Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 257 (2009) 659–682
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

On the sum of superoptimal singular values

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Received 23 October 2008; accepted 3 April 2009
Available online 29 April 2009
Communicated by N. Kalton

Abstract

In this paper, we study the following extremal problem and its relevance to the sum of the so-called superoptimal singular values of a matrix function: Given an $m \times n$ matrix function Φ, when is there a matrix function Ψ_* in the set $\mathcal{A}_k^{n,m}$ such that

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi_*(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = \sup_{\Psi \in \mathcal{A}_k^{n,m}} \left| \int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) \right| ?$$

The set $\mathcal{A}_k^{n,m}$ is defined by

$$\mathcal{A}_k^{n,m} \stackrel{\text{def}}{=} \bigl\{\Psi \in H_0^1(\mathbb{M}_{n,m})\colon \|\Psi\|_{L^1(\mathbb{M}_{n,m})} \leqslant 1,\ \operatorname{rank}\Psi(\zeta) \leqslant k \text{ a.e. } \zeta \in \mathbb{T}\bigr\}.$$

To address this extremal problem, we introduce Hankel-type operators on spaces of matrix functions and prove that this problem has a solution if and only if the corresponding Hankel-type operator has a maximizing vector. The main result of this paper is a characterization of the smallest number k for which

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta)$$

equals the sum of all the superoptimal singular values of an admissible matrix function Φ (e.g. a continuous matrix function) for some function $\Psi \in \mathcal{A}_k^{n,m}$. Moreover, we provide a representation of any such function Ψ when Φ is an admissible very badly approximable unitary-valued $n \times n$ matrix function.

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.04.002

Keywords: Best and superoptimal approximation; Badly and very badly approximable matrix functions; Hankel and Toeplitz operators

1. Introduction

The problem of best analytic approximation for a given $m \times n$ matrix-valued bounded function Φ on the unit circle $\mathbb{T}$ is to find a bounded analytic function Q such that

$$\|\Phi - Q\|_{L^\infty(\mathbb{M}_{m,n})} = \inf\bigl\{\|\Phi - F\|_{L^\infty(\mathbb{M}_{m,n})}\colon\ F \in H^\infty(\mathbb{M}_{m,n})\bigr\}.$$

Throughout,

$$\|\Psi\|_{L^\infty(\mathbb{M}_{m,n})} \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}}\bigl\|\Psi(\zeta)\bigr\|_{\mathbb{M}_{m,n}},$$

$\mathbb{M}_{m,n}$ denotes the space of $m \times n$ matrices equipped with the operator norm $\|\cdot\|_{\mathbb{M}_{m,n}}$ (of the space of linear operators from $\mathbb{C}^n$ to $\mathbb{C}^m$), and $H^\infty(\mathbb{M}_{m,n})$ denotes the space of bounded analytic $m \times n$ matrix-valued functions on $\mathbb{T}$.

It is well known that, unlike scalar-valued functions, a polynomial matrix function Φ may have many best analytic approximants. Therefore it is natural to impose additional conditions in order to distinguish a "very best" analytic approximant among all best analytic approximants. To do so here, we use the notion of superoptimal approximation by bounded analytic matrix functions.

1.1. Superoptimal approximation and very badly approximable matrix functions

Recall that for an $m \times n$ matrix A, the jth-*singular value* $s_j(A)$, $j \geqslant 0$, is defined to be the distance from A to the set of matrices of rank at most j under the operator norm. More precisely,

$$s_j(A) = \inf\bigl\{\|A - B\|_{\mathbb{M}_{m,n}}\colon\ B \in \mathbb{M}_{m,n} \text{ such that } \operatorname{rank} B \leqslant j\bigr\}.$$

Clearly, $s_0(A) = \|A\|_{\mathbb{M}_{m,n}}$.

Definition 1.1. Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$. For $k \geqslant 0$, we define the sets $\Omega_k = \Omega_k(\Phi)$ by

$$\Omega_0(\Phi) = \Bigl\{F \in H^\infty(\mathbb{M}_{m,n})\colon\ F \text{ minimizes } \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}}\bigl\|\Phi(\zeta) - F(\zeta)\bigr\|_{\mathbb{M}_{m,n}}\Bigr\} \quad \text{and}$$

$$\Omega_j(\Phi) = \Bigl\{F \in \Omega_{j-1}\colon\ F \text{ minimizes } \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_j\bigl(\Phi(\zeta) - F(\zeta)\bigr)\Bigr\} \quad \text{for } j > 0.$$

Any function $F \in \bigcap_{k\geqslant 0}\Omega_k = \Omega_{\min\{m,n\}-1}$ is called a *superoptimal approximation to Φ* by bounded analytic matrix functions. In this case, the *superoptimal singular values of Φ* are defined by

$$t_j = t_j(\Phi) = \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} s_j\bigl((\Phi - F)(\zeta)\bigr) \quad \text{for } j \geqslant 0.$$

Moreover, if the zero matrix function $\mathbb{O}$ belongs to $\Omega_{\min\{m,n\}-1}$, we say that Φ *is very badly approximable*.

Notice that any function $F \in \Omega_0$ is a best analytic approximation to Φ. Also, any very badly approximable matrix function is the difference between a bounded matrix function and its superoptimal approximant.

It turns out that Hankel operators on Hardy spaces play an important role in the study of superoptimal approximation. For a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$, we define the *Hankel operator H_Φ* by

$$H_\Phi f = \mathbb{P}_-\Phi f \quad \text{for } f \in H^2(\mathbb{C}^n),$$

where $\mathbb{P}_-$ denotes the orthogonal projection from $L^2(\mathbb{C}^m)$ onto $H^2_-(\mathbb{C}^m) \stackrel{\text{def}}{=} L^2(\mathbb{C}^m) \ominus H^2(\mathbb{C}^m)$.

When studying superoptimal approximation, we only consider bounded matrix functions that are admissible. A matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$ is said to be *admissible* if the essential norm $\|H_\Phi\|_{\mathrm{e}}$ of the Hankel operator H_Φ is strictly less than the smallest non-zero superoptimal singular value of Φ. As usual, the essential norm of a bounded linear operator T between Hilbert spaces is defined by

$$\|T\|_{\mathrm{e}} \stackrel{\text{def}}{=} \bigl\{\|T - K\|\colon\ K \text{ is compact}\bigr\}.$$

Note that any continuous matrix function Φ is admissible, as the essential norm of H_Φ equals zero in this case. Moreover, in the case of scalar-valued functions, to say that a function φ is admissible simply means that $\|H_\varphi\|_{\mathrm{e}} < \|H_\varphi\|$.

It is known that if Φ is an admissible matrix function, then Φ has a unique superoptimal approximation Q by bounded analytic matrix functions. Moreover, the functions $\zeta \mapsto s_j((\Phi - Q)(\zeta))$ equal $t_j(\Phi)$ a.e. on $\mathbb{T}$ for each $j \geqslant 0$. These results were first proved in [6] for the special case $\Phi \in (H^\infty + C)(\mathbb{M}_{m,n})$ (i.e. matrix functions which are a sum of a bounded analytic matrix function and a continuous matrix function), and shortly after proved for the class of admissible matrix functions in [5].

While it is possible to compute the superoptimal singular values of a given matrix function in concrete examples, it is not known how to verify if a matrix function that is not continuous is admissible or not. Thus a complete characterization of the smallest non-zero superoptimal singular value of a given matrix function is an important problem for superoptimal approximation. This remains an open problem.

We refer the reader to Chapter 14 of [2] which contains proofs to all of the previously mentioned results and many other interesting results concerning superoptimal approximation.

1.2. An extremal problem

Throughout this note, $\boldsymbol{m}$ denotes normalized Lebesgue measure on $\mathbb{T}$ so that $\boldsymbol{m}(\mathbb{T}) = 1$. For $1 \leqslant p \leqslant \infty$, $L^p(\mathbb{M}_{m,n})$ denotes the space of $m \times n$ matrix-valued functions on $\mathbb{T}$ whose entries belong to L^p. We equip $L^p(\mathbb{M}_{m,n})$ with the norm $\|\cdot\|_{L^p(\mathbb{M}_{m,n})}$, where

$$\|F\|_{L^p(\mathbb{M}_{m,n})}^p = \int_{\mathbb{T}} \|F(\zeta)\|_{\mathbb{M}_{m,n}}^p \, d\boldsymbol{m}(\zeta) \quad \text{for } 1 \leqslant p < \infty, \quad \text{and}$$

$$\|F\|_{L^\infty(\mathbb{M}_{m,n})} = \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} \|F(\zeta)\|_{\mathbb{M}_{m,n}}.$$

$H^p(\mathbb{M}_{m,n})$ and $H_0^p(\mathbb{M}_{m,n})$ consist of matrix-valued functions in $L^p(\mathbb{M}_{m,n})$ whose entries belong to the Hardy space H^p and H_0^p, respectively. (Recall that H^p and H_0^p denote the spaces of L^p functions on $\mathbb{T}$ whose Fourier coefficients of negative and non-positive index vanish, respectively.)

Definition 1.2. Let $m, n > 1$ and $1 \leqslant k \leqslant \min\{m, n\}$. For $\Phi \in L^\infty(\mathbb{M}_{m,n})$, we define $\sigma_k(\Phi)$ by

$$\sigma_k(\Phi) \stackrel{\text{def}}{=} \sup_{\Psi \in \mathcal{A}_k^{n,m}} \left| \int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr) \, d\boldsymbol{m}(\zeta) \right|, \tag{1.1}$$

where

$$\mathcal{A}_k^{n,m} = \bigl\{\Psi \in H_0^1(\mathbb{M}_{n,m}) \colon \|\Psi\|_{L^1(\mathbb{M}_{n,m})} \leqslant 1 \text{ and } \operatorname{rank} \Psi(\zeta) \leqslant k \text{ a.e. } \zeta \in \mathbb{T}\bigr\}.$$

Whenever $n = m$, we use the notation $\mathcal{A}_k^n \stackrel{\text{def}}{=} \mathcal{A}_k^{n,m}$.

