On the Weight Enumeration of Weights Less than 2.5d of Reed–Muller Codes

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Let \( P_r \) be the set of all polynomials of degree \( r \) in \( m \) variables over \( GF(2) \). Polynomial \( f \) in \( P_r \) is said to be affine equivalent to polynomial \( g \) in \( P_r \), if \( f \) is transformable to \( g \) by an invertible affine transformation of the variables. Any polynomial of weight less than \( 2^{m-r+1} + 2^{m-r-1} \) in \( P_r \) is shown to have a simple structure. By using this fact, we find out a set of representative polynomials such that any polynomial of weight less than \( 2^{m-r+1} + 2^{m-r-1} \) in \( P_r \) is affine equivalent to one and only one polynomial of the set. By counting the number of polynomials which are affine equivalent to each representative polynomial in the set, we derive explicit formulas for the enumerators of all weights less than 2.5\( d \) of Reed–Muller codes, where \( d \) is the minimum weight.

1. INTRODUCTION

Explicit weight enumerator formulas are known for the second-order Reed–Muller (RM) codes by Sloane and Berlekamp (1970), and for the codewords of weight less than 2\( d \) of any order RM codes by Kasami and Tokura (1970), where \( d \) is the minimum weight. The weight distribution of the (127, 64) RM code also was found by Sugino et al. (1971). We derived the weight enumerator formulas for the codewords of weight less than 2.25\( d \) of the third-order RM codes, and by using these results we found the complete weight distributions for the RM codes of length 256 (Kasami, Tokura, and Azumi, 1971), which were found independently by Sarwate and Berlekamp (1971, 1972) and by Tilborg (1971).

In this paper we characterize the codewords of weight less than 2.5\( d \) of RM codes of any order, and derive the weight enumerator formulas for those codewords (Kasami, Tokura, and Azumi, 1972, 1973a, b).

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2. Preliminaries

For $1 \leq r \leq m$, let $P_r$ be the set of polynomials over $GF(2)$ of $m$ variables of degree $r$ or less, and let $P_0 = \{0, 1\}$. Polynomials of degree 1 are said to be linear. Let $f = f(x_1, x_2, \ldots, x_m) \in P_r$ and let $V_m$ be an $m$-dimensional vector space over $GF(2)$. Let $|f|_m$ denote the number of those elements $(a_1, \ldots, a_m)$ in $V_m$ for which $f(a_1, \ldots, a_m) = 1$, and $|f|_m$ is called the weight of $f$. By Muller's definition (1954), each code vector in the $r$th order RM code corresponds to each polynomial in $P_r$, and the weight of a codeword is equal to that of the corresponding polynomial. Hence, we will consider the weight distribution of polynomials in $P_r$ instead of that of codewords.

The $m$ linear equations

$$y_i = a_{i0} + \sum_{j=1}^{m} a_{ij}x_j,$$

for $1 \leq i \leq m$ (1)

define an affine transformation. If the coefficient matrix $\| a_{ij} \|$ is nonsingular, this transformation is called an invertible affine transformation. If $f \in P_r$ can be transformed to $g \in P_r$ by an invertible affine transformation, $f$ is said to be affine equivalent to $g$ and we write $f \approx g$. If $f \approx g$, then $|f|_m = |g|_m$.

$P_r$ is partitioned into affine equivalent classes. For an affine equivalent class, we will choose a representative polynomial with the smallest number of variables. Let $m(f)$ denote the number of variables of the representative polynomial of the affine equivalent class containing $f$. For $f \in P_r$, we write $f \in P_{r,n}$ if there exist $n$ mutually independent linear polynomials $u_1, u_2, \ldots, u_n$ such that $u_1 = u_2 = \cdots = u_n = 0$ implies $f = 0$.

The following lemma summarizes some known results which are used below (see Sloane and Berlekamp, 1970; Kasami and Tokura, 1970).

**Lemma 1.** (1) If $f \in P_2$ and $0 < |f|_m < 2^m$, then $|f|_m$ is of the form

$$2^{m-1} + \epsilon 2^{m-1-l}$$

where $1 \leq l \leq m/2$ and $\epsilon$ is either $-1, 0,$ or $1$.

\begin{equation}
|f|_m = 2^{m-1} - 2^{m-1-l} \quad \text{(or } 2^{m-1} + 2^{m-1-l}) \quad \text{if and only if } f \approx x_1x_2 + x_3x_4 + \cdots + x_{2l-1}x_{2l} \quad \text{(or } f \approx x_1x_2 + x_3x_4 + \cdots + x_{2l-1}x_{2l} + 1). 
\end{equation}

\begin{equation}
|f|_m = 2^{m-1} \quad \text{if and only if } f \approx x_1x_2 + x_3x_4 + \cdots + x_{2l-1}x_{2l} + x_{2l+1} 
\end{equation}

