(t) x, \xi \rangle = \langle S_1(t) x, \xi \rangle$ for almost all $t > 0$: a consequence of uniqueness for Laplace transforms. The set for which equality holds may of course depend on x and ξ. An example of R. S. Phillips [9] shows that the righthand side of the equation may in fact be identically zero for all $t>0$, without S itself being the zero semigroup.

LEMMA 3.4. *Suppose that the resolvent $R(\lambda)$, $\lambda>\omega_S$ of the F-semigroup S has dense range in E.*

If S_1 is another F-semigroup with resolvent $R_1(\lambda)$, $\lambda>\omega_1$, and $R_1(\lambda)=R(\lambda)$ for all $\lambda>\max(\omega_S,\omega_1)$, then $S_1=S$.

Proof. According to Lemma 2.3, for each $x\in E$, $\xi\in F$, $\lambda>\omega_S$, $t>0$,

$$\langle S(t)\,R(\lambda)\,x,\xi\rangle=\langle R(\lambda)\,S(t)\,x,\xi\rangle=\int_0^\infty e^{-\lambda s}\langle S(s+t)\,x,\xi\rangle\,ds$$

$$=e^{\lambda t}\int_t^\infty e^{-\lambda u}\langle S(u)\,x,\xi\rangle\,du,$$

so for all $x\in E$ belonging to the common range of $R(\lambda)$ and $R_1(\lambda)$, the E-valued functions $S(\cdot)\,x$, $S_1(\cdot)\,x$ are continuous for the topology $\sigma(E,F)$. Therefore $S(t)\,x=S_1(t)\,x$ for all $t>0$. By assumption, $R(\lambda)\,E$ is dense in E, so equality holds everywhere.

Another possibility is that there may exist a dense subspace H of E and a subspace J of F separating points in E such that the functions $\langle S(\cdot)\,x,\xi\rangle$, $\langle S_1(\cdot)\,x,\xi\rangle$ are continuous on $(0,\infty)$ for every $x\in H$ and $\xi\in J$. Then of course the equality $S=S_1$ also holds.

Similarly, the E-generator, when it exists, only determines S up to equivalence in general. In applications to diffusion processes, the equality of subgenerators on subspaces much smaller than their respective domains can yield the equality of the semigroups; that is, a subgenerator may "generate" a semigroup with additional properties. Such examples are treated elsewhere [6].

A familiar technique of the standard theory of continuous semigroups on a Banach space is to restrict an arbitrary uniformly bounded semigroup to the subspace on which it is continuous, and to study its properties there. Next we deal with this method in the present setting.

Let $T\colon(0,\infty)\to L(E)$ be a semigroup of continuous linear operators on the locally convex space E. Let H be some T'-invariant subspace of E' and let J be a subspace of E' containing H. Set

$$E_H=\{x\in E\colon \lim_{t\to 0^+}\langle S(t)\,x,\xi\rangle=\langle x,\xi\rangle \text{ for all } \xi\in H\},$$

$$E_H^0=\{\xi\in E'\colon \langle x,\xi\rangle=0 \text{ for all } x\in E_H\},$$

$$J_H=J/(E_H^0\cap J),\qquad H_E=H/E_H^0\cap H).$$

Furthermore it is assumed that H separates points in E_H and J separates points in the closure $\bar{E}_H$ of E_H in E. It could be supposed for example, that $H \subset J \subset E'$ and H is $\sigma(E', E)$-dense in E.

The space E_H is endowed with relative topology of E. Since H is T'-invariant, E_H is a T-invariant subspace of E. The restriction of T to the subspace E_H is denoted by T_H.

PROPOSITION 3.5. *Suppose that for each $x \in E$ and $\xi \in J$ the function $\langle T(\cdot)\, x, \xi \rangle$ is Lebesgue measurable. Furthermore, suppose that there exists $\omega \geqslant 0$ such that for all $x \in E_H$ there exists a $\sigma(\bar{E}_H, J_H)$-compact convex subset K_x of $\bar{E}_H$, such that for almost all $t > 0$, $e^{-\omega t} T(t)\, x \in K_x$.*

Then T_H is a J_H-semigroup on E_H such that for each $x \in E_H$, $T_H(t)\, x \to x$ in $\sigma(E_H, H_E)$ as $t \to 0^+$. The domain of the E_H-generator of T_H is $\sigma(E_H, H_E)$-dense in E_H.