We are interested in the following extremal problem:

Extremal problem 1.1. For a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$, when is there a matrix function $\Psi \in \mathcal{A}_k^{n,m}$ such that

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr) \, d\boldsymbol{m}(\zeta) = \sigma_k(\Phi)?$$

The importance of this problem arose from the following observation due to Peller [3].

Theorem 1.3. *Let $1 \leqslant k \leqslant \min\{m, n\}$. If $\Phi \in L^\infty(\mathbb{M}_{m,n})$ is admissible, then*

$$\sigma_k(\Phi) \leqslant t_0(\Phi) + \cdots + t_{k-1}(\Phi). \tag{1.2}$$

Proof. Let $\Psi \in \mathcal{A}_k^{n,m}$. We may assume, without loss of generality, that Φ is very badly approximable. Indeed,

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr) \, d\boldsymbol{m}(\zeta) = \int_{\mathbb{T}} \operatorname{trace}\bigl((\Phi - Q)(\zeta)\Psi(\zeta)\bigr) \, d\boldsymbol{m}(\zeta)$$

holds for any $Q \in H^\infty(\mathbb{M}_{m,n})$, and so we may replace Φ with $\Phi - Q$ if necessary, where Q is the superoptimal approximation to Φ in $H^\infty(\mathbb{M}_{m,n})$.

Let $\boldsymbol{S}_1^m$ denote the collection of $m \times m$ matrices equipped with the *trace norm* $\|A\|_{\boldsymbol{S}_1^m} = \operatorname{trace}(A^*A)^{1/2} = \sum_{j \geqslant 0} s_j(A)$.

It follows from the well-known inequality $|\operatorname{trace}(A)| \leqslant \|A\|_{\boldsymbol{S}_1^m}$ that the inequalities

$$\left|\operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\right| \leqslant \bigl\|\Phi(\zeta)\Psi(\zeta)\bigr\|_{\boldsymbol{S}_1^m} \leqslant \left(\sum_{j=0}^{k-1} s_j\bigl(\Phi(\zeta)\bigr)\right)\bigl\|\Psi(\zeta)\bigr\|_{\mathbb{M}_{n,m}}$$

hold for a.e. $\zeta \in \mathbb{T}$. Thus,

$$\begin{aligned}
\left|\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta)\right| &\leqslant \int_{\mathbb{T}} \left(\sum_{j=0}^{k-1} s_j\bigl(\Phi(\zeta)\bigr)\right)\bigl\|\Psi(\zeta)\bigr\|_{\mathbb{M}_{n,m}}\, d\boldsymbol{m}(\zeta) \\
&\leqslant \int_{\mathbb{T}} \left(\sum_{j=0}^{k-1} t_j(\Phi)\right)\bigl\|\Psi(\zeta)\bigr\|_{\mathbb{M}_{n,m}}\, d\boldsymbol{m}(\zeta) \\
&\leqslant \left(\sum_{j=0}^{k-1} t_j(\Phi)\right)\|\Psi\|_{L^1(\mathbb{M}_{n,m})} \\
&\leqslant \sum_{j=0}^{k-1} t_j(\Phi), \qquad (1.3)
\end{aligned}$$

because the singular values of Φ satisfy $s_j(\Phi(\zeta)) = t_j(\Phi)$ for a.e. $\zeta \in \mathbb{T}$ since Φ is very badly approximable. $\square$

Before proceeding, let us observe that equality holds in (1.2) for some simple cases. Let r be a positive integer and $t_0, t_1, \ldots, t_{r-1}$ be positive numbers satisfying

$$t_0 \geqslant t_1 \geqslant \cdots \geqslant t_{r-1}.$$

Suppose Φ is an $n \times n$ matrix function of the form

$$\Phi \stackrel{\text{def}}{=} \begin{pmatrix} u_0 & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1 u_1 & \ldots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & t_{r-1}u_{r-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} & \Phi_{\#} \end{pmatrix}, \qquad (1.4)$$

where $\|\Phi_{\#}\|_{L^\infty} \leqslant t_{r-1}$ and u_j is a unimodular function of the form $u_j = \bar{z}\bar{\theta}_j \bar{h}/h$ with θ_j an inner function for $0 \leqslant j \leqslant r-1$ and h an outer function in H^2. Without loss of generality, we may assume that $\|h\|_{L^2} = 1$. It can be seen that if

$$\Psi \stackrel{\text{def}}{=} \begin{pmatrix} z\theta_0 h^2 & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & z\theta_1 h^2 & \ldots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \ldots & z\theta_{r-1}h^2 & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \ldots & \mathbb{O} & \mathbb{O} \end{pmatrix}, \qquad (1.5)$$

then $\Psi \in H_0^1(\mathbb{M}_n)$, $\operatorname{rank}\Psi(\zeta) = r$ a.e. on $\mathbb{T}$, $\|\Psi\|_{L^1(\mathbb{M}_n)} = 1$, and

$$\int_{\mathbb{T}} \operatorname{trace}\big(\Phi(\zeta)\Psi(\zeta)\big)\, d\boldsymbol{m}(\zeta) = t_0 + \cdots + t_{r-1}.$$

Thus we obtain that

$$\sigma_r(\Phi) = t_0(\Phi) + \cdots + t_{r-1}(\Phi).$$

On the other hand, one cannot expect the inequality (1.2) to hold with equality in general. After all, by the Hahn–Banach theorem,

$$\operatorname{dist}_{L^\infty(\boldsymbol{S}_1^n)}\big(\Phi, H^\infty(\mathbb{M}_n)\big) = \sigma_n(\Phi), \tag{1.6}$$

and there are admissible very badly approximable 2×2 matrix functions Φ for which the strict inequality

$$\operatorname{dist}_{L^\infty(\boldsymbol{S}_1^2)}\big(\Phi, H^\infty(\mathbb{M}_2)\big) < t_0(\Phi) + t_1(\Phi)$$

holds. For instance, consider the matrix function

$$\Phi = \begin{pmatrix} \bar{z} & \mathbb{O} \\ \mathbb{O} & \bar{z} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \bar{z} \\ -z & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{z} & \bar{z}^2 \\ -1 & \bar{z} \end{pmatrix}.$$

Clearly, Φ has superoptimal singular values $t_0(\Phi) = t_1(\Phi) = 1$. Let

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ -1 & \mathbb{O} \end{pmatrix}.$$

It is not difficult to verify that

$$s_0\big((\Phi - F)(\zeta)\big) = \frac{1}{2}\sqrt{3+\sqrt{5}} \quad \text{and} \quad s_1\big((\Phi - F)(\zeta)\big) = \frac{1}{2}\sqrt{3-\sqrt{5}}$$

for all $\zeta \in \mathbb{T}$. Therefore

$$\operatorname{dist}_{L^\infty(\boldsymbol{S}_1^2)}\big(\Phi, H^\infty(\mathbb{M}_2)\big) \leqslant \|\Phi - F\|_{L^\infty(\boldsymbol{S}_1^2)} < 2 = t_0(\Phi) + t_1(\Phi). \tag{1.7}$$

1.3. What is done in this paper?

In virtue of Theorem 1.3 and the remarks following it, one may ask whether it is possible to characterize the matrix functions Φ for which (1.2) becomes an equality. So let Φ be an admissible $n \times n$ matrix function with a superoptimal approximant Q in $H^\infty(\mathbb{M}_n)$ for which equality in Theorem 1.3 holds with $k = n$. In this case, it must be that

$$\operatorname{dist}_{L^\infty(\boldsymbol{S}_1^n)}\big(\Phi, H^\infty(\mathbb{M}_n)\big) = \sum_{j=0}^{n-1} t_j(\Phi) = \sum_{j=0}^{n-1} s_j\big((\Phi - Q)(\zeta)\big) = \|\Phi - Q\|_{L^\infty(\boldsymbol{S}_1^n)}$$

by (1.6) and thus the superoptimal approximant Q must be a best approximant to Φ under the $L^\infty(\boldsymbol{S}_1^n)$ norm as well. Hence, we are led to investigate the following problems:

1. For which matrix functions Φ does Extremal problem 1.1 have a solution?
2. If $Q_\$$ is a best approximant to Φ under the $L^\infty(\boldsymbol{S}_1^n)$-norm, when does it follow that $Q_\$$ is the superoptimal approximant to Φ in $L^\infty(\mathbb{M}_n)$?
3. Can we find necessary and sufficient conditions on Φ to obtain equality in (1.2) of Theorem 1.3?

Before addressing these problems, we recall certain standard principles of functional analysis in Section 2 that are used throughout the paper. In particular, we give their explicit formulation for the spaces $L^p(\boldsymbol{S}_q^{m,n})$.

In Section 3, we introduce the Hankel-type operators $H_\Phi^{\{k\}}$ on spaces of matrix functions and k-extremal functions, and prove that the number $\sigma_k(\Phi)$ equals the operator norm of $H_\Phi^{\{k\}}$. We also show that Extremal problem 1.1 has a solution if and only if the Hankel-type operator $H_\Phi^{\{k\}}$ has a maximizing vector, and thus answer question 1 in terms Hankel-type operators.

