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Coxeter systems with two-dimensional Davis–Vinberg complexes

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Dedicated to Professor Takao Hoshina on his 60th birthday

Abstract

In this paper, we study Coxeter systems with two-dimensional Davis–Vinberg complexes. We show that for a Coxeter group W, if (W,S) and (W,S') are Coxeter systems with two-dimensional Davis–Vinberg complexes, then there exists $S'' \subset W$ such that (W,S'') is a Coxeter system which is isomorphic to (W,S) and the sets of reflections in (W,S'') and (W,S') coincide. Hence, the Coxeter diagrams of (W,S) and (W,S') have the same number of vertices, the same number of edges and the same multiset of edge-labels. This is an extension of the results of A. Kaul and N. Brady, J.P. McCammond, B. Mühlherr and W.D. Neumann.

MSC: 20F55; 20F65; 57M07

1. Introduction and preliminaries

The purpose of this paper is to study Coxeter systems with two-dimensional Davis–Vinberg complexes. A *Coxeter group* is a group W having a presentation

$$\langle S \mid (st)^{m(s,t)} = 1 \quad \text{for } s,t \in S \rangle,$$

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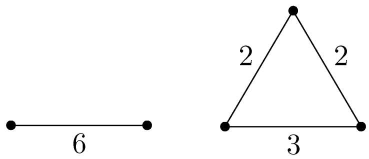

Fig. 1. Two distinct Coxeter diagrams for D_6.

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for each $s, t \in S$,
(2) $m(s,s) = 1$ for each $s \in S$, and
(3) $m(s,t) \geqslant 2$ for each $s, t \in S$ such that $s \neq t$.

The pair (W, S) is called a *Coxeter system*. For a Coxeter group W, a generating set S' of W is called a *Coxeter generating set for W* if (W, S') is a Coxeter system. In a Coxeter system (W, S), the conjugates of elements of S are called *reflections*. We note that the reflections depend on the Coxeter generating set S and not just on the Coxeter group W. Let (W, S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. If T is the empty set, then W_T is the trivial group.

A *diagram* is an undirected graph Γ without loops or multiple edges with a map $\mathrm{Edges}(\Gamma) \to \{2, 3, 4, \ldots\}$ which assigns an integer greater than 1 to each of its edges. Since such diagrams are used to define Coxeter systems, they are called *Coxeter diagrams*.

Let (W, S) and (W', S') be Coxeter systems. Two Coxeter systems (W, S) and (W', S') are said to be *isomorphic*, if there exists a bijection $\psi : S \to S'$ such that

$$m(s,t) = m'(\psi(s), \psi(t))$$

for each $s, t \in S$, where $m(s,t)$ and $m'(s', t')$ are the orders of st in W and $s't'$ in W', respectively.

In general, a Coxeter group does not always determine its Coxeter system up to isomorphism. Indeed some counterexamples are known.

Example 1 (*Bourbaki [1, p. 38, Exercise 8], Brady et al. [2]*). It is known that the Coxeter groups defined by the diagrams in Fig. 1 are isomorphic and D_6.

Example 2 (*Mühlherr [11], Brady et al. [2]*). In [11], Mühlherr showed that the Coxeter groups defined by the diagrams in Fig. 2 are isomorphic.

Here there exists the following natural problem:

Problem (*Brady et al. [2], Charney and Davis [4]*). When does a Coxeter group determine its Coxeter system up to isomorphism?

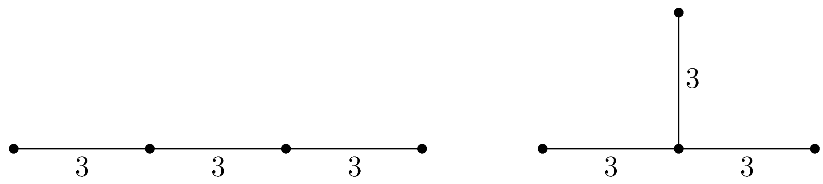

Fig. 2. Coxeter diagrams for isomorphic Coxeter groups.

Recently, Mühlherr and Weidmann proved that skew-angled Coxeter systems are reflection rigid up to diagram twisting [12].