for $0 \leq l \leq (m - 1)/2$, where $x_i$'s are mutually independent.
<table>
<thead>
<tr>
<th>Weight</th>
<th>[No.]</th>
<th>Representative polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^{m-r+1} + 2^{m-r-2} + 2^{m-r-4}$ (10.0101d)</td>
<td>(1) $X_1X_2X_3 + X_4X_2X_6 + X_7X_8X_9$</td>
<td></td>
</tr>
<tr>
<td>$2^{m-r+1} + 2^{m-r-2} + 2^{m-r-3}$ (10.0111d)</td>
<td>(2) $X_1(X_2X_3 + X_4X_5) + X_6(X_7X_8 + X_9X_2)$</td>
<td></td>
</tr>
<tr>
<td>$2^{m-r+1} + 2^{m-r-2} + 2^{m-r-3} + 2^{m-r-4}$ (10.0111d)</td>
<td>(3) $X_1X_2X_3 + X_4X_2X_6 + X_7X_8X_9 + X_1X_4$</td>
<td></td>
</tr>
<tr>
<td>$2^{m-r+1} + 2^{m-r-1} - 2^{m-r-s} + 3$ (10.01 $\cdots$ 1d) $\bar{s}$</td>
<td>(4) $X_1X_2X_3 + X_2X_4X_6 + X_7X_8X_9 + X_1X_4$</td>
<td></td>
</tr>
<tr>
<td>$2^{m-r+1} + 2^{m-r-1} - 2^{m-r-s} + 2^{m-r-s-1}$ (10.01 $\cdots$ 101d) $\bar{s}$</td>
<td>(5) $X_1X_2X_3 + X_4(X_5X_6 + X_7X_8 + \cdots + X_{2^{s+1}}X_{2^{s+2}})$ (s $\geq$ 4)</td>
<td></td>
</tr>
<tr>
<td>$2^{m-r+1} + 2^{m-r-1} - 2^{m-r-s}$ (10.01 $\cdots$ 1d) $\bar{s}$</td>
<td>(6) $X_1X_2X_3 + X_4(X_5X_6 + X_7X_8 + \cdots + X_{2^{s+2}}X_{2^{s+3}})$ (s $\geq$ 3)</td>
<td></td>
</tr>
<tr>
<td>$2^{m-r+1} + 2^{m-r-1} - 2^{m-r-s-1}$ (10.01 $\cdots$ 1d) $\bar{s}$</td>
<td>(7) $X_1X_2X_3 + X_4(X_5X_6 + \cdots + X_{2^{s+2}}X_{2^{s+4}})$ (s $\geq$ 3)</td>
<td></td>
</tr>
<tr>
<td>$2^{m-r+1} + 2^{m-r-1} - 2^{m-r-s}$ (10.01 $\cdots$ 1d) $\bar{s}$</td>
<td>(8) $X_1X_2X_3 + X_4((X_5 + 1)X_6 + X_7X_8 + \cdots + X_{2^{s+1}}X_{2^{s+2}})$ (s $\geq$ 3)</td>
<td></td>
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<tr>
<td>$2^{m-r+1} + 2^{m-r-1} - 2^{m-r-s}$ (10.01 $\cdots$ 1d) $\bar{s}$</td>
<td>(9) $X_1X_2X_3 + X_4(X_5X_6 + X_7X_8 + X_9X_9 + \cdots + X_{2^{s+2}}X_{2^{s+3}})$ (s $\geq$ 3)</td>
<td></td>
</tr>
<tr>
<td>$2^{m-r+1} + 2^{m-r-1} - 2^{m-r-s-1}$ (10.01 $\cdots$ 1d) $\bar{s}$</td>
<td>(10) $X_1X_2X_3 + X_4(X_5X_6 + \cdots + X_{2^{s+1}}X_{2^{s+4}} + X_2)$ (s $\geq$ 3)</td>
<td></td>
</tr>
</tbody>
</table>
\[2^{m-r+1} \quad (10d)\]
\[2^{m-r+1} + 2^{m-r-2} + 2^{m-r-5} \quad (10.011d)\]
\[2^{m-r+1} + 2^{m-r-2} + 2^{m-r-3} + 2^{m-r-4} \quad (10.0111d)\]
\[2^{m-r+1} + 2^{m-r-1} - 2^{m-2r+3} \quad (10.01 \cdots 1d) \quad r-4\]
\[2^{m-r+1} + 2^{m-r-1} - 2^{m-2r+2} \quad (10.01 \cdots 1d) \quad r-3\]
\[2^{m-r+1} + 2^{m-r-1} - 2^{m-2r+1} \quad (10.01 \cdots 1d) \quad r-2\]
\[2^{m-r+1} + 2^{m-r-1} - 2^{m-2r+3} + 2^{m-2r+1} \quad (10.01 \cdots 101d) \quad r-4\]
\[2^{m-r+1} + 2^{m-r-1} - 2^{m-2r+2} + 2^{m-2r} \quad (10.01 \cdots 101d) \quad r-3\]

(11) \[X_1 X_2 \cdots X_r + (X_1 + 1) X_{r+1} \cdots X_{m-1}\]

(12) \[X_1 X_2 X_3 X_4 + X_5 (X_7 X_9 X_7 + X_8 X_7 X_9)\]
(13) \[X_1 X_2 X_3 X_4 + X_5 X_6 X_7 X_2 + (X_2 + 1) X_7 + X_8 X_9\]
(14) \[X_1 X_2 X_3 X_4 + X_5 (X_7 X_9 X_7 + X_7 X_9)\]
(15) \[X_1 X_2 X_3 X_4 X_5 + X_9 X_7 (X_3 X_4 X_9 + X_2 (X_5 + X_8) X_9)\]

(16) \[X_1 X_2 X_3 X_4 + X_5 (X_7 X_9 X_7 + X_5 X_9 X_{10})\]
(17) \[X_1 X_2 X_3 X_4 + X_5 X_6 X_7 X_5 + X_7 X_8 + X_9 X_{10}\]
(18) \[X_1 X_2 X_3 X_4 X_5 + X_9 X_7 (X_5 X_4 X_9 + (X_5 + X_3) X_9 X_{10})\]