Proof. It is readily seen that T_H satisfies condition $(S1)$, because H is a T'-invariant subspace of J separating points in E_H.

To verify (S2), take $x \in E_H$ and $\lambda > \omega$. If $\xi \in J$ and $|\langle y, \xi \rangle| \leqslant 1$ for all $y \in K_x$, then

$$\int_0^\infty e^{-\lambda t} |\langle T(t)\, x, \xi \rangle|\, dt \leqslant (\lambda - \omega)^{-1}.$$

By the bipolar theorem [12, IV, 1.5] and the compactness of K_x, there exists $R(\lambda)\, x \in \bar{E}$ such that for all such ξ

$$\langle R(\lambda)\, x, \xi \rangle = \int_0^\infty e^{-\lambda t} \langle T(t)\, x, \xi \rangle\, dt.$$

Furthermore, $R(\lambda)\, x$ is uniquely defined.

To establish that $R(\lambda)\, x \in E_H$, let $\xi \in H$. Then

$$\begin{aligned} \langle T(t)\, R(\lambda)\, x - R(\lambda)\, x, \xi \rangle &= \int_0^\infty e^{-\lambda t} \langle T(s+t)\, x, \xi \rangle\, ds - \langle R(\lambda)\, x, \xi \rangle \\ &= e^{-\lambda t} \int_t^\infty e^{-\lambda u} \langle T(u)\, x, \xi \rangle\, du \\ &\quad - \int_0^\infty e^{-\lambda u} \langle T(u)\, x, \xi \rangle\, du. \end{aligned}$$

Here the assumption that H is a T'-invariant subspace of J has been used. Taking $t \to 0^+$, we see that $R(\lambda)\, x \in E_H$.

The remaining properties of the semigroup T_H follow from the definitions and the remark following Proposition 3.2.

For the case that E is a banach space and $H=J=E'$, the conditions of Propostion 3.5 are satisfied whenever $\limsup_{t\to\infty} e^{-\omega t}\|T(t\|<\infty$ for some $\omega\geqslant 0$, because the set $\{e^{-\lambda t}T(t)\,x: t>0\}$ is then a relatively weakly compact subset of E for each $x\in E_G$ and $\lambda>\omega$.

In this respect, the following remarkable result of W. Feller [3] is pertinent. Given a measurable function $f\colon (0,\infty)\to\mathbb{R}$, set lim ess. $\sup_{t\to\infty} f(t)=\lim_{r\to\infty}(\inf\{a: f(t)\leqslant a$ for almost all $t>r\})$.

PROPOSITION 3.6. (*Feller*). *Let X be a Banach space and Y a norming subspace of X'. Let $T\colon (0,\infty)\to L(X)$ be a semigroup such that*

(i) *the function $\langle T(\cdot)\,x, y\rangle$ is Lebesgue measurable for every $x\in X$ and $y\in Y$;*

(ii) *for each $y\in Y$, $T(t)'y\in Y$ for almost all $t>0$.*

Then there exists $\omega\geqslant 0$ such that for all $x\in X$ and $y\in Y$,

$$\lim_{t\to\infty} \operatorname{ess\,sup} e^{-\omega t}|\langle T(t)\,x, y\rangle|\leqslant \|x\|\ \|y\|.$$

Thus, whenever E is a Banach space, an F-semigroup on E is essentially bounded at infinity in the above sense.

4. INDUCTIVE LIMITS OF WEAKLY INTEGRABLE SEMIGROUPS

To deal with semigroups of unbounded operators, the inductive limit of a collection of weakly integrable semigroups needs to be considered. A resolvent will no longer be associated with the semigroup, but a meaning can still be given to its "generator;" the whole real line may be part of its spectrum.

Let X be a locally convex space. Suppose that $(A,\lesssim)$ is a directed set and E_α, $\alpha\in A$ is a family of locally convex spaces such that E_α is continuously included in E_β whenever $\alpha\lesssim\beta$, $\alpha,\beta\in A$, and for each $\alpha\in A$, E_α is continuously included in X. Furthermore, suppose that the space $\bigcup_{\alpha\in A} E$ is dense in X. Set $E=\varinjlim E$, the inductive limit of E_α, $\alpha\in A$ [12, II, 6] and put $F=X'$.