It is known that each Coxeter system (W, S) defines a CAT(0) geodesic space $\Sigma(W, S)$ called the Davis–Vinberg complex [5–7,10]. Here $\dim \Sigma(W, S) \geqslant 1$ by definition, and $\dim \Sigma(W, S) = 1$ if and only if the Coxeter group W is isomorphic to the free product of some $\mathbb{Z}_2$. Hence if $\dim \Sigma(W, S) = 1$, then the Coxeter group W is *rigid*, i.e., W determines its Coxeter system up to isomorphism. In this paper, we investigate Coxeter systems with two-dimensional Davis–Vinberg complexes.

Remark. Let (W, S) be a Coxeter system. We note that $\dim \Sigma(W, S) \leqslant 2$ if and only if W_T is infinite for each $T \subset S$ such that $|T| > 2$. It is known that for $\{s_1, s_2, s_3\} \subset S$ if

(1) $m(s_i, s_j) \geqslant 3$ for each $i, j \in \{1, 2, 3\}$ such that $i \neq j$, or
(2) $m(s_i, s_j) = \infty$ for some $i, j \in \{1, 2, 3\}$,

then the parabolic subgroup $W_{\{s_1,s_2,s_3\}}$ is infinite (see [1]). Hence, for example, if

(1) (W, S) is of type K_n (cf. [9]),
(2) all edge-labels of the Coxeter diagram of (W, S) are odd,
(3) all edge-labels of the Coxeter diagram of (W, S) are greater than 2 (i.e. (W, S) is skew-angled), or
(4) the Coxeter diagram of (W, S) is true,

then $\dim \Sigma(W, S) \leqslant 2$.

We first recall some basic properties of Coxeter groups and Davis–Vinberg complexes in Section 2. After some preliminaries in Section 3, we prove the following theorem in Section 4.

Theorem 1. *Let (W, S) and (W', S') be Coxeter systems with two-dimensional Davis–Vinberg complexes. Suppose that there exists an isomorphism $\phi : W \to W'$. For each $s \in S$, if $\phi(s)$ is not a reflection in (W', S'), then there exist unique elements $t \in S$ and $s', t' \in S'$ such that for some $w' \in W'$,*

(1) $m(s, t) = 2$,
(2) $m(s, u) = \infty$ *for each* $u \in S \setminus \{s, t\}$,
(3) $\phi(W_{\{s,t\}}) = w' W'_{\{s',t'\}} (w')^{-1}$,
(4) $m'(s', t') = 2$,

(5) $m'(s', u') = \infty$ *for each* $u' \in S' \backslash \{s', t'\}$,
(6) $\phi(s) = w's't'(w')^{-1}$ *and*
(7) $\phi(t) = w't'(w')^{-1}$.

Here we can define an automorphism ψ of W as follows: for each $s \in S$,

(1) if $\phi(s)$ is a reflection in (W', S'), then $\psi(s) = s$, and
(2) if $\phi(s)$ is not a reflection in (W', S'), then $\psi(s) = st$, where t is a unique element of S such that $m(s, t) = 2$.

Then the Coxeter systems (W, S) and $(W, \psi(S))$ are isomorphic and by Theorem 1 the isomorphism $\phi : W \to W'$ maps reflections in $(W, \psi(S))$ onto reflections in (W', S'). Thus we obtain the following theorem.

Theorem 2. *Let* (W, S) *and*(W, S') *be Coxeter systems with two-dimensional Davis–Vinberg complexes. Then there exists* $S'' \subset W$ *such that* (W, S'') *is a Coxeter system which is isomorphic to* (W, S) *and the sets of reflections in* (W, S'') *and* (W, S') *coincide.*

This implies the following corollary which is an extension of the results of Kaul [9] and Brady et al., [2, Lemma 5.3].

Corollary 3. *For a Coxeter group W, if* (W, S) *and* (W, S') *are Coxeter systems with two-dimensional Davis–Vinberg complexes, then the Coxeter diagrams of* (W, S) *and* (W, S') *have the same number of vertices, the same number of edges and the same multiset of edge-labels.*

Here a *multiset* is a collection in which the order of the entries does not matter, but multiplicities do. Thus the multisets $\{1, 1, 2\}$ and $\{1, 2, 2\}$ are different. In Corollary 3, we cannot omit the assumption “with two-dimensional Davis–Vinberg complexes” by Example 1.