In Section 4, we establish the main results of this paper concerning best approximation under the $L^\infty(\boldsymbol{S}_1^{m,n})$ norm (Theorem 4.7) and the sum of superoptimal singular values (Theorem 4.13). The latter result characterizes the smallest number k for which

$$\int_{\mathbb{T}} \operatorname{trace}\big(\Phi(\zeta)\Psi(\zeta)\big)\,d\boldsymbol{m}(\zeta)$$

equals the sum of all non-zero superoptimal singular values for some function $\Psi \in \mathcal{A}_k^{n,m}$. These results serve as partial solutions to problems 2 and 3.

Lastly, in Section 5, we restrict our attention to unitary-valued very badly approximable matrix functions. For any such matrix function U, we provide a representation of any function Ψ for which the formula

$$\int_{\mathbb{T}} \operatorname{trace}\big(U(\zeta)\Psi(\zeta)\big)\,d\boldsymbol{m}(\zeta) = n$$

holds.

2. Best approximation and dual extremal problems

We now provide explicit formulation of some basic results concerning best approximation in $H^q(\boldsymbol{S}_p^{m,n})$ for functions in $L^q(\boldsymbol{S}_p^{m,n})$ and the corresponding dual extremal problem. We first consider the general setting.

2.1. Best approximation

Definition 2.1. Let X be a normed space, M be a closed subspace of X, and $x_0 \in X$. We say that *m_0 is a best approximant to x_0 in M* if $m_0 \in M$ and

$$\|x_0 - m_0\|_X = \operatorname{dist}(x_0, M) \overset{\text{def}}{=} \inf\big\{\|x_0 - m\|_X\colon\ m \in M\big\}.$$

It is known that *if X is a reflexive Banach space and M is a closed subspace of X, then each $x_0 \in X \setminus M$ has a best approximant m_0 in M.*

Two standard principles from functional analysis are used throughout this note. Namely, if X is a normed space with a linear subspace M, then for any $\Lambda_0 \in X^*$ and $x_0 \in X$

$$\sup_{m\in M,\ \|m\|\leqslant 1} \bigl|\Lambda_0(m)\bigr| = \min\bigl\{\|\Lambda_0 - \Lambda\|\colon\ \Lambda \in M^{\perp}\bigr\} \quad \text{and}$$

$$\max_{\Lambda\in M^{\perp},\ \|\Lambda\|\leqslant 1} \bigl|\Lambda(x_0)\bigr| = \operatorname{dist}(x_0, M) \quad \text{whenever } M \text{ is closed.}$$

We now discuss these results in the case of the spaces $L^q(\boldsymbol{S}_p^{m,n})$.

2.2. *The spaces $L^q(\boldsymbol{S}_p^{m,n})$*

Let $1 \leqslant q < \infty$ and $1 \leqslant p \leqslant \infty$. *Let p' denote the conjugate exponent to p*, i.e. $p' = p/(p-1)$.

Let $\boldsymbol{S}_p^{m,n}$ denote the space of $m \times n$ matrices equipped with the Schatten–von Neumann norm $\|\cdot\|_{\boldsymbol{S}_p^{m,n}}$, i.e. for $A \in \mathbb{M}_{m,n}$

$$\|A\|_{\boldsymbol{S}_\infty^{m,n}} \stackrel{\text{def}}{=} \|A\|_{\mathbb{M}_{m,n}} \quad \text{and} \quad \|A\|_{\boldsymbol{S}_p^{m,n}} \stackrel{\text{def}}{=} \biggl(\sum_{j\geqslant 0} s_j^p(A)\biggr)^{1/p} \quad \text{for } 1 \leqslant p < \infty.$$

We also use the notation $\boldsymbol{S}_p^n \stackrel{\text{def}}{=} \boldsymbol{S}_p^{n,n}$.

If X is a normed space of functions on $\mathbb{T}$ with norm $\|\cdot\|_X$, then $X(\boldsymbol{S}_p^{m,n})$ denotes the space of $m \times n$ matrix functions whose entries belong to X. For $\Phi \in X(\boldsymbol{S}_p^{m,n})$, we define

$$\|\Phi\|_{X(\boldsymbol{S}_p^{m,n})} \stackrel{\text{def}}{=} \|\rho\|_X, \quad \text{where } \rho(\zeta) \stackrel{\text{def}}{=} \bigl\|\Phi(\zeta)\bigr\|_{\boldsymbol{S}_p^{m,n}} \text{ for } \zeta \in \mathbb{T}.$$

It is known that the dual space of $L^q(\boldsymbol{S}_p^{m,n})$ is isometrically isomorphic to $L^{q'}(\boldsymbol{S}_{p'}^{n,m})$ via the mapping $\Phi \mapsto \Lambda_\Phi$, where $\Phi \in L^{q'}(\boldsymbol{S}_{p'}^{n,m})$ and

$$\Lambda_\Phi(\Psi) = \int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) \quad \text{for } \Psi \in L^q\bigl(\boldsymbol{S}_p^{m,n}\bigr).$$

In particular, it follows that the annihilator of $H^q(\boldsymbol{S}_p^{m,n})$ in $L^q(\boldsymbol{S}_p^{m,n})$ is given by $H_0^{q'}(\boldsymbol{S}_{p'}^{n,m})$, and so

$$\operatorname{dist}_{L^q(\boldsymbol{S}_p^{m,n})}\bigl(\Phi, H^q\bigl(\boldsymbol{S}_p^{m,n}\bigr)\bigr) = \max_{\|\Psi\|_{H_0^{q'}(\boldsymbol{S}_{p'}^{n,m})}\leqslant 1} \biggl|\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta)\biggr|,$$

by our remarks in Section 2.1. Moreover, if $1 < q < \infty$, then $\Phi \in L^q(\boldsymbol{S}_p^{m,n})$ has a best approximant Q in $H^q(\boldsymbol{S}_p^{m,n})$ (as $L^q(\boldsymbol{S}_p^{m,n})$ is reflexive); that is,

$$\|\Phi - Q\|_{L^q(\boldsymbol{S}_p^{m,n})} = \operatorname{dist}_{L^q(\boldsymbol{S}_p^{m,n})}\bigl(\Phi, H^q\bigl(\boldsymbol{S}_p^{m,n}\bigr)\bigr).$$

The situation is similar in the case of $L^\infty(\boldsymbol{S}_p^{m,n})$. Indeed, $L^\infty(\boldsymbol{S}_p^{m,n})$ is a dual space, and so there is a $Q \in H^\infty(\boldsymbol{S}_p^{m,n})$ such that

$$\|\Phi - Q\|_{L^\infty(\boldsymbol{S}_p^{m,n})} = \operatorname{dist}_{L^\infty(\boldsymbol{S}_p^{m,n})}\bigl(\Phi, H^\infty\bigl(\boldsymbol{S}_p^{m,n}\bigr)\bigr).$$

Again, it also follows from our remarks in Section 2.1 that

$$\operatorname{dist}_{L^\infty(\boldsymbol{S}_p^{m,n})}\bigl(\Phi, H^\infty\bigl(\boldsymbol{S}_p^{m,n}\bigr)\bigr) = \sup_{\|\Psi\|_{H_0^1(S_{p'}^{n,m})} \leqslant 1} \left| \int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) \right|.$$

However, an extremal function may fail to exist in this case even if Φ is a scalar-valued function. An example can be deduced from Section 1 of Chapter 1 in [2].

3. $\sigma_k(\Phi)$ as the norm of a Hankel-type operator and k-extremal functions

We now introduce the Hankel-type operators $H_\Phi^{\{k\}}$ which act on spaces of matrix functions. We prove that the number $\sigma_k(\Phi)$ equals the operator norm of $H_\Phi^{\{k\}}$ and characterize when $H_\Phi^{\{k\}}$ has a maximizing vector. Recall that for an operator $T : X \to Y$ between normed spaces X and Y, a vector $x \in X$ is called *a maximizing vector of T* if x is non-zero and

$$\|Tx\|_Y = \|T\| \cdot \|x\|_X.$$

We begin by establishing the following lemma.

Lemma 3.1. *Let $1 \leqslant k \leqslant \min\{m, n\}$. If $\Psi \in H^1(\mathbb{M}_{n,m})$ is such that* $\operatorname{rank} \Psi(\zeta) = k$ *for a.e. $\zeta \in \mathbb{T}$, then there are functions $R \in H^2(\mathbb{M}_{n,k})$ and $Q \in H^2(\mathbb{M}_{k,m})$ such that $R(\zeta)$ has rank equal to k for almost every $\zeta \in \mathbb{T}$,*

$$\Psi = RQ \quad \textit{and} \quad \bigl\|R(\zeta)\bigr\|_{\mathbb{M}_{n,k}}^2 = \bigl\|Q(\zeta)\bigr\|_{\mathbb{M}_{k,m}}^2 = \bigl\|\Psi(\zeta)\bigr\|_{\mathbb{M}_{n,m}} \quad \textit{for a.e. } \zeta \in \mathbb{T}.$$

Proof. Consider the set

$$\mathscr{A} = \operatorname{clos}_{L^1(\mathbb{C}^n)}\bigl\{f \in H^1\bigl(\mathbb{C}^n\bigr)\colon\ f(\zeta) \in \operatorname{Range} \Psi(\zeta) \text{ a.e. on } \mathbb{T}\bigr\}.$$

Since $\mathscr{A}$ is a non-trivial invariant subspace of $H^1(\mathbb{C}^n)$ under multiplication by z, there is an $n \times r$ inner function Θ such that $\mathscr{A} = \Theta H^1(\mathbb{C}^r)$. We first show that $r = k$. Let $\{e_j\}_{j=1}^r$ be an orthonormal basis for $\mathbb{C}^r$. Then for almost every $\zeta \in \mathbb{T}$, we have that $\{\Theta(\zeta)e_j\}_{j=1}^r$ is a linearly independent set, since Θ is inner. Moreover, $\{\Theta(\zeta)e_j\}_{j=1}^r$ is a basis for $\operatorname{Range}\Theta(\zeta) = \operatorname{Range}\Psi(\zeta)$ for a.e. $\zeta \in \mathbb{T}$. Since $\dim \operatorname{Range}\Psi(\zeta) = k$ a.e. on $\mathbb{T}$, it follows that $r = \dim \operatorname{Range}\Theta(\zeta) = \dim \operatorname{Range}\Psi(\zeta) = k$. In particular, we obtain that

$$\mathscr{A} = \Theta H^1\bigl(\mathbb{C}^k\bigr).$$

Therefore, $\Psi = \Theta F$ for some $k \times m$ matrix function $F \in H^1(\mathbb{M}_{k,m})$, because the columns of Ψ belong to $\mathscr{A}$.