(19) \[X_1 \cdots X_r + X_{r-1} \cdots X_{2r-3} (X_{r-2} X_{r-3} + X_{r-3} (X_r + 1)) \quad (r \geq 6)\]
(20) \[X_1 \cdots X_r + X_{r+1} \cdots X_{2r-3} (X_{r-1} + 1) + X_{r-3} X_{r-1} \quad (r \geq 5)\]

(21) \[X_1 \cdots X_r + X_{r+1} \cdots X_{2r-3} (X_{r-2} X_{r-3} + X_{r-3} X_r) \quad (r \geq 5)\]
(22) \[X_1 \cdots X_r + X_{r+1} \cdots X_{2r-3} (X_{r-2} X_{r-3} + X_{r-2} X_r) \quad (r \geq 5)\]
(23) \[X_1 \cdots X_r + X_{r+1} \cdots X_{2r-3} (X_{r-2} X_{r-3} + X_{r-2} X_r) \quad (r \geq 5)\]

(24) \[X_1 \cdots X_r + X_{r+1} \cdots X_{2r-3} (X_{r-2} X_{r-3} X_{r-4} + X_{r-3} (X_r + 1)) \quad (r \geq 5)\]
(25) \[X_1 \cdots X_r + X_{r+1} \cdots X_{2r-3} (X_{r-2} X_{r-3} X_{r-4} + X_{r-2} X_{r-3} (X_r + 1)) \quad (r \geq 5)\]

(26) \[X_1 \cdots X_r + X_{r+1} \cdots X_{2r-3} (X_{r-2} X_{r-3} X_{r-4} + X_{r-2} X_{r-3} X_r) \quad (r \geq 5)\]
(27) \[X_1 \cdots X_r + X_{r+1} \cdots X_{2r-3} (X_{r-2} X_{r-3} X_{r-4} + X_{2r-1} X_{r-1})\]

Where \(X_1, \ldots\) are mutually independent linear polynomials.
(2) Suppose that $f \in P_r$ and $0 < |f|_m < 2^{m-r+1}$. Then, $|f|_m$ is of the form

$$2^{m-r+1} - 2^{m-r-l}, \quad 0 \leq l \leq \max(m - r, (m - r + 2)/2).$$

(2.1) $|f|_m = 2^{m-r}$ if and only if $f \approx x_1 \cdots x_r$.

(2.2) $|f|_m = 2^{m-r+1} - 2^{m-r-l}$ if and only if

$$f \approx x_1 \cdots x_{r-l}(x_{r-l+1} \cdots x_r + x_{r+1} \cdots x_{r+l}) \quad \text{for } m \geq r + l$$

or

$$f \approx x_1 \cdots x_{r-2}(x_{r-1}x_r + x_{r+1}x_{r+2} + \cdots + x_{r+2l-3}x_{r+2l-2}) \quad \text{for } m \geq r + 2l - 2.$$

3. REPRESENTATIVE POLYNOMIALS AND THEIR COUNTING

The following theorem characterizes the polynomials of weight less than $2.5d$, and the proof is omitted.

**Theorem 1.** Suppose that $f \in P_r$ and $|f|_m < 2^{m-r+1} + 2^{m-r-1}$.

(1) If $r \geq 4$, then $f \in P_{r,2}$.

(2) If $r = 3$, then either $f \in P_{3,2}$ or

$$f \approx x_1x_2x_3 + x_4x_5x_6 + x_7g,$$

where $x_i$'s are mutually independent linear polynomials and $g \in P_2$ is independent of $x_7$. Furthermore, if $m(f) \geq 9$, then

$$f \approx x_1x_2x_3 + x_4x_5x_6 + x_7(x_8x_9 + ax_1x_4),$$

where $a \in GF(2)$.

(3) If $r = 3$ and $|f|_m < 2^{m-r+1} + 2^{m-r-3}$, then $f \in P_{3,2}$.

We present a set of representative polynomials such that any polynomial of weight $2d < |f|_m < 2.5d$ in $P_r - P_{r,2}$ with $r \geq 3$ is affine equivalent to one and only one polynomial of the set. For each representative polynomial, we count the number of those polynomials which are affine equivalent to the polynomial. As shall be explained in Section 4, we need not consider the polynomials such that $m(f) < 9$ and $|f|_m > 2d$. By Theorem 1, it is sufficient to consider the polynomials in $P_{r,2}$. The polynomial $f$ is expressed as

$$f = xg + yh + xyk,$$
where \( f \in P_r, g, h \in P_{r-1}, k \in P_{r-2} \) and \( g, h, k \) are independent of \( x, y \). Then,
\[
|f_m| = |g|_{m-2} + |h|_{m-2} + |g + h + k|_{m-2}.
\] (5)

There is no loss of generality in assuming that
\[
|g|_{m-2} \leq |h|_{m-2} \leq |g + h + k|_{m-2}.
\] (6)

It follows from Lemma 1 that we can consider the following two cases.

1. \( |g|_m = 2^{m-r-1}, \quad |h|_m = 2^{m-r} - 2^{m-r-s} \),
\[
|g + h + k|_{m-2} < 2^{m-r} + 2^{m-r-s} \quad \text{for} \quad s \geq 2,
\] (7)

2. \( |g|_{m-2} = |h|_{m-2} = 2^{m-r-1} + 2^{m-r-2}, \quad |g + h + k|_{m-2} < 2^{m-r}. \) (8)

Since the weights of \( g \) and \( h \) are less than twice the minimum weight, the form of these polynomials are given by Lemma 1. Furthermore, some close relation between \( g \) and \( h \) holds by the restriction of \( |g + h + k|_m \) and the following theorem is derived.