2. Basics on Coxeter groups and Davis–Vinberg complexes

In this section, we introduce some basic properties of Coxeter groups and Davis–Vinberg complexes.

Definition 2.1. Let (W, S) be a Coxeter system and $T \subset S$. The subset T is called a *spherical subset of S*, if the parabolic subgroup W_T is finite.

Definition 2.2. Let (W, S) be a Coxeter system and $w \in W$. A representation $w = s_1 \cdots s_l$ $(s_i \in S)$ is said to be reduced, if $\ell(w) = l$, where $\ell(w)$ is the minimum length of a word in S which represents w.

The following lemmas are known.

Lemma 2.3 (*Bourbaki [1], Brown [3], Davis [5], Hymphreys [8]*). *Let* (W, S) *be a Coxeter system.*

(i) *Let* $w \in W$ *and let* $w = s_1 \cdots s_l$ *be a representation. If* $\ell(w) < l$, *then* $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_l$ *for some* $1 \leqslant i < j \leqslant l$.
(ii) *Let* $w \in W$ *and let* $w = s_1 \cdots s_l$ *be a representation. Then the length* $\ell(w)$ *is even if and only if l is even.*
(iii) *For each* $w \in W$, *there exists a unique subset* $S(w) \subset S$ *such that* $S(w) = \{s_1, \ldots, s_l\}$ *for every reduced representation* $w = s_1 \cdots s_l$ $(s_i \in S)$.
(iv) *Let* $w \in W$ *and* $T \subset S$. *Then* $w \in W_T$ *if and only if* $S(w) \subset T$.
(v) *For each subset* $T \subset S$, (W_T, T) *is a Coxeter system.*
(vi) *For all subsets* $T_1, T_2 \subset S$, $W_{T_1} = W_{T_2}$ *if and only if* $T_1 = T_2$.
(vii) *If W is finite, then there exists a unique element* $w_0 \in W$ *of longest length.*

Lemma 2.4 (*Bourbaki [1], Davis [5, Lemma 7.11]*). *Let* (W, S) *be a Coxeter system, let* $T \subset S$ *and let* $w \in W_T$. *Then the following statements are equivalent*:

(1) W_T *is finite and w is the element of longest length in* W_T;
(2) $\ell(wt) < \ell(w)$ *for each* $t \in T$.

Lemma 2.5 (*Bourbaki [1, p. 12, Proposition 3]*). *Let* (W, S) *be a Coxeter system and let* $s, t \in S$. *Then s is conjugate to t if and only if there exists a sequence* $s_1, \ldots, s_n \in S$ *such that* $s_1 = s$, $s_n = t$ *and* $m(s_i, s_{i+1})$ *is odd for each* $i \in \{1, \ldots, n-1\}$.

Lemma 2.6 (*Brady et al. [2, Results 3.7 and 3.8]*). *Let* (W, S) *and* (W, S') *be Coxeter systems. Then*

(1) *if* $S \subset R_{S'}$ *then* $R_S = R_{S'}$, *and*
(2) *if* $R_S = R_{S'}$ *then* $|S| = |S'|$,
where R_S *and* $R_{S'}$ *are the sets of all reflections in* (W, S) *and* (W, S'), *respectively.*

By Results 1.8, 1.9 and 1.10 in [2], we obtain the following theorem.

Theorem 2.7 (*cf. Brady et al. [2]*). *Let* (W, S) *and* (W', S') *be Coxeter systems. Suppose that there exists an isomorphism* $\phi : W \to W'$. *Then for each maximal spherical subset* $T \subset S$, *there exists a unique maximal spherical subset* $T' \subset S'$ *such that* $\phi(W_T) = w' W'_{T'} (w')^{-1}$ *for some* $w' \in W'$.

We introduce a definition of the Davis–Vinberg complex of a Coxeter system.