The polar E_α^0 of E_α in F is the set of all $\xi\in F$ such that $\langle x,\xi\rangle=0$ for all $x\in E_\alpha$. Let F_α denote the quotient space F/E_α^0 for each $\alpha\in A$.

DEFINITION 4.1. A semigroup S of continuous linear operators on E is called the *inductive limit* of the F_α-semigroups S_α, $\alpha\in A$, if for each $\alpha\in A$, E_α is an S-invariant subspace of E and $S_\alpha=S(\cdot)|E_\alpha$ is an F_α-semigroup on E_α.

Only the dual F of the space X plays a rôle in the definition, but it has been formulated in terms of X for motivation.

The following fact follows easily from the properties of inductive limit topologies [10, II, 6].

LEMMA 4.2. *Let $H = \varinjlim H_\alpha$ be the inductive limit of an increasing family $\{H_\alpha : \alpha \in A\}$ of locally convex spaces. For each $\alpha \in A$, suppose that J_α is a subspace of H'_α separating points in H_α, such that $\langle J_\alpha : \alpha \in A\}$ forms a projective system of vector spaces via the adjoints of the inclusion maps associated with the inductive limit H. Furthermore, suppose that J_β is mapped onto J_α whenever $\alpha, \beta \in A$ and $\alpha \lesssim \beta$.*

Let X be the set H endowed with the topology $\sigma(H, \varprojlim J_\alpha)$.

If for each $\alpha \in A$, S_α is a J_α-semigroup on H_α and $S_\beta(\cdot)| H_\alpha = S_\alpha$ for every $\alpha \lesssim \beta$, then there exists a unique semigroup $\varinjlim S_\alpha$ of continuous linear operators on H such that $\varinjlim S_\alpha$ is the inductive limit of the J_α-semigroups S_α, $\alpha \in A$.

As a consequence of Definition 4.1, condition (S1) of Section 2 holds; that is, there exists an S'-invariant subspace of F separating points in E. However, condition (S2) holds only "locally."

Let S be the inductive limit of F_α-semigroups S_α, $\alpha \in A$, with $R_\alpha(\lambda) = \int_0^\infty e^{-\lambda t} S_\alpha(t)\,dt$, $\lambda > \omega_\alpha$ for each $\alpha \in A$.

Suppose that $P: \mathscr{D}(P) \to E$ is a linear operator with domain $\mathscr{D}(P) \subset E$. If for each $x \in \mathscr{D}(P)$ there exists $\alpha \in A$ such that x, $Px \in E_\alpha$ and $R_\alpha(\lambda)(\lambda I - P)\,x = x$ for all $\lambda > \omega_\alpha$, then P is called a *subgenerator* of S. When A is a singleton set, the term agrees with that previously defined.

The next assertion is a direct consequence of Proposition 2.5.

PROPOSITION 4.3. *Suppose that $P: \mathscr{D}(P) \to E$ is a linear operator with domain $\mathscr{D}(P) \subset E$.*

If P is a subgenerator of S, then for every $x \in \mathscr{D}(P)$, $\xi \in F$ and $t > 0$

$$S(t)\,x, \xi\rangle = \langle x, \xi\rangle + \int_0^t \langle S(s)\,Px, \xi\rangle\,ds. \tag{5}$$

Conversely, if for every $x \in \mathscr{D}(P)$ the set of all $\xi \in F$ for which (5) holds for almost all $t > 0$ separates points in E, then P is a subgenerator of S.