**Theorem 2.** Suppose that (1) \( f \in P_r - P_{r+1} \) with \( r \geq 3 \), (2) \( 2^{m-r+1} \leq |f|_m < 2^{m-r+1} \), and (3) \( m(f) \geq 9 \) if either \( r \geq 4 \) or \( |f|_m = 2^{m-r+1} + 2^{m-r+2} \). Then, \( f \) is affine equivalent to one of the polynomials listed in Table 1.

**Corollary 1.** Suppose that \( f \in P_3 - P_{a+1} \).

1. If \( |f|_m = 2^{m-2} \), then \( f \) is affine equivalent to one of the following forms:
\[
\begin{align*}
    x_1 x_2 x_3 + (x_1 + 1) x_4 x_5, \\
x_1 x_2 x_3 + x_4 (x_2 x_5 + x_3 x_6), \\
x_1 x_2 x_3 + x_4 (x_3 x_6 + x_5 x_7).
\end{align*}
\]

2. If \( 2^{m-2} \leq |f|_m < 2^{m-2} + 2^{m-5} \), then \( |f_m| = 2^{m-2} + 2^{m-6} \) and \( f \) is affine equivalent to \( x_1 x_2 x_3 + x_4 (x_3 x_6 + x_5 x_7) \).

The proofs of these results are lengthy and are omitted here. For the details, refer to the report of Kasami, Tokura, and Azumi (1974).

Let \( N(f) \) denote the number of polynomials \( f' \) such that \( f' \approx f \) in \( V_m \),

\footnote{By \( d \), we mean the minimum weight \( 2^{m-r} \) and \((10.0101d)\) is a binary representation.}

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### TABLE II

$N(f)$ of Representative Polynomials in Table I

<table>
<thead>
<tr>
<th>Weight</th>
<th>[No.]</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>17</th>
<th>31</th>
<th>73</th>
<th>127</th>
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<tbody>
<tr>
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<td>(1)</td>
<td>35</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$10.011d$</td>
<td>(2)</td>
<td>29</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(12)</td>
<td>29</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(13)</td>
<td>28</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(14)</td>
<td>29</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
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<tr>
<td></td>
<td>(15)</td>
<td>29</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$10.0111d$</td>
<td>(3)</td>
<td>35</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
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<td></td>
<td>(4)</td>
<td>38</td>
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<td>1</td>
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<td></td>
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<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td></td>
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<td>36</td>
<td>1</td>
<td>1</td>
<td>3</td>
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<td>1</td>
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<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(18)</td>
<td>34</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

$$
\begin{align*}
10.01 & \cdots 1d & \frac{s-2}{s-2} & (5) & 2s^{2}+2s+2 & \cdot \beta_{2s+1}^{3} & \cdot \gamma_{s}^{2} \\
10.01 & \cdots 1d & \frac{s-1}{s-1} & (6) & 2s^{2}+5s+2 & \cdot \beta_{2s+1}^{3} & \cdot \gamma_{s-1} \\
10.01 & \cdots 1d & \frac{s}{s} & (7) & 2s^{2}+7s+7 & \cdot \beta_{2s+1}^{3} & \cdot \gamma_{s} \\
10.01 & \cdots 101d & \frac{s-2}{s-2} & (8) & 2s^{2}+13s+3 & \cdot \beta_{2s+1}^{3} & \cdot \gamma_{s-2} \\
10d & & \frac{r-4}{r-4} & (9) & 2s^{2}+13s+3 & \cdot \beta_{r+1}^{3} & \cdot \gamma_{s-2} \\
10.01 & \cdots 1d & \frac{r-3}{r-3} & (10) & 2s^{2}+13s+3 & \cdot \beta_{r+1}^{3} & \cdot \gamma_{s-3} \\
10.01 & \cdots 1d & \frac{r-2}{r-2} & (11) & 2s^{2}+13s+3 & \cdot \beta_{r+1}^{3} & \cdot \gamma_{s-4} \\
10.01 & \cdots 101d & \frac{r-4}{r-4} & (12) & 2s^{2}+13s+3 & \cdot \beta_{r+1}^{3} & \cdot \gamma_{s-3} \\
10.01 & \cdots 101d & \frac{r-3}{r-3} & (13) & 2s^{2}+13s+3 & \cdot \beta_{r+1}^{3} & \cdot \gamma_{s-2} \\
10d & & \frac{r-3}{r-3} & (14) & 2s^{2}+13s+3 & \cdot \beta_{r+1}^{3} & \cdot \gamma_{s-3} \\
\end{align*}
$$

*Where $\beta_{m}=\prod_{i=0}^{m-1}(2^{m-i}-1)$, $\gamma_{s}=\prod_{i=0}^{m-1}(4^{s+i}-1)$, and assuming that the multiplication is preceded by the division.*
and let $I(f)$ be the number of the affine transformations which keep $f$ invariant. Then,

$$N(f) = 2^{m(m+1)/2} \prod_{i=0}^{m-1} (2^{m-i} - 1)^{I(f)}.$$  

(9)

As the arguments for finding $I(f)$ for each representative polynomial are straightforward but lengthy, they are omitted (see Kasami, Tokura, and Azumi, 1974). For some cases, $I(f)$ were found by a computer. By (9), we obtain $N(f)$ from $I(f)$. The results are tabulated in Table II.

For each polynomial $f$ in Table I, it is easy to check that the number of variables occurring in its expression is identical with $m(f)$.

If $f_1 \approx f_2$, then (1) $f_1$ and $f_2$ have the same degree, (2) $|f_1|_m = |f_2|_m$, (3) $m(f_1) = m(f_2)$, and (4) $I(f_1) = I(f_2)$. At least one of these conditions does not hold for any two polynomials in Table I. Hence, we have the following theorem.