Definition 2.8 (*Davis [5–7]*). Let (W, S) be a Coxeter system and let $W\mathcal{S}^f$ denote the set of all left cosets of the form wW_T, with $w \in W$ and a spherical subset $T \subset S$. The set $W\mathcal{S}^f$ is partially ordered by inclusion. The Davis–Vinberg complex $\Sigma(W, S)$ is defined as the geometric realization of the partially ordered set $W\mathcal{S}^f$ [5,6]. Here it is known that $\Sigma(W, S)$ has a structure of a PE (i.e. piecewise euclidean) cell complex whose 1-skeleton is the Cayley graph of W with respect to S [7]. Then the vertex set of each cell of the PE cell complex $\Sigma(W, S)$ is wW_T for some $w \in W$ and some spherical subset T of S. The Coxeter group W acts properly, discontinuously and cocompactly as isometries on the PE cell complex $\Sigma(W, S)$ with the natural metric [5,7].

Remark. For a Coxeter system (W, S), by the definition of $\Sigma(W, S)$,

$$\dim \Sigma(W, S) = \max\{|T| : T \text{ is a spherical subset of } S\}.$$

Theorem 2.7 implies the following lemma.

Lemma 2.9. *Let (W, S) and (W', S') be Coxeter systems with two-dimensional Davis–Vinberg complexes. Suppose that there exists an isomorphism $\phi : W \to W'$.*

(i) *For each two elements $s, t \in S$ such that $m(s, t) < \infty$, there exist unique two elements $s', t' \in S'$ such that $\phi(W_{\{s,t\}}) = w' W'_{\{s',t'\}}(w')^{-1}$ (hence $m(s, t) = m'(s', t')$) for some $w' \in W'$.*
(ii) *The multisets of edge-labels of the Coxeter diagrams of (W, S) and (W', S') coincide.*

3. Lemmas on Coxeter groups

We show some lemmas needed later.

Lemma 3.1. *Let (W, S) be a Coxeter system and let $w \in W$. Suppose that $w^2 = 1$ and $\ell(w) = \min\{\ell(vwv^{-1}) : v \in W\}$. Then $W_{S(w)}$ is finite and w is the element of longest length in $W_{S(w)}$, where $S(w)$ is the subset of S defined in Lemma* 2.3(iii).

Proof. Let $w = s_1 \cdots s_l$ be a reduced representation. Since $w^2 = 1$,

$$s_1 \cdots s_l = w = w^{-1} = s_l \cdots s_1.$$

Hence $\ell(ws_1) < \ell(w)$. By Lemma 2.3(i),

$$ws_1 = (s_1 \cdots s_l)s_1 = s_1 \cdots \hat{s}_i \cdots s_l$$

for some $i \in \{1, \ldots, l\}$. Suppose that $1 < i \leqslant l$. Then

$$s_1 w s_1 = s_2 \cdots \hat{s}_i \cdots s_l,$$

and $\ell(s_1 w s_1) < \ell(w)$. This contradicts the assumption

$$\ell(w) = \min\{\ell(vwv^{-1}) : v \in W\}.$$

Thus $i = 1$ and $ws_1 = s_2 \cdots s_l$. Hence $w = (s_2 \cdots s_l)s_1$ is reduced.

By iterating the above argument,

$$w = (s_{i+1} \cdots s_l)(s_1 \cdots s_i)$$

is reduced for each $i \in \{1, \ldots, l-1\}$. Hence $\ell(ws_i) < \ell(w)$ for each $i \in \{1, \ldots, l\}$, i.e., $\ell(ws) < \ell(w)$ for each $s \in S(w)$. By Lemma 2.4, $W_{S(w)}$ is finite and w is the element of longest length in $W_{S(w)}$. $\square$

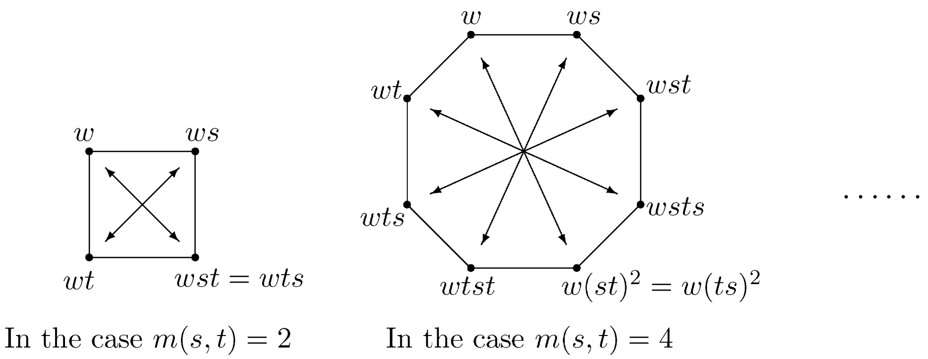

Fig. 3. Isometry v on the 2-cell C.