Let h be an outer function in H^2 such that $|h(\zeta)| = \|\Psi(\zeta)\|^{1/2}_{\mathbb{M}_{n,m}}$ for a.e. $\zeta \in \mathbb{T}$. (The existence of h is a consequence of the fact that $\log \|\Psi(\zeta)\|_{\mathbb{M}_{n,m}} \in L^1$ as $\Psi \in H^1(\mathbb{M}_{n,m})$.) Thus, the matrix functions

$$R = h\Theta \quad \text{and} \quad Q = h^{-1}F$$

have the desired properties. □

Definition 3.2. Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$, $1 \leqslant k \leqslant \min\{m,n\}$, and $\rho : L^2(\boldsymbol{S}_1^{m,k}) \to L^2(\boldsymbol{S}_1^{m,k})/H^2(\boldsymbol{S}_1^{m,k})$ denote the natural quotient map. We define the *Hankel-type operator* $H_\Phi^{\{k\}} : H^2(\mathbb{M}_{n,k}) \to L^2(\boldsymbol{S}_1^{m,k})/H^2(\boldsymbol{S}_1^{m,k})$ by setting

$$H_\Phi^{\{k\}} F \stackrel{\text{def}}{=} \rho(\Phi F) \quad \text{for } F \in H^2(\mathbb{M}_{n,k}).$$

The norm in the quotient space $L^2(\boldsymbol{S}_1^{m,k})/H^2(\boldsymbol{S}_1^{m,k})$ is the natural one; that is, the norm of a coset equals the infimum of the $L^2(\boldsymbol{S}_1^{m,k})$-norms of its elements.

Theorem 3.3. *Let* $1 \leqslant k \leqslant \min\{m,n\}$. *If* $\Phi \in L^\infty(\mathbb{M}_{m,n})$, *then*

$$\sigma_k(\Phi) = \big\| H_\Phi^{\{k\}} \big\|_{H^2(\mathbb{M}_{n,k}) \to L^2(\boldsymbol{S}_1^{m,k})/H^2(\boldsymbol{S}_1^{m,k})}.$$

Proof. Consider the collection

$$\mathcal{B}_k^{n,m} = \big\{ RQ \colon \|R\|_{H^2(\mathbb{M}_{n,k})} \leqslant 1,\ \|Q\|_{H_0^2(\mathbb{M}_{k,m})} \leqslant 1 \big\}.$$

We claim that $\mathcal{B}_k^{n,m} = \mathcal{A}_k^{n,m}$. Indeed if $\Psi \in \mathcal{A}_k$ satisfies $\operatorname{rank} \Psi(\zeta) = j$ for $\zeta \in \mathbb{T}$, where $1 \leqslant j \leqslant k$, then by Lemma 3.1 there are functions $R \in H^2(\mathbb{M}_{n,j})$ and $Q \in H_0^2(\mathbb{M}_{j,m})$ such that $R(\zeta)$ has rank equal to j for almost every $\zeta \in \mathbb{T}$,

$$\Psi = RQ \quad \text{and} \quad \big\|R(\zeta)\big\|^2_{\mathbb{M}_{n,j}} = \big\|Q(\zeta)\big\|^2_{\mathbb{M}_{j,m}} = \big\|\Psi(\zeta)\big\|_{\mathbb{M}_{n,m}} \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

We may now add zeros, if necessary, to obtain $n \times k$ and $k \times m$ matrix functions

$$R_\# = \begin{pmatrix} R & \mathbb{O} \end{pmatrix} \quad \text{and} \quad Q_\# = \begin{pmatrix} Q \\ \mathbb{O} \end{pmatrix},$$

respectively, from which it follows that $\Psi = R_\# Q_\# \in \mathcal{B}_k^{n,m}$. Therefore $\mathcal{A}_k^{n,m} \subset \mathcal{B}_k^{n,m}$. The reverse inclusion is trivial and so these sets are equal.

Hence

$$\begin{aligned}
\sigma_k(\Phi) &= \sup_{\|R\|_{H^2(\mathbb{M}_{n,k})} \leqslant 1} \ \sup_{\|Q\|_{H_0^2(\mathbb{M}_{k,m})} \leqslant 1} \left| \int_{\mathbb{T}} \operatorname{trace}\big(\Phi(\zeta)R(\zeta)Q(\zeta)\big)\, d\boldsymbol{m}(\zeta) \right| \\
&= \sup_{\|R\|_{H^2(\mathbb{M}_{n,k})} \leqslant 1} \operatorname{dist}_{L^2(\boldsymbol{S}_1^{m,k})}\big(\Phi R, H^2(\mathbb{M}_{m,k})\big) \\
&= \big\| H_\Phi^{\{k\}} \big\|_{H^2(\mathbb{M}_{n,k}) \to L^2(\boldsymbol{S}_1^{m,k})/H^2(\boldsymbol{S}_1^{m,k})}. \qquad \square
\end{aligned}$$

Definition 3.4. Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$ and $1 \leqslant k \leqslant \min\{m,n\}$. We say that Ψ is *a k-extremal function for Φ* if $\Psi \in \mathcal{A}_k^{n,m}$ and

$$\sigma_k(\Phi) = \int_{\mathbb{T}} \operatorname{trace}\big(\Phi(\zeta)\Psi(\zeta)\big)\, d\boldsymbol{m}(\zeta).$$

Thus a matrix function Φ has a k-extremal function if and only if Extremal problem 1.1 has a positive solution.

We can now describe matrix functions that have a k-extremal function in terms of Hankel-type operators.

Theorem 3.5. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$. The matrix function Φ has a k-extremal function if and only if the Hankel-type operator $H_\Phi^{\{k\}}$ has a maximizing vector.*

Proof. To simplify notation, let

$$\big\|H_\Phi^{\{k\}}\big\| \stackrel{\text{def}}{=} \big\|H_\Phi^{\{k\}}\big\|_{H^2(\mathbb{M}_{n,k}) \to L^2(\boldsymbol{S}_1^{m,k})/H^2(\boldsymbol{S}_1^{m,k})}.$$

Suppose Ψ is a k-extremal function for Φ. Let $j \in \mathbb{N}$ be such that $j \leqslant k$ and

$$\operatorname{rank}\Psi(\zeta) = j \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

By Lemma 3.1, there is an $R \in H^2(\mathbb{M}_{n,j})$ and a $Q \in H_0^2(\mathbb{M}_{j,m})$ such that

$$\Psi = RQ \quad \text{and} \quad \big\|R(\zeta)\big\|_{\mathbb{M}_{n,j}}^2 = \big\|Q(\zeta)\big\|_{\mathbb{M}_{j,m}}^2 = \big\|\Psi(\zeta)\big\|_{\mathbb{M}_{n,m}} \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

As before, adding zeros if necessary, we obtain $n \times k$ and $k \times m$ matrix functions

$$R_\# = (\,R \quad \mathbb{O}\,) \quad \text{and} \quad Q_\# = \begin{pmatrix} Q \\ \mathbb{O} \end{pmatrix},$$

respectively, so that $\Psi = R_\# Q_\#$ and

$$\big\|Q_\#(\zeta)\big\|_{\mathbb{M}_{k,m}}^2 = \big\|Q(\zeta)\big\|_{\mathbb{M}_{j,m}}^2 = \big\|\Psi(\zeta)\big\|_{\mathbb{M}_{n,m}} \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

Let us show that $R_\#$ is a maximizing vector for $H_\Phi^{\{k\}}$. Since $Q_\#$ belongs to $H_0^2(\mathbb{M}_{k,m})$, we have that for any $F \in H^2(\boldsymbol{S}_1^{m,k})$

$$\begin{aligned}\sigma_k(\Phi) &= \int_{\mathbb{T}} \operatorname{trace}\big(\Phi(\zeta)\Psi(\zeta)\big)\, d\boldsymbol{m}(\zeta) = \int_{\mathbb{T}} \operatorname{trace}\big(\Phi(\zeta)R_\#(\zeta)Q_\#(\zeta)\big)\, d\boldsymbol{m}(\zeta) \\ &= \int_{\mathbb{T}} \operatorname{trace}\big((\Phi R_\# - F)(\zeta)Q_\#(\zeta)\big)\, d\boldsymbol{m}(\zeta),\end{aligned}$$

and so

$$\begin{aligned}
\sigma_k(\Phi) &= \left| \int_{\mathbb{T}} \operatorname{trace}\bigl((\Phi R_\# - F)(\zeta) Q_\#(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) \right| \\
&\leqslant \int_{\mathbb{T}} \bigl|\operatorname{trace}\bigl((\Phi R_\# - F)(\zeta) Q_\#(\zeta)\bigr)\bigr|\, d\boldsymbol{m}(\zeta) \\
&\leqslant \int_{\mathbb{T}} \bigl\|(\Phi R_\# - F)(\zeta) Q_\#(\zeta)\bigr\|_{\boldsymbol{S}_1^m}\, d\boldsymbol{m}(\zeta) \\
&\leqslant \int_{\mathbb{T}} \bigl\|(\Phi R_\# - F)(\zeta)\bigr\|_{\boldsymbol{S}_1^{m,k}} \bigl\|Q_\#(\zeta)\bigr\|_{\mathbb{M}_{k,m}}\, d\boldsymbol{m}(\zeta) \\
&\leqslant \|\Phi R_\# - F\|_{L^2(\boldsymbol{S}_1^{m,k})} \|Q_\#\|_{L^2(\mathbb{M}_{k,m})} \\
&= \|\Phi R_\# - F\|_{L^2(\boldsymbol{S}_1^{m,k})} \|\Psi\|_{L^1(\mathbb{M}_{n,m})} \\
&\leqslant \|\Phi R_\# - F\|_{L^2(\boldsymbol{S}_1^{m,k})}.
\end{aligned}$$