**Theorem 3.** Any two polynomials in Table I for which $m(f) \geq 9$ are not affine equivalent.

4. **Enumerator Formulas**

Let $N_{m,r,w}$ be the number of codewords of weight $w$ of the $r$th order RM code of length $2^m$. For $r \geq 2$, let $N'_{m,r,w}$ be the number of polynomials $f$ such that $f \in P_r - P_{r-1}$ and $|f|_m = w$, and let $M_{m,r,w}$ be the number of polynomials $f$ such that $f \in P_r - P_{r-1} - P_{r-1}$, $|f|_m = w$ and $m(f) = m$. Let $N_{m,r,w} = N'_{m,r,w}$ for $r < 2$.

For simplicity, let

$$\alpha_{m,j} = \prod_{i=0}^{j-1} (2^{m-i} - 1)/(2^{j-i} - 1) \quad \text{for } 0 < j \leq m,$$

$$\alpha_{m,0} = 1 \quad \text{and} \quad \alpha_{m,j} = 0 \quad \text{for } j > m.$$ 

For $f \in P_r$, there exist mutually independent linear polynomials with no constant term $u_1, u_2, \ldots, u_{m(f)}$ such that

$$f = F(u_1, u_2, \ldots, u_{m(f)}),$$  

(10)

where $F$ is binary polynomial of $m(f)$ variables in $P_r$. Since the number of subspaces of dimension $j$ in $V_m$ is $\alpha_{m,j}$, we have that

$$N(f) = N_j(F) \alpha_{m,j},$$  

(11)
where \( j = m(f) \) and \( N_j(F) \) denotes the number of polynomials \( F' \) such that \( F' \equiv F \mod V_{m(f)} \). By definition, we have

\[
|f|_m = 2^{m-m(f)} |F|_{m(f)}.
\]  

(12)

It follows from (11), (12), and the results of Kasami-Tokura (1970) that

\[
N_{m,r,w} = \sum_{j=0}^{r} N'_{m-r+j,j,w} 2^{r-j} \xi_{m,r-j},
\]

(13)

\[
N'_{m,r,w} - N'_{m,r-1,w} = \sum_{j=r}^{m} M_{j,r,2^{-m}w} \xi_{m,j}
\]

for \( r \geq 2 \).

(14)

Now, consider \( N_{m,r,w} \) with \( 2^{m-r+1} \leq w < 2^{m-r+1} + 2^{m-r-1} \). By definition,

\[
N'_{m-r,0,w} = 0;
\]

(15)

\[
N'_{m-r+1,1,2^{m-r+1}} = 1,
\]

(16)

and if

\[
w > 2^{m-r+1},
\]

then

\[
N'_{m-r+1,1,w} = 0.
\]

(17)

Let \( f \in P_2 \). If \( |f|_{m-r+2} \) is of the form \( 2^m \ r+1 + \epsilon 2^m \ r+1 \), where \( l \geq 2 \) and \( \epsilon \) is either \(-1\) or \(1\), then by Lemma 1 \( f \) has no linear factor. That is, for \( w = 2^{m-r+1} + 2^{m-r-2} \) or \( 2^{m-r+1} + 2^{m-r-3} \), we have that

\[
N'_{m-r+2,2,w} = N'_{m-r+2,2,w}.
\]

(18)

By Lemma 1.1.2 and Sloane-Berlekamp formula (1970), we have

\[
N'_{m-r+2,2,2^{m-r+1}}
\]

\[
= 2 \left( 2^{(m-r+2)(m-r+3)/2} - 2^{m-r+5} \right) - \sum_{l=1}^{[m-r+2/2]} 2^{l(1+l)} \prod_{i=1}^{l} \left( 2^{m-r+4-2i} - 1 \right) \left( 2^{m-r+3-2i} - 1 \right)/(4^i - 1).
\]

(19)

\(^2\) Letting \([x]\) denote the integer part of \(x\).
For the remaining weights of Table I, we have

\[ N'_{m-r+2,2,w} = 0. \quad (20) \]

By Lemma 1.1, we have that

\[ N'_{m-r+i,i-1,w} = 0, \quad (21) \]

for \( 3 \leq i \leq r \) and \( 2^{m-r+1} < w < 2^{m-r+1} + 2^{m-r-1} \). Since \( f \in P_{i-1} \) such that \( |f|_{m-r+i} = 2^{m-r+1} \) has linear factors by Lemma 1.2.1, we have that

\[ N'_{m-r+i,i-1,2^{m-r+1}} = 0 \quad \text{for} \quad i \geq 3. \quad (22) \]

Hence, it follows from (13)-(22) that

\[ N_{m,r,w} = 2^{r-1} \alpha_{m,r-1} + N'_{m-r+2,2,w} 2^{r-3} \alpha_{m,r-2} \]

\[ = \sum_{i=3}^{r} \left( \sum_{j=i}^{m-r+i} M_{i,j,2^{i-1}} 2^{r-i} \alpha_{m-r+i,j} \right) 2^{r-i} \alpha_{m,r-1}, \]

where the first term is zero except for the case \( w = 2^{m-r+1} \), and the second term is zero except for the cases \( w = 2^{m-r+1} + 2^{m-r-1} \), or \( 2^{m-r+1} + 2^{m-r-2} \).

Then, for finding \( N_{m,r,w} \), it is sufficient to find \( M_{m,r,w} \) with \( r \geq 3 \). If \( N_{m',r',w'} \) with \( m' \leq m, r' \leq r, w' \leq w \) are known, then \( M_{m,r,w} \) can be found by (13) and (14).