Remark. Let (W, S) and (W', S') be Coxeter systems with two-dimensional Davis–Vinberg complexes. Suppose that there exists an isomorphism $\phi : W \to W'$. Let $s \in S$. Since $(\phi(s))^2 = 1$, by Lemma 3.1, either

(1) $\phi(s)$ is a reflection in (W', S'), or
(2) $\phi(s) = w'(s't')^{m'}(w')^{-1}$ for some $w' \in W'$ and $s', t' \in S'$, where $m'(s', t')$ is even and $m' = m'(s', t')/2$.

Lemma 3.2. *Let (W, S) be a Coxeter system with two-dimensional Davis–Vinberg complex, let $s, t, a, b \in S$ and let $w, x \in W$. Suppose that $m(s, t)$ is even, $m(a, b)$ is finite and $w(st)^m w^{-1} \in xW_{\{a,b\}}x^{-1}$, where $m = m(s, t)/2$. Then $wW_{\{s,t\}}w^{-1} = xW_{\{a,b\}}x^{-1}$ and $\{s, t\} = \{a, b\}$.*

Proof. Suppose that $m(s, t)$ is even, $m(a, b)$ is finite and $w(st)^m w^{-1} \in xW_{\{a,b\}}x^{-1}$, where $m = m(s, t)/2$. Let $v = w(st)^m w^{-1}$ and let C and D be the 2-cells in $\Sigma(W, S)$ such that

$$C^{(0)} = wW_{\{s,t\}} \quad \text{and} \quad D^{(0)} = xW_{\{a,b\}}.$$

Then v is an isometry of $\Sigma(W, S)$ and the barycenter of C is the unique fixed point of v because $m(s, t) = 2m$ and $\dim \Sigma(W, S) = 2$ (cf. Fig. 3). Since

$$v = w(st)^m w^{-1} \in xW_{\{a,b\}}x^{-1},$$

there exists $u \in W_{\{a,b\}}$ such that $v = xux^{-1}$. Then

$$v(xW_{\{a,b\}}) = xux^{-1}(xW_{\{a,b\}}) = x(uW_{\{a,b\}}) = xW_{\{a,b\}}.$$

Hence $vD = D$. In general, for each cell E of $\Sigma(W, S)$ and each $y \in W$, if $yE = E$ then the isometry y fixes the barycenter of E by the definition of $\Sigma(W, S)$. Thus, the barycenter of D is a fixed point of v. On the other hand, the barycenter of C is the unique fixed point of v. Hence $C = D$ and

$$wW_{\{s,t\}} = C^{(0)} = D^{(0)} = xW_{\{a,b\}}.$$

Since $x^{-1}wW_{\{s,t\}} = W_{\{a,b\}}$, we have that

$$x^{-1}w \in x^{-1}wW_{\{s,t\}} = W_{\{a,b\}}.$$

Hence, $W_{\{s,t\}} = W_{\{a,b\}}$ and $\{s, t\} = \{a, b\}$ by Lemma 2.3(vi). Since

$$x^{-1}w, (x^{-1}w)^{-1} \in W_{\{s,t\}} = W_{\{a,b\}},$$

$x^{-1}wW_{\{s,t\}}w^{-1}x = W_{\{a,b\}}$. Hence, we obtain that $wW_{\{s,t\}}w^{-1} = xW_{\{a,b\}}x^{-1}$. □

Lemma 3.3. *Let (W, S) and (W', S') be Coxeter systems with two-dimensional Davis–Vinberg complexes such that there exists an isomorphism $\phi : W \to W'$, let $s \in S$, let $s', t' \in S'$ and let $w' \in W'$. Suppose that $m'(s', t')$ is even and $\phi(s) = w'(s't')^{m'}(w')^{-1}$, where $m' = m'(s', t')/2$. Then there exists a unique element $t \in S$ such that*

(1) $\phi(W_{\{s,t\}}) = w'W'_{\{s',t'\}}(w')^{-1}$,
(2) *$\phi(t)$ is a reflection in (W', S'), and*
(3) $m(s, t) = m'(s', t') = 2$.

