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Some recent results on niche graphs

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Abstract

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In an earlier paper entitled "Niche graphs" written by Cable, Jones, Lundgren and Seager, niche graphs were introduced and examples were provided of graphs which have niche number 0, 1, 2, and ∞. However, no examples were found of a niche graph having finite niche number 3 or larger. We still have had no success in our efforts to find such a graph. Nevertheless we have gotten some interesting results. For example, we show in this paper that if there is such a graph, then there must be one which is connected. We also show that the niche number of a graph which has a finite niche number is $\leq \frac{2}{3}|V(G)|$. In addition we determine the niche number of all "wheel" graphs.

Keywords. Niche number, wheel.

1. Introduction

Niche graphs were introduced by Cable, Jones, Lundgren and Seager [1] in 1988. In this paper we extend the theory of niche graphs in our efforts to find a niche graph whose niche number is finite and greater than 2.

If $D=(V,A)$ is a digraph, assumed throughout to be weakly connected, and acyclic, and if $[x,z]\in A$ we will say that x is a predator of z and z is a prey of x. The *niche graph* corresponding to D is the undirected graph $G=(V,E)$ with an edge between distinct elements x and y of V if and only if for some $z\in V$, there are arcs $[x,z]$ and $[y,z]$ in D or there are arcs $[z,x]$ and $[z,y]$ in D. In other words, $[x,y]\in E$ if and only if x and y have either a common predator or a common prey in D. In this connection, an element x of V (in the context of either D or G) will be called *isolated* if x is isolated (in the usual sense) as a vertex of G. Since all digraphs considered here are weakly connected, this usage should cause no confusion.

While it is easy to determine the niche graph corresponding to any digraph, it was shown in [1] that not all graphs are niche graphs. It is fairly easy to show in any niche graph G there exists a pair of nonadjacent vertices by simply using the fact that in an acyclic digraph there exists a vertex y with $\text{indeg}(y)=0$ and a vertex z such that $\text{outdeg}(z)=0$. It follows then that if $m \geq 2$, then K_m is not a niche graph. However if x is an isolated vertex not in K_m, then it is easy to see that $K_m \cup \{x\}$ is a niche graph. We show this for $m=3$ in Fig. 1.

This gives rise to the idea of the niche number of a graph [1]. We restate the definition here. Let I_k denote k isolated vertices. Then the *niche number* of G is the smallest number, k, of isolated vertices such that $G \cup I_k$ is the niche graph of an acyclic digraph. It is denoted $n(G)$. It is obvious from this definition and our discussion above that $n(K_m)=1$ for $m \geq 2$. From [1] we know that $n(C_4)=2$ where C_4 is a cycle on 4 vertices and that $n(P_m)=0$ if $m \geq 3$, where P_m is a path on m vertices. It is also shown in [1] that there is an infinite class of graphs (called novas in [1]) which cannot be made into niche graphs by the addition of any finite number of isolated vertices. For such graphs we say that the niche number is infinite and write $n(G)=\infty$. Interestingly enough, we have been unable to find any graphs with finite niche number greater than 2, nor have we been able to prove such graphs do not exist. However we have obtained some results in our efforts to settle this matter. We show that if there is a graph having a finite niche number greater than 2, then there is one which is connected. In order to accomplish this we first show that if D is an acyclic digraph associated with G which has the fewest number of isolated vertices and if x is an isolated vertex of G, then in D either $\text{indeg}(x)=0$ or $\text{outdeg}(x)=0$. Moreover, we have been able to show that if G has a finite niche number, then its niche number is at most $\frac{2}{3}|V(G)|$ (improving the bound of $|V(G)|$ obtained in [1]).

2. Niche number and connectedness

In this section we show that if $G_1, G_2, \ldots, G_r$ is a set of disjoint graphs each having a niche number less than or equale to 2, then the graph G which is the union of $G_1, \ldots, G_r$ is one which has niche number less than or equal to 2. We say that an

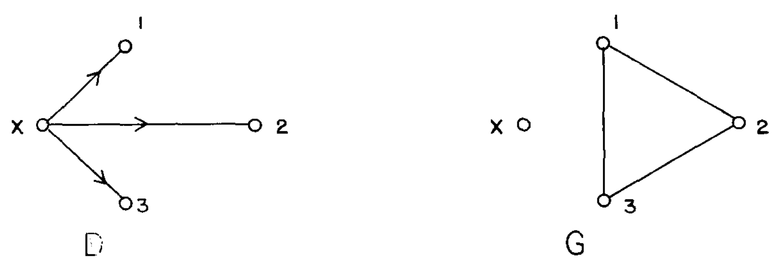

Fig. 1.

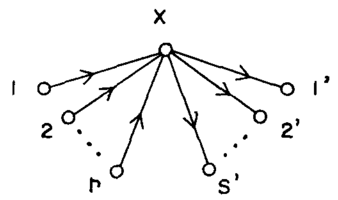

Fig. 2.

acyclic digraph D is *niche minimal* for the undirected graph G if (1) the niche graph associated with D is $G \cup I_k$ and (2) if D' is an acyclic digraph with niche graph $G \cup I_{k'}$, then $k' \geq k$.

Lemma 2.1. *If G is a graph such that $n(G) = k < \infty$, D is an acyclic digraph whose niche graph is $G \cup I_k$ and if x is an isolated vertex in I_k, then in D,* $\mathrm{outdeg}(x) = 0$ *or* $\mathrm{indeg}(x) = 0$.