By Theorem 3.3, we obtain that

$$\sigma_k(\Phi) \leqslant \bigl\|H_\Phi^{\{k\}} R_\#\bigr\|_{L^2(\boldsymbol{S}_1^{m,k})/H^2(\boldsymbol{S}_1^{m,k})} \leqslant \bigl\|H_\Phi^{\{k\}}\bigr\| = \sigma_k(\Phi),$$

and therefore

$$\bigl\|H_\Phi^{\{k\}}\bigr\| = \bigl\|H_\Phi^{\{k\}} R_\#\bigr\|_{L^2(\boldsymbol{S}_1^{m,k})/H^2(\boldsymbol{S}_1^{m,k})}.$$

Thus, $R_\#$ is a maximizing vector of H_Φ.

Conversely, suppose the Hankel-type operator $H_\Phi^{\{k\}}$ has a maximizing vector $R \in H^2(\mathbb{M}_{n,k})$. Without loss of generality, we may assume that $\|R\|_{L^2(\mathbb{M}_{n,k})} = 1$. Then

$$\operatorname{dist}_{L^2(\boldsymbol{S}_1^{m,k})}\bigl(\Phi R, H^2\bigl(\boldsymbol{S}_1^{m,k}\bigr)\bigr) = \bigl\|H_\Phi^{\{k\}}\bigr\|.$$

By the remarks in Section 2.2, there is a function $G \in H_0^2(\mathbb{M}_{k,m})$ such that $\|G\|_{L^2(\mathbb{M}_{k,m})} \leqslant 1$ and

$$\int_{\mathbb{T}} \operatorname{trace}\bigl((\Phi R)(\zeta) G(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = \operatorname{dist}_{L^2(\boldsymbol{S}_1^{m,k})}\bigl(\Phi R, H^2\bigl(\boldsymbol{S}_1^{m,k}\bigr)\bigr).$$

On the other hand, since R is a maximizing vector of $H_\Phi^{\{k\}}$, it follows from Theorem 3.3 that

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)(RG)(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = \bigl\|H_\Phi^{\{k\}}\bigr\| = \sigma_k(\Phi).$$

Hence $\Psi \stackrel{\text{def}}{=} RG$ is a k-extremal function for Φ. $\square$

Before stating the next result, let us recall that the Hankel operator $H_\Phi : H^2(\mathbb{C}^n) \to H^2_-(\mathbb{C}^m)$ is defined by $H_\Phi f = \mathbb{P}_- \Phi f$ for $f \in H^2(\mathbb{C}^n)$. The following is an immediate consequence of the previous theorem when $k = 1$.

Corollary 3.6. *Let $\Phi \in L^\infty(\mathbb{M}_{m,n})$. The Hankel operator H_Φ has a maximizing vector if and only if Φ has a 1-extremal function.*

Proof. By Theorem 3.5, Φ has a 1-extremal function if and only if the Hankel-type operator $H_\Phi^{\{1\}} : H^2(\mathbb{C}^n) \to L^2(\mathbb{C}^m)/H^2(\mathbb{C}^m)$ has a maximizing vector. The conclusion now follows by considering the "natural" isometric isomorphism between the spaces $H^2_-(\mathbb{C}^m) = L^2(\mathbb{C}^m) \ominus H^2(\mathbb{C}^m)$ and $L^2(\mathbb{C}^m)/H^2(\mathbb{C}^m)$. $\square$

Remark 3.7. It is worth mentioning that if a matrix function Φ is such that the Hankel operator H_Φ has a maximizing vector (e.g. $\Phi \in (H^\infty + C)(\mathbb{M}_n)$), then any 1-extremal function Ψ of Φ satisfies

$$\int_{\mathbb{T}} \operatorname{trace}\big(\Phi(\zeta)\Psi(\zeta)\big)\, d\boldsymbol{m}(\zeta) = \|H_\Phi\| = t_0(\Phi).$$

This is a consequence of Corollary 3.6 and Theorem 3.3.

Remark 3.8. There are other characterizations of the class of bounded matrix functions Φ such that the Hankel operator H_Φ has a maximizing vector. These involve "dual" extremal functions and "thematic" factorizations. We refer the interested reader to [4] for details.

Corollary 3.9. *Let $1 \leqslant k \leqslant \ell \leqslant n$ and $\Phi \in L^\infty(\mathbb{M}_n)$. Suppose that $\sigma_k(\Phi) = \sigma_\ell(\Phi)$. If $H_\Phi^{\{k\}}$ has a maximizing vector, then $H_\Phi^{\{\ell\}}$ also has a maximizing vector.*

Proof. This is an immediate consequence of Theorem 3.5. $\square$

4. How about the sum of superoptimal singular values?

In this section, we prove in Theorem 4.7 that equality is obtained in (1.2) under some natural conditions.

For the rest of this note, we assume that $m = n$.

Consider the non-decreasing sequence $\sigma_1(\Phi), \ldots, \sigma_n(\Phi)$. Recall that

$$\sigma_n(\Phi) = \operatorname{dist}_{L^\infty(\boldsymbol{S}_1^n)}\big(\Phi, H^\infty(\mathbb{M}_n)\big)$$

and the distance on the right-hand side is in fact always attained, i.e. a best approximant Q to Φ under the $L^\infty(\boldsymbol{S}_1^n)$ norm always exists as explained in Section 2.2.

Theorem 4.1. *Let $\Phi \in L^\infty(\mathbb{M}_n)$ and $1 \leqslant k \leqslant n$. Suppose Q is a best approximant to Φ in $H^\infty(\mathbb{M}_n)$ under the $L^\infty(\boldsymbol{S}_1^n)$-norm. If the Hankel-type operator $H_\Phi^{\{k\}}$ has a maximizing vector $\mathcal{F}$ in $H^2(\mathbb{M}_{n,k})$ and $\sigma_k(\Phi) = \sigma_n(\Phi)$, then*

1. *$Q\mathcal{F}$ is a best approximant to $\Phi\mathcal{F}$ in H^2 under the $L^2(\boldsymbol{S}_1^{n,k})$-norm,*
2. *for each $j \geqslant 0$,*

$$s_j\big((\Phi - Q)(\zeta)\mathcal{F}(\zeta)\big) = s_j\big((\Phi - Q)(\zeta)\big)\big\|\mathcal{F}(\zeta)\big\|_{\mathbb{M}_{n,k}} \quad \textit{for a.e. } \zeta \in \mathbb{T},$$

3. *$\sum_{j=0}^{k-1} s_j((\Phi - Q)(\zeta)) = \sigma_k(\Phi)$ holds for a.e. $\zeta \in \mathbb{T}$, and*
4. *$s_j((\Phi - Q)(\zeta)) = 0$ holds for a.e. $\zeta \in \mathbb{T}$ whenever $j \geqslant k$.*

Proof. By our assumptions,

$$\begin{aligned}
\big\|H_\Phi^{\{k\}}\big\|^2\|\mathcal{F}\|^2_{L^2(\mathbb{M}_{n,k})} &= \big\|H_\Phi^{\{k\}}\mathcal{F}\big\|^2_{L^2(\boldsymbol{S}_1^{n,k})/H^2(\boldsymbol{S}_1^{n,k})} = \big\|\rho(\Phi\mathcal{F})\big\|^2 \\
&= \big\|\rho\big((\Phi - Q)\mathcal{F}\big)\big\|^2 \\
&\leqslant \big\|(\Phi - Q)\mathcal{F}\big\|^2_{L^2(\boldsymbol{S}_1^{n,k})} = \int_{\mathbb{T}} \big\|(\Phi - Q)(\zeta)\mathcal{F}(\zeta)\big\|^2_{\boldsymbol{S}_1^{n,k}}\, d\boldsymbol{m}(\zeta) \\
&\leqslant \int_{\mathbb{T}} \big\|(\Phi - Q)(\zeta)\big\|^2_{\boldsymbol{S}_1^n}\big\|\mathcal{F}(\zeta)\big\|^2_{\mathbb{M}_{n,k}}\, d\boldsymbol{m}(\zeta) \\
&\leqslant \|\Phi - Q\|^2_{L^\infty(\boldsymbol{S}_1^n)}\|\mathcal{F}\|^2_{L^2(\mathbb{M}_{n,k})} = \sigma_k(\Phi)^2\|\mathcal{F}\|^2_{L^2(\mathbb{M}_{n,k})}.
\end{aligned}$$