Proof. Suppose that $m'(s', t')$ is even and $\phi(s) = w'(s't')^{m'}(w')^{-1}$, where $m' = m'(s', t')/2$. By Lemma 2.9, there exist $r, t \in S$ and $x \in W$ such that

$$\phi^{-1}(W'_{\{s',t'\}}) = xW_{\{r,t\}}x^{-1}.$$

Here we note that $m(r, t) = m'(s', t')$.

We first show that we may suppose $r = s$.

Since $\phi(s) = w'(s't')^{m'}(w')^{-1}$,

$$(\phi^{-1}(w'))^{-1}s\phi^{-1}(w') = \phi^{-1}((s't')^{m'}) \in \phi^{-1}(W'_{\{s',t'\}}) = xW_{\{r,t\}}x^{-1}.$$

Hence, $(\phi^{-1}(w'))^{-1}s\phi^{-1}(w') = xyx^{-1}$ for some $y \in W_{\{r,t\}}$. Since the length of $(\phi^{-1}(w'))^{-1}s\phi^{-1}(w')$ is odd, the length of y is also odd. Hence, y is conjugate to either r or t because $y \in W_{\{r,t\}}$. Here we may suppose that y is conjugate to r. Then s is conjugate to r, since $(\phi^{-1}(w'))^{-1}s\phi^{-1}(w') = xyx^{-1}$.

Now we show that $s = r$. If $s \neq r$, then there exists $a \in S\setminus\{s\}$ such that $m(s, a)$ is odd by Lemma 2.5. By Lemma 2.9, there exist $a', b' \in S'$ such that $\phi(W_{\{s,a\}}) = x'W'_{\{a',b'\}}(x')^{-1}$ for some $x' \in W'$. Here we note that $m(s, a) = m'(a', b')$. Then

$$w'(s't')^{m'}(w')^{-1} = \phi(s) \in \phi(W_{\{s,a\}}) = x'W'_{\{a',b'\}}(x')^{-1}.$$

Hence $\{s', t'\} = \{a', b'\}$ by Lemma 3.2 and $m'(a', b') = m'(s', t')$. Then

$$m(s, a) = m'(a', b') = m'(s', t').$$

Here $m(s, a)$ is odd. This contradicts the assumption $m'(s', t') = 2m'$ is even. Thus $s = r$.

Then $\phi^{-1}(W'_{\{s',t'\}}) = xW_{\{s,t\}}x^{-1}$, and

$$w'(s't')^{m'}(w')^{-1} = \phi(s) \in \phi(W_{\{s,t\}}) = (\phi(x))^{-1}W'_{\{s',t'\}}\phi(x).$$

By Lemma 3.2, $(\phi(x))^{-1} W'_{\{s',t'\}} \phi(x) = w' W'_{\{s',t'\}} (w')^{-1}$. Hence

$$\phi(W_{\{s,t\}}) = w' W'_{\{s',t'\}} (w')^{-1}.$$

Here we note that such $t \in S$ is unique by Lemma 2.9.

Next we show that $\phi(t)$ is a reflection. Here $\phi(t)$ is a reflection if and only if the length $\ell(\phi(t))$ is odd, because $\phi(t) \in \phi(W_{\{s,t\}}) = w' W'_{\{s',t'\}} (w')^{-1}$. Now we suppose that the length $\ell(\phi(t))$ is even. Then the lengths of $\phi(s) = w'(s't')^{m'}(w')^{-1}$ and $\phi(t)$ are even and the set $\{\phi(s), \phi(t)\}$ generates $\phi(W_{\{s,t\}}) = w' W'_{\{s',t'\}} (w')^{-1}$. In general, for $f, g \in W$ if $\ell(f)$ and $\ell(g)$ are even, then the length $\ell(fg)$ is even by Lemma2.3(ii). Hence the length of each element of $\phi(W_{\{s,t\}})$ is even. On the other hand, the length of $w's'(w')^{-1} \in w' W'_{\{s',t'\}} (w')^{-1} = \phi(W_{\{s,t\}})$ is odd. This is a contradiction. Thus, the length of $\phi(t)$ is odd and $\phi(t)$ is a reflection.