Corollary 1.1 implies that

\[ M_{i,3,2i-2} = 0 \quad \text{for} \quad i \leq 4 \quad \text{and} \quad i \geq 8. \]

As \( N_{5,3,2^3}, N_{6,3,2^4} \) and \( N_{7,3,2^5} \) are known (Peterson and Weldon, 1972; Sloane and Berlekamp, 1970; Sugino et al., 1971), we can calculate \( M_{i,3,2i-2} \) with \( 5 \leq i \leq 7 \). Hence, we can find \( N_{8,3,2^6} \).

The weight distribution for the third-order RM code of length 256 can be calculated by using the MacWilliams–Pless identities (1963), \( N_{8,3,2^6} \), and previously known results. At the same time, the weight distribution of the dual code, that is, the fourth-order RM code of the same length can be found. The results\(^3\) are tabulated in Tables III and IV.

Then, \( M_{m,r,w} \) with \( m \leq 8, 2 \leq r \leq 5 \) and \( w \leq 2^n \) are calculated from (13), (14) and the known weight distributions of the RM codes (Sloane and

---

\(^3\) They are found independently by D. V. Sarwate and E. R. Berlekamp, and by H. C. A. Van Tilborg.
TABLE III
Weight Distribution of the (256, 93) Reed-Muller Code

<table>
<thead>
<tr>
<th>Weight</th>
<th>Number of codewords</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>256</td>
</tr>
<tr>
<td>32</td>
<td>224</td>
</tr>
<tr>
<td>48</td>
<td>208</td>
</tr>
<tr>
<td>56</td>
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<tr>
<td>64</td>
<td>192</td>
</tr>
<tr>
<td>68</td>
<td>188</td>
</tr>
<tr>
<td>72</td>
<td>184</td>
</tr>
<tr>
<td>76</td>
<td>180</td>
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<td>80</td>
<td>176</td>
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<td>88</td>
<td>168</td>
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<tr>
<td>120</td>
<td>136</td>
</tr>
<tr>
<td>124</td>
<td>132</td>
</tr>
<tr>
<td>128</td>
<td>130</td>
</tr>
</tbody>
</table>

Berlekamp, 1970; Sugino et al., 1971; Azumi, 1972; Peterson and Weldon, 1972). For the weights of nonzero $N_{m,r,w}$, McEliece’s theorem (1967) implies that

$$w \equiv 0 \mod 2^{m/2-1}. \quad (24)^4$$

Hence, it follows from Lemma 1.1 that

$$w \equiv 0 \mod 2^{m-r-1} \quad \text{for nonzero } N_{m,r,w},$$

that is,

$$M_{m,r,w} = 0, \quad (25)$$

where $6 \leq r \leq m \leq 8$ and $2^{m-r+1} \leq w < 2^{m-r+1} + 2^{m-r-1}$. Thus, $M_{m,r,w}$ with $m \leq 8$ are found. This is the reason why we omit representative polynomials such that $m(f) < 9$ and $|f|_m > 2d$.

---

$^4 [x]$ is the smallest integer not less than $x$. 
## TABLE IV

Weight Distribution of the (256, 163) Reed–Muller Code

<table>
<thead>
<tr>
<th>Weight</th>
<th>Number of codewords</th>
</tr>
</thead>
<tbody>
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<td>240</td>
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<td>30</td>
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<td>128</td>
<td>128</td>
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</tbody>
</table>
By Table II, \( M_{m,r,w} \) with \( m \geq 9, r \geq 3, \) and \( 2^{m-r+1} \leq w < 2^{m-r+1} + 2^{m-r-1} \) are found too.

Then, by using (23), (25) and known \( M_{m,r,w} \) (Azumi, 1972), the weight enumerator formulas for \( N_{m,r,w} \) of any order with

\[
2^{m-r+1} \leq w < 2^{m-r+1} + 2^{m-r-1}
\]

are obtained. Formulas for nonzero \( N_{m,r,w} \) are listed below and that for other weights in the interval is \( N_{m,r,w} = 0. \)

**Enumerator Formulas for**

\[
2^{m-r+1} \leq w < 2^{m-r+1} + 2^{m-r-1}
\]

(a) \( N_{m,r,2^{m-r+1}} \)

\[
= 2^{r-1} \cdot \alpha_{m,r-1} + 2^{r-1} \cdot \left( 2^{(m-r+2)(m-r+3)}/2 - 2^{m-r+2} \right)
- \sum_{l=1}^{[m-r+2/2]} 2^{l(l+1)} \prod_{i=1}^{l} \left( 2^{m-r+4-2i} - 1 \right)
\times \left( 2^{m-r+3-2i} - 1 \right) \cdot \left( 4^{i} - 1 \right) \cdot \alpha_{m,r-2}
+ 2^{r+5} \cdot 5 \cdot 7 \cdot 31 \cdot (\alpha_{m-r+3,5} + 2^{3} \cdot 3^{3} \cdot 7 \cdot \alpha_{m-r+3,6})
+ 2^{3} \cdot 3^{3} \cdot 7 \cdot 127 \cdot \alpha_{m-r+3,2} \cdot \alpha_{m,r-3}
+ 2^{r+10} \cdot 3^{2} \cdot 7 \cdot 31 \cdot (\alpha_{m-r+4,6} + 2 \cdot 5 \cdot 7 \cdot 127 \cdot \alpha_{m-r+4,7}) \cdot \alpha_{m,r-4}
+ \sum_{k=4}^{r} 2^{r-k+2} \cdot \beta_{m-r+k} \cdot \alpha_{m,r-k} \cdot (\beta_{k-1} / \beta_{k-1})
\]