Now assume that each operator $R_\alpha(\lambda)$, $\lambda > \omega_\alpha$, $\alpha \in A$ is injective. Let $G_\alpha: \mathscr{D}(G_\alpha) \to E$ be the E_α-generator of S_α, $\alpha \in A$. Since $R_\beta(\lambda)| E = R_\alpha(\lambda)$ for $\alpha \lesssim \beta$, we have the inclusion $G_\alpha \subset G_\beta$ also. Define the E-generator $G_S: \mathscr{D}(G_S) \to E$ of S by

$$\mathscr{D}(G_S) = \bigcup_{\alpha \in A} \mathscr{D}(G_\alpha), \qquad G_S x = G_\alpha x \quad \text{for } x \in \mathscr{D}(G_\alpha),\ \alpha \in A.$$

It is easily verified that G_S commutes with $S(t)$ for each $t>0$. The proofs of the various properties of F-semigroups established in Section 3 go over by localization to inductive limits. In particular, the following analogues of Propositions 3.1 and 3.2 are worth mentioning.

Let ρ be a locally convex topology on E with a fundamental system of $\sigma(E, F)$-closed convex neighbourhoods of zero.

PROPOSITION 4.4. *Let $P: \mathscr{D}(P) \to E$ be a subgenerator of S. If for all $y \in E$, the E-valued function $S(\cdot)\, y$ is ρ-bounded in a neighbourhood of $t=0$, then $S(t)\, x \to x$ in ρ as $t \to 0^+$ for every $x \in \mathscr{D}(P)$.*

Furthermore, if $\mathscr{D}(P)$ is ρ-dense in E and S is ρ-equicontinuous in a neighbourhood of $t=0$, then $S(t)\, x \to x$ in ρ as $t \to 0^+$ for all $x \in E$.

PROPOSITION 4.5. *If $S(t)\, x \to x$ in ρ as $t \to 0^+$ for every $x \varepsilon E$, then the E-generator, G_S of S exists and $D(G_S)$ is ρ-dense in E.*

The E-generator of S can be compared with the "differential" generator of S by an argument along the lines of the proof of Proposition 3.3. To treat semigroups of unbounded operators on a Banach space, it is essential *not* to assume that each of the operators $S(t)$, $t>0$ is ρ-continuous. That condition can be avoided in the following manner.

Define $D_S^\rho x$ to be the limit $\lim_{t \to 0^+} (S(t)\, x - x)/t$ for all $x \in E$ for which the limit exists in ρ, and for which the function $S(\cdot)\, x$ is ρ-differentiable in E; that is, $\lim_{h \to 0} (S(t+h)\, x - S(t)\, x)/h$ exists in the topology ρ for all $t>0$.

PROPOSITION 4.6. *Suppose that for each $x \in E$, $S(t)\, x \to x$ in ρ as $t \to 0^+$ and $S(\cdot)\, x$ is ρ-continuous.*

Then the E-generator G_S of S exists and $G_S = D_S^\rho$.

Proof. We know that G_S exists by Proposition 4.5. First, consider the case of $\rho = \sigma(E, F)$.

By Proposition 4.3, $\mathscr{D}(G_S) \subset \mathscr{D}(D_S^\rho)$, and G_S and D_S^ρ agree on $\mathscr{D}(G_S)$. It remains to prove that $\mathscr{D}(D_S^\rho) \subset \mathscr{D}(G_S)$.

According to the remark after Lemma 4.2, there exists an S'-invariant subspace H of F separating points in E. Take $x \in \mathscr{D}(D_S^\rho)$ and $\zeta \in H$. Then for each $t>0$

$$\begin{aligned}\langle (S(t+h)\, x - S(t)\, x)/h, \zeta \rangle &= \langle S(t)(S(h) - I)\, x/h, \zeta \rangle \\ &= \langle (S(h) - I)\, x/h, S(t) \rangle \\ &\to \langle D_S x, S(t)'\zeta \rangle = \langle S(t)\, D_S x, \zeta \rangle\end{aligned}$$

as $h \to 0^+$.

By the definition of D_S^ρ, $S(\cdot)x$ is even ρ-differentiable in E, so its derivative at the point $t>0$ is precisely $S(t)D_Sx$, because H separates points in E.

An appeal to the fundamental theorem of calculus gives

$$\langle S(t)x, \xi\rangle = \langle x, \xi\rangle + \int_0^t \langle S(s)D_Sx, \xi\rangle\, ds$$

for every $x\in\mathscr{D}(D_S)$, $\xi\in F$ and $t>0$. Finally, Proposition 4.3 shows that $\mathscr{D}(D_S)\subset\mathscr{D}(G_S)$.