Since $\phi(t)$ is a reflection, $\phi(t) = w'(s't')^k s'(w')^{-1}$ for some $0 \leqslant k < 2m'$. Then

$$\begin{aligned}\phi(s)\phi(t) &= (w'(s't')^{m'}(w')^{-1})(w'(s't')^k s'(w')^{-1}) \\ &= w'(s't')^{m'}(s't')^k s'(w')^{-1} \\ &= w'(s't')^{m'+k} s'(w')^{-1}.\end{aligned}$$

Hence $\phi(s)\phi(t)$ is a reflection and $(\phi(s)\phi(t))^2 = 1$, i.e., $(st)^2 = 1$. This means that $m(s,t) = m'(s',t') = 2$. □

Lemma 3.4. *Let (W, S) be a Coxeter system and let $s, t \in S$. Suppose that $m(s,t) = 2$ and $m(s,u) = \infty$ for each $u \in S \backslash \{s,t\}$. Let $S' = (S \backslash \{s\}) \cup \{st\}$. Then (W, S') is a Coxeter system which is isomorphic to (W, S).*

Proof. The map $\psi : S \to S'$ defined by $\psi(s) = st$ and $\psi(u) = u$ for each $u \in S \backslash \{s\}$ induces an automorphism $\psi : W \to W$, and (W, S) and (W, S') are isomorphic. □

4. Proof of the main results

Using some lemmas in Sections 2 and 3, we prove the main results.

Theorem 4.1. *Let (W, S) and (W', S') be Coxeter systems with two-dimensional Davis–Vinberg complexes. Suppose that there exists an isomorphism $\phi : W \to W'$. For each $s \in S$,* if *$\phi(s)$ is not a reflection in (W', S'), then there exist unique $t \in S$ and $s', t' \in S'$ such that for some $w' \in W'$,*

(1) $m(s,t) = 2$,
(2) $m(s,u) = \infty$ *for each* $u \in S \backslash \{s,t\}$,
(3) $\phi(W_{\{s,t\}}) = w' W'_{\{s',t'\}} (w')^{-1}$,
(4) $m'(s',t') = 2$,
(5) $m'(s',u') = \infty$ *for each* $u' \in S' \backslash \{s',t'\}$,
(6) $\phi(s) = w's't'(w')^{-1}$ *and*
(7) $\phi(t) = w't'(w')^{-1}$.

Proof. Suppose that $s \in S$ and $\phi(s)$ is not a reflection in (W', S'). Since $s^2=1$, $(\phi(s))^2=1$. By Lemma 3.1, there exist $w', v' \in W'$ such that $\phi(s) = w'v'(w')^{-1}$ and v' is the element of longest length in $W'_{S'(v')}$. Since $\phi(s)$ is not a reflection, $v' \notin S'$, i.e., $|S'(v')| > 1$. Hence $|S'(v')|=2$ because $\dim \Sigma(W', S')=2$. Let $S'(v')=\{s', t'\}$. Since v' is the element of longest length in $W'_{S'(v')}=W'_{\{s',t'\}}$ and v' is not a reflection, $m'(s', t')$ is even and $v'=(s't')^{m'}$, where $m' = m'(s', t')/2$. Hence $\phi(s) = w'(s't')^{m'}(w')^{-1}$. By Lemma 3.3, there exists a unique element $t \in S$ such that

(i) $\phi(W_{\{s,t\}}) = w'W'_{\{s',t'\}}(w')^{-1}$,
(ii) $\phi(t)$ is a reflection in (W', S'), and
(iii) $m(s, t) = m'(s', t') = 2$.

Then $\phi(s) = w's't'(w')^{-1}$ by (iii).