(b) \( N_{m,r,2^{m-r+1}+2^{m-r+1}} \)

\[
= 2 \cdot 2 \cdot N_{m-r+2,2^{m-r+1}+2^{m-r+1}} \cdot \alpha_{m,r-2}
+ 2^{r+22} \cdot 3^{2} \cdot 5 \cdot 7 \cdot 17 \cdot 31 \cdot 127 \cdot \alpha_{m-r+3,8} \cdot \alpha_{m,r-3}
+ 2^{r+19} \cdot 3 \cdot 5^{2} \cdot 7^{2} \cdot 17 \cdot 31 \cdot 127 \cdot \alpha_{m-r+4,8} \cdot \alpha_{m,r-4}
\]

(c) \( N_{m,r,2^{m-r+1}+2^{m-r+1}} \)

\[
= 2^{r-2} \cdot N_{m-r+2,2^{m-r+1}+2^{m-r+1}} \cdot \alpha_{m,r-2}
+ 2^{r+11} \cdot 3 \cdot 7 \cdot 31 \cdot (7 \cdot 23 \cdot \alpha_{m-r+3,8} + 2^{2} \cdot 5 \cdot 127 \cdot 661 \cdot \alpha_{m-r+3,7})
+ 2^{8} \cdot 5^{3} \cdot 7^{3} \cdot 13 \cdot 17 \cdot 127 \cdot \alpha_{m-r+3,8}
\]
WEIGHTS OF REED–MULLER CODES

\[ + 2^{18} \cdot 3 \cdot 5 \cdot 7^2 \cdot 17 \cdot 73 \cdot 127 \cdot \alpha_{m-r+3, 9} \cdot \alpha_{m-r-3} \]
\[ + 2^{r+15} \cdot 3 \cdot 7^2 \cdot 31 \cdot 127 \cdot (3^2 \cdot 5 \cdot \alpha_{m-r+14, 7} + 2^3 \cdot 17 \cdot 19 \cdot 53 \cdot \alpha_{m-r+4, 8} \]
\[ + 2^9 \cdot 3 \cdot 5^2 \cdot 17 \cdot 73 \cdot \alpha_{m-r+4, 9} \cdot \alpha_{m-r-4} \]
\[ + 2^{r+23} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 31 \cdot 127 \cdot (11 \cdot \alpha_{m-r+5, 8} \]
\[ + 3 \cdot 5 \cdot 7 \cdot 73 \cdot \alpha_{m-r+5, 9} \cdot \alpha_{m-r-5} \cdot \]

(d) \[ N_{m, r, 2^m-r+1; 2^m-r+2; 2^m-r-4} \]
\[ = 2^{r+32} \cdot 3 \cdot 5 \cdot 17 \cdot 31 \cdot 73 \cdot 127 \cdot (5 \cdot \alpha_{m-r+3, 9} \]
\[ + 2^9 \cdot 7 \cdot 11 \cdot 31 \cdot \alpha_{m-r+3, 10} \cdot \alpha_{m-r-3} \]
\[ + 2^{r+32} \cdot 3 \cdot 7^2 \cdot 11 \cdot 17 \cdot 31^2 \cdot 73 \cdot 127 \cdot \alpha_{m-r+4, 11} \cdot \alpha_{m-r-4} \]
\[ + 2^{r+30} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31^2 \cdot 73 \cdot 127 \cdot \alpha_{m-r+5, 10} \cdot \alpha_{m-r-5} \cdot \]

(e) \[ N_{m, r, 2^m-r+1; 2^m-r+2; 2^m-r-3} \]
\[ = 2^{r+31} \cdot 3 \cdot 5 \cdot 7^2 \cdot 17 \cdot 31 \cdot 127 \cdot (2 \cdot 463 \cdot \alpha_{m-r+3, 8} \]
\[ + 3 \cdot 11 \cdot 73 \cdot 317 \cdot \alpha_{m-r+3, 9} + 2^6 \cdot 3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 73 \cdot \alpha_{m-r+3, 10} \]
\[ + 2^{14} \cdot 11 \cdot 23 \cdot 31 \cdot 73 \cdot 89 \cdot \alpha_{m-r+3, 11} \cdot \alpha_{m-r-3} \]
\[ + 2^{r+24} \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 31 \cdot 127 \cdot (3 \cdot 181 \cdot \alpha_{m-r+4, 8} \]
\[ + 2^9 \cdot 5 \cdot 7 \cdot 41 \cdot 73 \cdot \alpha_{m-r+4, 9} \cdot \alpha_{m-r+3, 9} \]
\[ + 2^{r+24} \cdot 3^2 \cdot 5 \cdot 7 \cdot 17 \cdot 31 \cdot 73 \cdot 127 \cdot (5 \cdot 7 \cdot 23 \cdot \alpha_{m-r+5, 8} \]
\[ + 2^9 \cdot 3^2 \cdot 5 \cdot 11 \cdot 31 \cdot \alpha_{m-r+5, 10} \]
\[ + 2^{14} \cdot 11 \cdot 23 \cdot 31 \cdot 89 \cdot \alpha_{m-r+5, 11} \cdot \alpha_{m-r-5} \]
\[ + 2^{r+31} \cdot 3^2 \cdot 7 \cdot 11 \cdot 17 \cdot 31^2 \cdot 73 \cdot 127 \cdot (7 \cdot \alpha_{m-r+6, 10} \]
\[ + 2^9 \cdot 5 \cdot 23 \cdot 89 \cdot \alpha_{m-r+6, 11} \cdot \alpha_{m-r-6} \cdot \]