The case for general ρ follows from estimates of the form (4).

It is not hard to see that the *projective limit* of F-semigroups can also be defined in such a way that the preceding properties are still valid, but we have no need of this fact at present.

5. Semigroups of Unbounded Operators on a Banach Space

The approach of R. J. hughes [5] to semigroups of unbounded linear operators on a Banach space will be adapted to the purpose of illustrating the properties of weakly integrable semigroups and their inductive limits. To this end, Hughes' method is particularly suitable because in [5] the "infinitesimal generator" is defined directly in terms of resolvents.

Let $(X, \|\cdot\|)$ be a Banach space. For each $t>0$, suppose that the linear operator $T(t)$: $\mathscr{D}(T(t))\to X$ on X is given. Denote by $\mathscr{D}$ the collection of all $x\in\bigcap_{s,t>0}\mathscr{D}(T(s)T(t))$ for which: $T(s)T(t)x=T(s+t)x$ for all $s,t>0$, the X-valued function $T(\cdot)x$ is strongly continuous on $(0,\infty)$, and $\|T(t)x-x\|\to 0$ as $t\to 0^+$.

If $\mathscr{D}\neq\{0\}$, then $\{T(t): t>0\}$ is called a *semigroup of unbounded operators* on X.

For $n=0, 1,\ldots$, $x\in\bigcap_{t>0}\mathscr{D}(T(t))$, let

$$N_n(x)=\sup\{e^{-nt}\|T(t)x\|: t>0\},$$

set $E_n=\{x\in\mathscr{D}: N_n(x)<\infty\}$ and equip E_n with the topology defined by the norm N_n.

It is readily shown (see [5] for the details) that $E_n\subset E_{n+1}$, $T(t)E_n\subset E_n$ for $t>0$ and $n=0, 1,\ldots$, and that the inductive limit $E=\varinjlim E_n$ can be formed. The locally convex space E is continuously included in X.

Put $T_E(t)x=T(t)x$ for every $x\in E$ and $t>0$. Now assume that E is dense in X. We want to show that T_E is the inductive limit of weakly integrable semigroups. Some conditions additional to those of [5] are needed.

For each $t>0$, the adjoint of $T(t)$ for the topology of X is denoted by $T(t)^*$. Because $E\subset\mathscr{D}(T(t))$ is assumed to be dense in X, $T(t)^*$ is well defined.

Let $F=X'$, $E_n^0=\{\xi\in F\colon \langle x,\xi\rangle=0 \text{ for all } x\in E_n\}$, and $F_n=F/E_n^0$ for each $n=0,1,\ldots$.

The following result is to be compared with Proposition 3.5; the basic difference is that the number $\omega\geqslant 0$ there now depends on $x\in E$.

PROPOSITION 5.1. *Let $\{T(t)\colon t>0\}$ be a semigroup of closed unbounded operators on X. Suppose further that the space E is dense in X and $\bigcap_{t>0}\mathscr{D}(T(t)^*)$ separates points in E.*

Then T_E is the inductive limit of F_n-semigroups on E_n, $n=0,1,\ldots$. Moreover, the E-generator G_E of T_E exists and $D(G_E)$ is dense in X.

Proof. By virtue of [5, Theorem 2.8], for each $n=0,1,\ldots$, and $x\in E$, the Bochner integral $R_n(\lambda)x=\int_0^\infty e^{-\lambda t}T(t)\,x\,dt$, $\lambda>n$ converges in X and $R_n(\lambda)E_n\subset E_n$. Condition (S2) is therefore satisfied by the semigroup $T_n=T(\cdot)|E_n$ on E_n for each $n=0,1,\ldots$.

That condition (S1) is satisfied by T_n follows from the assumption that the set $\bigcap_{t>0}\mathscr{D}(T(t)^*)$ separates points in E; obviously it must also separate points in each space E_n, $n=0,1,\ldots$. An appeal to Lemma 4.2 and Proposition 4.5 completes the proof.

Under the above assumptions, many of the results of [5] now follow from the general approach presented here. In particular, we note that Proposition 4.6 provides the equality between the E-generator of T_E (which corresponds to the "infinitesimal operator" of [5] Definition 2.10) and various "infinitesimal operators" defined by differentiation. The exact statements may be gleaned from [5].