It follows from Theorem 3.3 that all inequalities are equalities. In particular, we obtain that $Q\mathcal{F}$ is a best approximant to ΦQ under the $L^2(\boldsymbol{S}_1^{n,k})$-norm since the first inequality is actually an equality. For almost every $\zeta \in \mathbb{T}$,

$$\begin{gathered}
\big\|(\Phi - Q)(\zeta)\mathcal{F}(\zeta)\big\|_{\boldsymbol{S}_1^n} = \big\|(\Phi - Q)(\zeta)\big\|_{\boldsymbol{S}_1^n}\big\|\mathcal{F}(\zeta)\big\|_{\mathbb{M}_{n,k}} \quad \text{and} \\
\big\|(\Phi - Q)(\zeta)\big\|_{\boldsymbol{S}_1^n} = \|\Phi - Q\|_{L^\infty(\boldsymbol{S}_1^n)} = \sigma_k(\Phi),
\end{gathered} \tag{4.1}$$

because the second and third inequalities are equalities as well. It follows from (4.1) that for each $j \geqslant 0$,

$$s_j\big((\Phi - Q)(\zeta)\mathcal{F}(\zeta)\big) = s_j\big((\Phi - Q)(\zeta)\big)\big\|\mathcal{F}(\zeta)\big\|_{\mathbb{M}_{n,k}} \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

We claim that if $j \geqslant k$, then $s_j((\Phi - Q)(\zeta)) = 0$ for a.e. $\zeta \in \mathbb{T}$. By Theorem 3.5, we can choose a k-extremal function, say Ψ, for Φ. Since Ψ belongs to $H_0^1(\mathbb{M}_n)$,

$$\begin{aligned}
\sigma_k(\Phi) &= \int_{\mathbb{T}} \operatorname{trace}\big(\Phi(\zeta)\Psi(\zeta)\big)\, d\boldsymbol{m}(\zeta) = \int_{\mathbb{T}} \operatorname{trace}\big((\Phi - Q)(\zeta)\Psi(\zeta)\big)\, d\boldsymbol{m}(\zeta) \\
&\leqslant \int_{\mathbb{T}} \big\|(\Phi - Q)(\zeta)\Psi(\zeta)\big\|_{\boldsymbol{S}_1^n}\, d\boldsymbol{m}(\zeta) \leqslant \int_{\mathbb{T}} \big\|(\Phi - Q)(\zeta)\big\|_{\boldsymbol{S}_1^n}\big\|\Psi(\zeta)\big\|_{\mathbb{M}_n}\, d\boldsymbol{m}(\zeta) \\
&\leqslant \|\Phi - Q\|_{L^\infty(\boldsymbol{S}_1^n)}\|\Psi\|_{L^1(\mathbb{M}_n)} \leqslant \|\Phi - Q\|_{L^\infty(\boldsymbol{S}_1^n)} = \sigma_k(\Phi),
\end{aligned}$$

and so all inequalities are equalities. It follows that

$$\left|\operatorname{trace}\bigl((\Phi - Q)(\zeta)\Psi(\zeta)\bigr)\right| = \bigl\|(\Phi - Q)(\zeta)\bigr\|_{S_1^n}\bigl\|\Psi(\zeta)\bigr\|_{\mathbb{M}_n} \quad \text{for a.e. } \zeta \in \mathbb{T}. \tag{4.2}$$

In order to complete the proof, we need the following lemma.

Lemma 4.2. *Let $A \in \mathbb{M}_n$ and $B \in \mathbb{M}_n$. Suppose that A and B satisfy*

$$\left|\operatorname{trace}(AB)\right| = \|A\|_{\mathbb{M}_n}\|B\|_{S_1^n}.$$

If $\operatorname{rank} A \leqslant k$, *then* $\operatorname{rank} B \leqslant k$ *as well.*

We first finish the proof of Theorem 4.1 before proving Lemma 4.2.
It follows from (4.2) and Lemma 4.2 that

$$\operatorname{rank}\bigl((\Phi - Q)(\zeta)\bigr) \leqslant k \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

In particular, if $j \geqslant k$, then

$$s_j\bigl((\Phi - Q)(\zeta)\bigr) = 0 \quad \text{for a.e. } \zeta \in \mathbb{T},$$

and so

$$\sum_{j=0}^{k-1} s_j\bigl((\Phi - Q)(\zeta)\bigr) = \bigl\|(\Phi - Q)(\zeta)\bigr\|_{S_1^n} = \sigma_k(\Phi) \quad \text{for a.e. } \zeta \in \mathbb{T}.$$

This completes the proof. □

Remark 4.3. Lemma 4.2 is a slight modification of Lemma 4.6 in [1]. Although the proof of Lemma 4.2 given below is almost the same as that given in [1] for Lemma 4.6, we include it for the convenience of the reader.

Proof of Lemma 4.2. Let B have polar decomposition $B = UP$ and set $C = AU$, where $P = (B^*B)^{1/2}$. Let $e_1, \dots, e_n$ be an orthonormal basis of eigenvectors for P and $Pe_j = \lambda_j e_j$. It is easy to see that the following inequalities hold:

$$\begin{aligned}\left|\operatorname{trace}(AB)\right| &= \left|\operatorname{trace}(CP)\right| = \left|\sum_{j=1}^{n}\bigl(Pe_j, C^*e_j\bigr)\right| = \left|\sum_{j=1}^{n}\lambda_j\bigl(e_j, C^*e_j\bigr)\right| \\ &= \left|\sum_{j=1}^{n}\lambda_j(Ce_j, e_j)\right| \leqslant \sum_{j=1}^{n}\lambda_j\left|(Ce_j, e_j)\right| \leqslant \sum_{j=1}^{n}\lambda_j\|Ce_j\| \\ &\leqslant \|C\|_{\mathbb{M}_n}\sum_{j=1}^{n}\lambda_j.\end{aligned}$$

On the other hand,

$$\|A\|_{\mathbb{M}_n}\|B\|_{\boldsymbol{S}_1^n} = \|C\|_{\mathbb{M}_m}\|P\|_{\boldsymbol{S}_1^n} = \|C\|_{\mathbb{M}_n}\sum_{j=1}^{n}\lambda_j$$

and so, by the assumption $|\operatorname{trace}(AB)| = \|A\|_{\mathbb{M}_n}\|B\|_{\boldsymbol{S}_1^n}$, it follows that

$$\sum_{j=1}^{n}\lambda_j\|Ce_j\| = \|C\|_{\mathbb{M}_n}\sum_{j=1}^{n}\lambda_j.$$

Therefore $\lambda_j\|Ce_j\| = \|C\|_{\mathbb{M}_n}\lambda_j$ for each j. However, if $\operatorname{rank} A \leqslant k$, then $\operatorname{rank} C \leqslant k$. Thus there are at most k vectors e_j such that $\|Ce_j\| = \|C\|_{\mathbb{M}_n}$. In particular, there are at least $n-k$ vectors e_j such that $\|Ce_j\| < \|C\|_{\mathbb{M}_n}$. Thus, $\lambda_j = 0$ for those $n-k$ vectors e_j, $\operatorname{rank} P \leqslant k$, and so $\operatorname{rank} B \leqslant k$. $\square$

Remark 4.4. Note that the distance function d_Φ defined on $\mathbb{T}$ by

$$d_\Phi(\zeta) \stackrel{\text{def}}{=} \big\|(\Phi - Q)(\zeta)\big\|_{\boldsymbol{S}_1^n}$$

equals $\sigma_k(\Phi)$ for almost every $\zeta \in \mathbb{T}$ and is therefore independent of the choice of the best approximant Q. This is an immediate consequence of Theorem 4.1. A similar phenomenon occurs in the case of matrix functions $\Phi \in L^p(\mathbb{M}_n)$ for $2 < p < \infty$. We refer the reader to [1] for details.

Corollary 4.5. *Let $\Phi \in L^\infty(\mathbb{M}_n)$ be an admissible matrix function and $1 \leqslant k \leqslant n$. If the Hankel-type operator $H_\Phi^{\{k\}}$ has a maximizing vector and $\sigma_k(\Phi) = \sigma_n(\Phi)$, then*

$$\sum_{j=0}^{k-1} s_j\big((\Phi - Q)(\zeta)\big) \leqslant \sum_{j=0}^{k-1} t_j(\Phi)$$

for any best approximation Q of Φ in $H^\infty(\mathbb{M}_n)$ under the $L^\infty(\boldsymbol{S}_1^n)$-norm.

Proof. This is an immediate consequence of Theorems 1.3 and 4.1. $\square$

Definition 4.6. A matrix function $\Phi \in L^\infty(\mathbb{M}_n)$ is said to have *order* ℓ if ℓ is the smallest number such that $H_\Phi^{\{\ell\}}$ has a maximizing vector and

$$\sigma_\ell(\Phi) = \operatorname{dist}_{L^\infty(\boldsymbol{S}_1^n)}\big(\Phi, H^\infty(\mathbb{M}_n)\big).$$

If no such number ℓ exists, we say that Φ is *inaccessible*.

The interested reader should compare this definition of “order” with the one made in [1] for matrix functions in $L^p(\mathbb{M}_n)$ for $2 < p < \infty$. Also, due to Corollary 3.9, it is clear that if $\Phi \in L^\infty(\mathbb{M}_n)$ has order ℓ, then the Hankel-type operator $H_\Phi^{\{k\}}$ has a maximizing vector and

$$\sigma_k(\Phi) = \operatorname{dist}_{L^\infty(\boldsymbol{S}_1^n)}\big(\Phi, H^\infty(\mathbb{M}_n)\big)$$

holds for each $k \geqslant \ell$.