Now $\phi(t)$ is a reflection by (ii) and

$$\begin{aligned}\phi(t) \in \phi(W_{\{s,t\}}) &= w'W'_{\{s',t'\}}(w')^{-1} \\ &= \{1, w's'(w')^{-1}, w't'(w')^{-1}, w's't'(w')^{-1}\}.\end{aligned}$$

Hence, either $\phi(t) = w's'(w')^{-1}$ or $\phi(t) = w't'(w')^{-1}$. Here we may suppose that

$$\phi(t) = w't'(w')^{-1}.$$

Finally we show that $m(s, u) = \infty$ for each $u \in S\backslash\{s, t\}$ and $m'(s', u') = \infty$ for each $u' \in S'\backslash\{s', t'\}$.

We suppose that there exists $u \in S\backslash\{s, t\}$ such that $m(s, u) < \infty$. By Lemma 2.9, $\phi(W_{\{s,u\}}) = x'W'_{\{a',b'\}}(x')^{-1}$ for some $x' \in W'$ and $a', b' \in S'$. Then

$$w's't'(w')^{-1} = \phi(s) \in \phi(W_{\{s,u\}}) = x'W'_{\{a',b'\}}(x')^{-1}.$$

By Lemma 3.2, $x'W'_{\{a',b'\}}(x')^{-1} = w'W'_{\{s',t'\}}(w')^{-1}$. Hence

$$\begin{aligned}\phi(W_{\{s,u\}}) &= x'W'_{\{a',b'\}}(x')^{-1} \\ &= w'W'_{\{s',t'\}}(w')^{-1} \\ &= \phi(W_{\{s,t\}}).\end{aligned}$$

Thus $W_{\{s,u\}} = W_{\{s,t\}}$ and $\{s, u\} = \{s, t\}$ by Lemma 2.3(vi). Hence $u = t$. This contradicts the assumption $u \in S\backslash\{s, t\}$. Thus $m(s, u) = \infty$ for each $u \in S\backslash\{s, t\}$.

We note that

$$\phi(st) = (w's't'(w')^{-1})(w't'(w')^{-1}) = w's'(w')^{-1}$$

and

$$\phi^{-1}(s') = (\phi^{-1}(w'))^{-1}st\phi^{-1}(w').$$

By applying the above argument to $\phi^{-1} : W' \to W$, we can prove that $m'(s', u') = \infty$ for each $u' \in S'\backslash\{s', t'\}$. □

We obtain the following theorem from Theorem 4.1.

Theorem 4.2. *Let (W, S) and (W, S') be Coxeter systems with two-dimensional Davis–Vinberg complexes. Then there exists $S'' \subset W$ such that (W, S'') is a Coxeter system which is isomorphic to (W, S) and $R_{S'} = R_{S''}$.*

Proof. Let

$$S_0 = \{s \in S : s \text{ is not a reflection in } (W, S')\} = \{s_1, \dots, s_n\}.$$

For each $i \in \{1, ..., n\}$, there exists a unique element $t_i \in S \setminus S_0$ such that $m(s_i, t_i) = 2$ by Theorem 4.1. Then $s_i t_i$ is a reflection in (W, S') by Theorem 4.1. Let

$$S'' = (S \setminus S_0) \cup \{s_1 t_1, ..., s_n t_n\}.$$

Then (W, S'') is a Coxeter system which is isomorphic to (W, S) by Lemma 3.4. Since $S'' \subset R_{S'}$ by the construction of S'', $R_{S''} = R_{S'}$ by Lemma 2.6(1). □

Theorem 4.2 implies the following corollary.

Corollary 4.3. *For a Coxeter group W, if (W, S) and (W, S') are Coxeter systems with two-dimensional Davis–Vinberg complexes, then the Coxeter diagrams of (W, S) and (W, S') have the same number of vertices, the same number of edges and the same multiset of edge-labels.*

Proof. By Lemma 2.9, the Coxeter diagrams of (W, S) and (W, S') have the same number of edges and the same multiset of edge-labels. By Theorem 4.2, there exists $S'' \subset W$ such that (W, S'') is a Coxeter system which is isomorphic to (W, S) and $R_{S'} = R_{S''}$. Hence $|S| = |S''| = |S'|$ by Lemma 2.6(2). □

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