(f) \[ N_{m, r, 2^m-r+1; 2^m-r+2; 2^m-r-2; 2^m-r-4} \]
\[ = 2^{r+31} \cdot 3 \cdot 7^2 \cdot 31 \cdot 73 \cdot 127 \cdot (2 \cdot 5^2 \cdot 7 \cdot 17 \cdot \alpha_{m-r+3, 9} \]
\[ + 2^9 \cdot 11 \cdot 17 \cdot 31 \cdot 47 \cdot \alpha_{m-r+3, 10} \]
\[ + 5 \cdot 11 \cdot 17 \cdot 23 \cdot 31 \cdot 73 \cdot 89 \cdot \alpha_{m-r+3, 11} \]
\[ + 2^8 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 31 \cdot 89 \cdot \alpha_{m-r+3, 12} \]
\[ + 2^{18} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 31 \cdot 89 \cdot 8191 \cdot \alpha_{m-r+3, 13} \cdot \alpha_{m-r-3} \]
\[ \begin{align*}
+ 2^{r+32} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 31^2 \cdot 73 \cdot 127 \cdot 139 \cdot \alpha_{m-r+4,10} \cdot \alpha_{m,r-4} \\
+ 2^{r+20} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 31^2 \cdot 73 \cdot 127 \cdot (5^2 \cdot 11 \cdot \alpha_{m-r+6,10} \\
+ 2^6 \cdot 23 \cdot 89 \cdot \alpha_{m-r+5,11} \cdot \alpha_{m,r-6} \\
+ 2^{r+136} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 23 \cdot 31 \cdot 73 \cdot 89 \cdot 127 \cdot (31 \cdot \alpha_{m-r+6,11} \\
+ 2^4 \cdot 3^2 \cdot 7 \cdot 13 \cdot 31 \cdot \alpha_{m-r+6,12} \\
+ 2^{11^h} \cdot 3 \cdot 7 \cdot 13 \cdot 8191 \cdot \alpha_{m-r+6,13} \cdot \alpha_{m,r-6} \\
+ 2^{r+43} \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 31 \cdot 73 \cdot 89 \cdot 127 \cdot (3 \cdot \alpha_{m-r+7,12} \\
+ 2^4 \cdot 8191 \cdot \alpha_{m-r+7,13} \cdot \alpha_{m,r-7}.
\end{align*} \]

\[(g) \quad N_{m,r,2^{m-r+1},2^{m-r-1},2^{m-r-1},2^{m-r-k-1}} \quad (w = 10.01 \cdots 1d, m - r - \lfloor m/r \rfloor \geq k \geq 4)
\]

\[= 2^{r+k^2+58+7} \cdot \beta_{m-r+3} \cdot (1/3 \cdot 7 \cdot \gamma_k^{k-1} + 2^{k-3} \cdot 41/3 \cdot \gamma_k^{k-3} + 2^{k+6} \cdot \gamma_k^{k+1}) \quad \alpha_{m,r-3} \\
+ 2^{r+k^2+58+6} \cdot \beta_{m-r+k+2} \cdot \gamma_k^{k+1} \cdot \beta_{k+1} \cdot \beta_{k+2} \cdot \alpha_{m,r-k-2} \\
+ 2^{r+k^2+58+9} \cdot \beta_{m-r+k+3} \cdot \gamma_k^{k+1} \cdot \beta_{k+1} \cdot \beta_{k+2} \cdot \alpha_{m,r-k-3} \\
+ 2^{r+k^2+58+13} \cdot \beta_{m-r+k+4} \cdot (1/3 \cdot 5 \cdot \beta_{k+1} \cdot \beta_{k+2} + 2^{k+1} \cdot \beta_{k+3} \cdot \gamma_k^{k+1} \\
+ 2^{r+k^2+58+16} \cdot \beta_{k+1} \cdot \beta_{k+2} \cdot \alpha_{m,r-k-4}.
\]

\[(h) \quad N_{m,r,2^{m-r+1},2^{m-r-1},2^{m-r-k-1},2^{m-r-k-1}} \quad (w = 10.01 \cdots 101d, m - r - \lfloor m/r \rfloor - 2 \geq k \geq 2)
\]

\[= 2^{r+k^2+11k+22} \cdot \beta_{m-r+3} \cdot \alpha_{m,r-3} \cdot (3 \cdot 7 \cdot \gamma_k^{k+2} + 2^{k+2} \cdot 3 \cdot \beta_{k+1} \cdot \beta_{k+2} \cdot \alpha_{m,r-k-3} \\
+ 2^{r+k^2+19k+22} \cdot \beta_{m-r+k+3} \cdot \alpha_{m,r-k-3} \cdot 3^3 \cdot \beta_{k+1} \cdot \beta_{k+3} \\
+ 2^{r+k^2+19k+29} \cdot \beta_{m-r+k+4} \cdot \alpha_{m,r-k-4} \cdot 3^2 \cdot \beta_{k+2}.
\]

where

\[
\alpha_{m,i} = \prod_{i=0}^{i=0} (2^{m-i} - 1)/(2^{i-1} - 1) \quad \text{for } 0 < j \leq m,
\]

\[
\alpha_{m,0} = 1, \quad \alpha_{m,j} = 0 \quad \text{for } j < 0 \text{ or } j > m,
\]

\[
\beta_m = \prod_{i=0}^{i=0} (2^{m-i} - 1), \quad \text{and} \quad \gamma_k = \prod_{i=0}^{i=0} (4^{i+1} - 1).
\]

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WEIGHTS OF REED–MULLER CODES

REFERENCES