Theorem 4.7. *Let $\Phi \in L^\infty(\mathbb{M}_n)$ be an admissible matrix function of order k. The following statements are equivalent.*

(1) *$Q \in H^\infty$ is a best approximant to Φ under the $L^\infty(\boldsymbol{S}_1^n)$-norm and the functions*

$$\zeta \mapsto s_j\big((\Phi - Q)(\zeta)\big), \quad 0 \leqslant j \leqslant k-1,$$

are constant almost everywhere on $\mathbb{T}$.

(2) *Q is the superoptimal approximant to Φ, $t_j(\Phi) = 0$ for $j \geqslant k$, and*

$$\sigma_k(\Phi) = t_0(\Phi) + \cdots + t_{k-1}(\Phi).$$

Proof. We first prove that (1) implies (2). By Corollary 4.5, we have that, for almost every $\zeta \in \mathbb{T}$,

$$\sum_{j=0}^{k-1} s_j\big((\Phi - Q)(\zeta)\big) \leqslant \sum_{j=0}^{k-1} t_j(\Phi) \leqslant \sum_{j=0}^{k-1} \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big((\Phi - Q)(\zeta)\big) = \sum_{j=0}^{k-1} s_j\big((\Phi - Q)(\zeta)\big).$$

This implies that

$$t_j(\Phi) = \operatorname*{ess\,sup}_{\zeta \in \mathbb{T}} s_j\big((\Phi - Q)(\zeta)\big) = s_j\big((\Phi - Q)(\zeta)\big) \quad \text{for } 0 \leqslant j \leqslant k-1,$$

$Q \in \Omega_{k-1}(\Phi)$, and

$$\sum_{j=0}^{k-1} t_j(\Phi) = \sum_{j=0}^{k-1} s_j\big((\Phi - Q)(\zeta)\big) = \sigma_k(\Phi).$$

Moreover, Theorem 4.1 gives that $s_j((\Phi - Q)(\zeta)) = 0$ a.e. on $\mathbb{T}$ for $j \geqslant k$, and so $t_j(\Phi) = 0$ for $j \geqslant k$, as $Q \in \Omega_{k-1}(\Phi)$. Hence, Q is the superoptimal approximant to Φ.

Let us show that (2) implies (1). Clearly, it suffices to show that if (2) holds, then Q is a best approximant to Φ under the $L^\infty(\boldsymbol{S}_1^n)$-norm. Suppose (2) holds. In this case, we must have that

$$\sigma_k(\Phi) = \sum_{j=0}^{k-1} t_j(\Phi) = \sum_{j=0}^{k-1} s_j\big((\Phi - Q)(\zeta)\big) = \|\Phi - Q\|_{L^\infty(\boldsymbol{S}_1^n)}.$$

Since Φ has order k, it follows that

$$\sigma_n(\Phi) = \|\Phi - Q\|_{L^\infty(\boldsymbol{S}_1^n)}$$

and so the proof is complete. $\square$

For the rest of this section, we restrict ourselves to admissible matrix functions Φ which are also very badly approximable. Recall that, in this case, the function $\zeta \mapsto s_j(\Phi(\zeta))$ equals $t_j(\Phi)$ a.e. on $\mathbb{T}$ for $0 \leqslant j \leqslant n-1$, as mentioned in Section 1.1. The next result follows at once from Theorem 4.7.

Corollary 4.8. *Let Φ be an admissible very badly approximable $n \times n$ matrix function of order k. The zero matrix function is a best approximant to Φ under the $L^\infty(\boldsymbol{S}_1^n)$-norm if and only if $t_j(\Phi) = 0$ for $j \geqslant k$ and*

$$\sigma_k(\Phi) = t_0(\Phi) + \cdots + t_{k-1}(\Phi).$$

It is natural to question at this point whether or not the collection of admissible very badly approximable matrix functions of order k is non-empty. It turns out that one can easily construct examples of admissible very badly approximable matrix functions of order k (see Examples 4.14 and 4.15). Theorem 4.10 below gives a simple sufficient condition for determining when a very badly approximable matrix function has order k. We first need the following lemma.

Lemma 4.9. *Let $\Phi \in L^\infty(\mathbb{M}_n)$. Suppose there is $\Psi \in \mathcal{A}_k^n$ such that*

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = \|\Phi\|_{L^\infty(\boldsymbol{S}_1^n)}.$$

Then Ψ is a k-extremal function for Φ, $\sigma_k(\Phi) = \sigma_n(\Phi)$, and the zero matrix function is a best approximant to Φ under the $L^\infty(\boldsymbol{S}_1^n)$-norm.

Proof. By the assumptions on Ψ, we have

$$\|\Phi\|_{L^\infty(\boldsymbol{S}_1^n)} = \int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) \leqslant \sigma_k(\Phi).$$

On the other hand,

$$\sigma_k(\Phi) \leqslant \operatorname{dist}_{L^\infty(\boldsymbol{S}_1^n)}\bigl(\Phi, H^\infty\bigr) \leqslant \|\Phi\|_{L^\infty(\boldsymbol{S}_1^n)}$$

always holds. Since all the previously mentioned inequalities are equalities, the conclusion follows. □

Theorem 4.10. *Let $\Phi \in L^\infty(\mathbb{M}_n)$ be an admissible very badly approximable matrix function. Suppose there is $\Psi \in \mathcal{A}_k^n$ such that*

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = t_0(\Phi) + \cdots + t_{n-1}(\Phi).$$

If $t_{k-1}(\Phi) > 0$, then Φ has order k and the zero matrix function is a best approximant to Φ under the $L^\infty(\boldsymbol{S}_1^n)$-norm.

Proof. By the remarks preceding Corollary 4.8, it is easy to see that

$$\|\Phi\|_{L^\infty(S_1^n)} = t_0(\Phi) + \cdots + t_n(\Phi).$$

It follows from Lemma 4.9 that Ψ is a k-extremal function for Φ, $\sigma_k(\Phi) = \sigma_n(\Phi)$, and the zero matrix function is a best approximant to Φ under the $L^\infty(\boldsymbol{S}_1^n)$-norm. Thus $\|\Phi\|_{L^\infty(S_1^n)} = \sigma_k(\Phi)$. Moreover, by Theorem 1.3,

$$\sigma_{k-1}(\Phi) \leqslant t_0(\Phi) + \cdots + t_{k-2}(\Phi) < t_0(\Phi) + \cdots + t_{k-1}(\Phi) \leqslant \|\Phi\|_{L^\infty(S_1^n)}.$$

Therefore $\sigma_{k-1}(\Phi) < \sigma_k(\Phi)$. $\square$

Remark 4.11. Notice that under the hypotheses of Theorem 4.10, one also obtains that $t_{k-1}(\Phi)$ is the smallest non-zero superoptimal singular value of Φ. This is an immediate consequence of Corollary 4.8.

We now formulate the corresponding result for admissible very badly approximable unitary-valued matrix functions. These functions are considered in greater detail in Section 5.

Corollary 4.12. *Let $U \in L^\infty(\mathbb{M}_n)$ be an admissible very badly approximable unitary-valued matrix function. If there is $\Psi \in \mathcal{A}_n^n$ such that*

$$\int_{\mathbb{T}} \operatorname{trace}\big(U(\zeta)\Psi(\zeta)\big)\, d\boldsymbol{m}(\zeta) = n,$$

then U has order n and the zero matrix function is a best approximant to U under the $L^\infty(\boldsymbol{S}_1^n)$-norm.

Proof. This is a trivial consequence of Theorem 4.10 and the fact that

$$t_j(U) = 1 \quad \text{for } 0 \leqslant j \leqslant n-1. \qquad \square$$

We are now ready to state the main result of this section.

Theorem 4.13. *Let Φ be an admissible very badly approximable $n \times n$ matrix function. The following statements are equivalent*:

(1) *k is the smallest number for which there exists $\Psi \in \mathcal{A}_k^n$ such that*

$$\int_{\mathbb{T}} \operatorname{trace}\big(\Phi(\zeta)\Psi(\zeta)\big)\, d\boldsymbol{m}(\zeta) = t_0(\Phi) + \cdots + t_{n-1}(\Phi);$$

(2) *Φ has order k, $t_j(\Phi) = 0$ for $j \geqslant k$ and*

$$\sigma_k(\Phi) = t_0(\Phi) + \cdots + t_{k-1}(\Phi).$$

Proof. Let

$$\kappa(\Phi) \stackrel{\text{def}}{=} \inf\left\{ j \geqslant 0\text{: there exists a } \Psi \in \mathcal{A}_j^n \text{ such that } \int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = t_0(\Phi) + \cdots + t_{n-1}(\Phi) \right\}.$$

Clearly, $\kappa(\Phi)$ may be infinite for arbitrary Φ.

Suppose $\kappa = \kappa(\Phi)$ is finite. Then Lemma 4.9 implies that Φ has a κ-extremal function, $\sigma_\kappa(\Phi) = \sigma_n(\Phi)$, and the zero matrix function is a best approximant to Φ under the $L^\infty(\boldsymbol{S}_1^n)$-norm. In particular, Φ has order $k \leqslant \kappa(\Phi)$, $t_j(\Phi) = 0$ for $j \geqslant k$, and

$$\sigma_k(\Phi) = t_0(\Phi) + \cdots + t_{k-1}(\Phi),$$

by Corollary 4.8.