It is readily seen that the same procedure works for any sequentially complete locally convex space X, provided the norm $\|\cdot\|$ is replaced by a collection of seminorms defining the topology of X.

Fractional powers of certain closed operators offer a wide class of examples of inductive limits of weakly integrable semigroups.

Let A be a closed linear operator acting in a Banach space X, and suppose that the resolvent of A satisfies

$$\|\lambda R(\lambda;A)\|\leqslant M,\qquad \lambda>0.$$

For $n=1,2,\ldots$, and $\alpha\in\mathbb{C}$ with $n-1<\operatorname{Re}\alpha<n$, define the linear operator J^α on $\mathscr{D}(A^n)$, the domain of the operator A^n, by

$$J^\alpha x=-\frac{\sin\alpha\pi}{\pi}\int_0^\infty \lambda^{\alpha-n}R(\lambda;A)\,A^n x\,d\lambda,\qquad x\in\mathscr{D}(A^n).$$

The principal value of λ^α is taken so that $\lambda^\alpha>0$ for $\alpha>0$.

Set $\mathscr{D}(A^{\infty}) = \bigcap_{n=1}^{\infty} \mathscr{D}(A^n)$. For $\alpha = 1, 2, ..., (-A)^{\alpha}$ denotes the αth power of the operator $(-A)$ defined in the usual sense. For $\alpha \in \mathbb{C}$ with $n-1 < \operatorname{Re} \alpha < n = 1, 2, ...$, let $(-A)^{\alpha}$ denote the closure of the operator J^{α}. Then for $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re} \alpha$, $\operatorname{Re} \beta > 0$ and $x \in \mathscr{D}(A^{\infty})$,

$$(-A)^{\alpha}(-A)^{\beta}x = (-A)^{\alpha+\beta}x$$

(see [5] for the references).

Now suppose that A is the infinitesimal generator (X-generator) of a C_0-semigroup on X. Then $\mathscr{D}(A^{\infty})$ is dense in X and $\bigcap_{n=1}^{\infty} \mathscr{D}((A^n)^*)$ is weak $*$-dense in X'; that is, it separates points in X. Replacing the space X by the closure of E (defined previously) in X if necessary, the semigroup $\{(-A)^{\alpha} : \alpha > 0\}$ satisfies the conditions of Proposition 5.1, so $(-A)_E$ is the inductive limit of weakly integrable semigroups.

Another class of examples of the inductive limit of weakly integrable semigroups are the spectral local semigroups associated with Klein–Landau systems of unbounded linear operators on a Banach space X (briefly, KL-systems); for the terminology and references, see [7].

Given a KL-system $\{T(t) : t > 0\}$ for which the associated local semigroup T_E defined previously is a *spectral* local semigroup, then there exists a scalar-type operator G with

$$T_E(t) = e^{-tG} | E, \qquad t > 0.$$

Here e^{-tG} is defined by means of the operational calculus for scalar-type spectral operators. The adjoint G^* of G is also a scalar-type spectral operator for the weak$*$-topology. It follows that $T_E(t)^* = e^{-tG^*}$, $t > 0$ and $\bigcap_{t>0} \mathscr{D}(T_E(t)^*)$ separates points in X.

According to Propostion 5.1, the semigroup T_E on E is therefore the inductive limit of F_n-semigroups on E_n, $n = 1, 2, ...$, with $F = X'$. It turns out that the E-generator G_E of T_E agrees with G on $D(G_E)$.

Conditions for which a KL-system on an arbitrary Banach space defines a spectral local semigroup, and so an inductive limit of weakly integrable semigroups are given by W. Ricker [11].

Finally, we remark that in the example of fractional derivatives $(D^{\alpha})_{\alpha>0}$ discussed by Hughes [5], $\bigcap_{\alpha>0} \mathscr{D}((D^{\alpha})^*) = \{0\}$; the condition imposed in Proposition 5.1 that $\bigcap_{t>0} \mathscr{D}(T(t)^*)$ should separate points in E is therefore a severe restriction, and the treatment given in [5] is more general than ours, at least for applications to semigroups of unbounded operators on a Banach space.

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