On the other hand, if Φ has order k, $t_j(\Phi) = 0$ for $j \geqslant k$, and

$$\sigma_k(\Phi) = t_0(\Phi) + \cdots + t_{k-1}(\Phi),$$

then Φ has a k-extremal function $\Psi \in \mathcal{A}_k^n$ such that

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = \sigma_k(\Phi) = t_0(\Phi) + \cdots + t_{k-1}(\Phi).$$

Since $t_j(\Phi) = 0$ for $j \geqslant k$, it follows that

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = t_0(\Phi) + \cdots + t_{n-1}(\Phi).$$

Thus $\kappa(\Phi) \leqslant k$.

Hence, if either $\kappa(\Phi)$ is finite or Φ satisfies (2), then $k = \kappa(\Phi)$. $\square$

We end this section by illustrating existence of very badly approximable matrix functions of order k by giving two simple examples; a 2×2 matrix function of order 2 and a 3×3 matrix function of order 2.

Example 4.14. Let

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{z}^2 & \mathbb{O} \\ \mathbb{O} & \bar{z} \end{pmatrix}.$$

It is easy to see that Φ is a continuous (and hence admissible) unitary-valued very badly approximable matrix function with superoptimal singular values $t_0(\Phi) = t_1(\Phi) = 1$. We claim that Φ has order 2. Indeed, the matrix function

$$\Psi = \begin{pmatrix} z^2 & \mathbb{O} \\ \mathbb{O} & z \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

satisfies

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = 2,$$

and so Φ has order 2 by Corollary 4.12.

Example 4.15. Let t_0 and t_1 be two positive numbers satisfying $t_0 \geqslant t_1$. Let

$$\Phi = \begin{pmatrix} t_0\bar{z}^a & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & t_1\bar{z}^b & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix},$$

where a and b are positive integers. It is easy to see that Φ is a continuous (and hence admissible) very badly approximable matrix function with superoptimal singular values $t_0(\Phi) = t_0$, $t_1(\Phi) = t_1$, and $t_2(\Phi) = 0$. Again, we have that Φ has order 2. After all, the matrix function

$$\Psi = \begin{pmatrix} z^a & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & z^b & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix}$$

satisfies

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(\Phi(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = t_0 + t_1 = t_0(\Phi) + t_1(\Phi) + t_2(\Phi),$$

and so Φ has order 2 by Theorem 4.10, since $t_1(\Phi) = t_1 > 0$.

5. Unitary-valued very badly approximable matrix functions

We lastly consider the class $\mathcal{U}_n$ of admissible very badly approximable unitary-valued matrix functions of size $n \times n$ and provide a representation of any n-extremal function Ψ for a function $U \in \mathcal{U}_n$ such that

$$\int_{\mathbb{T}} \operatorname{trace}\bigl(U(\zeta)\Psi(\zeta)\bigr)\, d\boldsymbol{m}(\zeta) = t_0(U) + \cdots + t_{n-1}(U) \tag{5.1}$$

holds. Note that for any such U we have that $t_j(U) = 1$ for $0 \leqslant j \leqslant n-1$.

For a matrix function $\Phi \in L^\infty(\mathbb{M}_{m,n})$, we define the *Toeplitz operator* T_Φ by

$$T_\Phi f = \mathbb{P}_+ \Phi f \quad \text{for } f \in H^2(\mathbb{C}^n),$$

where $\mathbb{P}_+$ denotes the orthogonal projection from $L^2(\mathbb{C}^n)$ onto $H^2(\mathbb{C}^m)$.

It is well known that, for any function $U \in \mathcal{U}_n$, the Toeplitz operator T_U is Fredholm and $\operatorname{ind} T_U > 0$. (As usual, for a Fredholm operator T, its index, $\operatorname{ind} T$, is defined by $\dim\ker T - \dim\ker T^*$.) In particular,

$$\operatorname{ind} T_{\det U} = \operatorname{ind} T_U.$$

We refer the reader to Chapter 14 in [2] for more information concerning functions in $\mathcal{U}_n$.

Theorem 5.1. *Suppose $U \in \mathcal{U}_n$ has an n-extremal function Ψ such that* (5.1) *holds. Then Ψ admits a representation of the form*

$$\Psi = zh^2\Theta,$$

where $h \in H^2$ is an outer function such that $\|h\|_{L^2} = 1$ and Θ is a finite Blaschke–Potapov product. Moreover, the scalar functions $\det(U\Theta)$ *and* $\operatorname{trace}(U\Theta)$ *are admissible badly approximable functions that admit the factorizations*

$$\det(U\Theta) = \bar{z}^n\frac{\bar{h}^n}{h^n} \quad \textit{and} \quad \operatorname{trace}(U\Theta) = n\bar{z}\frac{\bar{h}}{h}.$$

We refer the reader to Section 5 of Chapter 2 in [2] for the definition and other results concerning Blaschke–Potapov products.

Proof. It follows from (5.1) that all inequalities in (1.3) are equalities and so

$$\operatorname{trace}\big(U(\zeta)\Psi(\zeta)\big) = \big\|U(\zeta)\Psi(\zeta)\big\|_{\boldsymbol{S}_1^n} = n\big\|\Psi(\zeta)\big\|_{\mathbb{M}_n} \tag{5.2}$$

holds for a.e. $\zeta \in \mathbb{T}$. Since U is unitary-valued, then

$$\big\|U(\zeta)\Psi(\zeta)\big\|_{\boldsymbol{S}_1^n} = \big\|\Psi(\zeta)\big\|_{\boldsymbol{S}_1^n},$$

and so

$$\big\|\Psi(\zeta)\big\|_{\boldsymbol{S}_1^n} = n\big\|\Psi(\zeta)\big\|_{\mathbb{M}_n}$$

must hold for a.e. $\zeta \in \mathbb{T}$. Therefore we must have

$$\Psi(\zeta) = \big\|\Psi(\zeta)\big\|_{\mathbb{M}_n}V(\zeta) \quad \text{for a.e. } \zeta \in \mathbb{T}, \tag{5.3}$$

for some unitary-valued matrix function V, because

$$s_j\big(\Psi(\zeta)\big) = \big\|\Psi(\zeta)\big\|_{\mathbb{M}_n} \quad \text{for a.e. } \zeta \in \mathbb{T},\ 0 \leqslant j \leqslant n-1.$$

Let $h \in H^2$ be an outer function such that

$$\big|h(\zeta)\big| = \big\|\Psi(\zeta)\big\|_{\mathbb{M}_n}^{1/2} \quad \text{on } \mathbb{T}.$$

Consider also the matrix function $\Xi \stackrel{\text{def}}{=} h^{-2}\Psi$. It follows from (5.3) that

$$\big(\Xi^*\Xi\big)(\zeta) = \frac{1}{|h(\zeta)|^4}\big(\Psi^*\Psi\big)(\zeta) = I_n \quad \text{for a.e. } \zeta \in \mathbb{T},$$

and so Ξ is an inner function. Thus Ψ admits the factorization

$$\Psi = zh^2\Theta$$

for some $n \times n$ unitary-valued inner function Θ and an outer function $h \in H^2$ such that $\|h\|_{L^2} = 1$.

Note that the first equality in (5.2) indicates that the scalar function $\varphi \stackrel{\text{def}}{=} \operatorname{trace}(U\Theta)$ satisfies

$$zh^2\varphi = n|h|^2 \quad \text{on } \mathbb{T},$$

or equivalently

$$\varphi = n\bar{z}\frac{\bar{h}}{h}.$$

Moreover, $\|H_{U\Theta}\|_{\mathrm{e}} \leqslant \|H_U\|_{\mathrm{e}} < 1$, hence $\|H_\varphi\|_{\mathrm{e}} < n = \|H_\varphi\|$ implying that φ is an admissible badly approximable scalar function on $\mathbb{T}$. We conclude that the Toeplitz operator T_φ is Fredholm and $\operatorname{ind} T_\varphi > 0$ (cf. Theorem 7.5.5 in [2]).

Returning to (5.2), it also follows that each eigenvalue of $U(\zeta)\Psi(\zeta)$ equals $\|\Psi(\zeta)\|_{\mathbb{M}_n} = |h(\zeta)|^2$ for a.e. $\zeta \in \mathbb{T}$. In particular,

$$\big|h(\zeta)\big|^{2n} = \det U(\zeta)\Psi(\zeta) = \big(z^n h^{2n}\big)(\zeta) \cdot \det U(\zeta) \cdot \det \Theta(\zeta)$$

holds a.e. $\zeta \in \mathbb{T}$. By setting

$$\theta \stackrel{\text{def}}{=} \det \Theta \quad \text{and} \quad u \stackrel{\text{def}}{=} \det U,$$

we have that u admits the factorization

$$u = \bar{\theta}\bar{z}^n\frac{\bar{h}^n}{h^n} = \bar{\theta}\omega^n,$$

where $\omega \stackrel{\text{def}}{=} \bar{z}\bar{h}/h = \varphi/n$. Since the Toeplitz operator T_ω is Fredholm with positive index, $T_{u\bar{\omega}^n}$ is Fredholm as well. We conclude now that

$$\dim\big(H^2 \ominus \theta H^2\big) = \dim\ker T_\theta^* = \dim\ker T_{\bar{\theta}} = \operatorname{ind} T_{\bar{\theta}} < \infty$$

because $\ker T_\theta = \{\mathbb{O}\}$ and $u\bar{\omega}^n = \bar{\theta}$. Therefore Θ is a Blaschke–Potapov product, because Θ is a unitary-valued inner function and $\det \Theta$ is a finite Blaschke product. $\square$

Acknowledgments

This article is based in part on the author's PhD dissertation at Michigan State University. I would like to thank Professor Vladimir V. Peller for communicating Theorem 1.3 and for suggesting corrections on earlier versions of this paper. I also thank the reviewer for remarks that led to improvements in the paper's presentation.

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