Proof. Assume that there is an acyclic digraph D for G satisfying conditions stated in the lemma, and that there is an isolated vertex x of D such that $\mathrm{indeg}(x) > 0$ and $\mathrm{outdeg}(x) > 0$. See Fig. 2. By definition, D is niche minimal. We notice that there are no arcs leaving the vertices $1, 2, \ldots, t$ except the ones indicated which lead to x, since x is an isolated vertex. By the same argument there are no arcs leading into $1', 2', \ldots, s'$ except those which originate at x.

In general D will have additional vertices and arcs but we have pictured in Fig. 2 only the vertices adjacent in D to the vertex x and their corresponding arcs, since these are the items which are germane to the argument.

We construct D' from D as follows. D' consists of the vertex set of D with x removed and the same arc set as D, except we replace the arcs shown in Fig. 2 by those shown in Fig. 3. Specifically we replace $\{[1,x], \ldots, [t,x], [x,1'], \ldots, [x,s']\}$ by $\{[1,1'], \ldots, [1,s'], [2,1'], \ldots, [t,1']\}$. It is easy to see that D' is acyclic (since D is) and that D' has niche graph $G \cup (I_k - \{x\})$. The existence of this D' contradicts the niche minimality of D. $\square$

The following corollary is an immediate consequence of Lemma 2.1 and is used later.

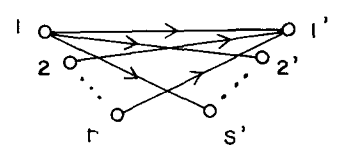

Fig. 3.

Corollary 2.2. *No path in a niche-minimal digraph D can pass through an isolated vertex.*

We now use the preceding lemma to prove the following theorem.

Theorem 2.3. *If $G_1, G_2, \dots, G_r$ are graphs such that $n(G_i) \leq 2$ for $1 \leq i \leq r$ and G is the disjoint union of $G_1, G_2, \dots, G_r$, then $n(G) \leq 2$.*

Proof. Let G_1 and G_2 be graphs. If $n(G_1)=0$ or $n(G_2)=0$, it is clear that $n(G_1 \cup G_2) \leq 2$. Suppose that $n(G_1)>0$ and $n(G_2)>0$. Let D_1 and D_2 be niche-minimal acyclic digraphs corresponding to G_1 and G_2 respectively. Let x be an isolated vertex of G_1 and y be an isolated vertex of G_2. By Lemma 2.1 (and possibly reversing all arcs in one or both digraphs) we can assume without loss of generality that $\text{outdeg}(x)=0$ and $\text{indeg}(y)=0$. See Fig. 4. If we identify vertices x and y we obtain a new digraph D' which is acyclic and has one fewer isolated vertex than the sum $n(G_1)+n(G_2)$. Since $\text{outdeg}(xy)>0$ and $\text{indeg}(xy)>0$, by Lemma 2.1, D' is not niche minimal. Therefore a niche-minimal digraph for $G_1 \cup G_2$ has at most $n(G_1)+n(G_2)-2$ isolated vertices. This implies that $n(G_1 \cup G_2) \leq 2$. Repeating this argument we see that if G is the union of a finite number of disjoint components $G_1, G_2, \dots, G_r$, then $n(G) \leq 2$. □

3. Niche number of wheels and an upper bound theorem

We first refine a result from [1] and use this to obtain the niche numbers of the class of graphs called wheels. We then concentrate on determining an upper bound for the niche number of an arbitrary graph in terms of its order. In the remainder of the paper we will use "clique" to mean a (not necessarily maximal) complete subgraph.

Notation. Let $\omega(G)$ represent the clique number of G, for $x \in V(G)$ let $\omega_x(G)$ = size of a largest clique in G containing x, let $G \setminus \{x\}$ be the subgraph of G generated by $V(G)-\{x\}$, and let $d(x)$ be the degree of vertex x in G.

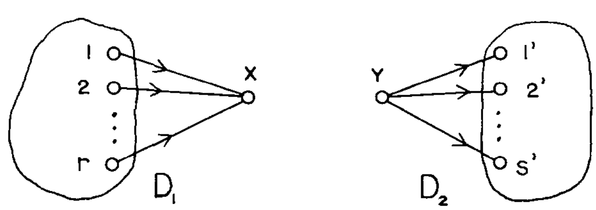

Fig. 4.

Theorem 3.1. *If G is a graph with finite niche number and x is any vertex of G, then $d(x) \le 2\omega(G \setminus \{x\})[\omega_x(G)-1]$.*

Proof. Since G has finite niche number, there is an acyclic digraph D with niche graph $G \cup I_K$. Fix $x \in V(G)$, let P be the set of predators of x and let Q be the set of vertices of D (other than x) having a common predator with x in D. Since $P \subseteq V(G \setminus \{x\})$ and since P generates a clique in G, it generates a clique in $G \setminus \{x\}$. It follows that

$$|P| \le \omega(G \setminus \{x\}). \tag{1}$$

For each $y \in P$, let $Q(y)$ be the set of prey of y other than x, if any; then

$$Q = \bigcup_{y \in P} Q(y). \tag{2}$$

Since $Q(y) \cup \{x\}$ generates a clique in G containing x, it follows that

$$|Q(y)| \le \omega_x(G) - 1. \tag{3}$$

Combining (1), (2), and (3)

$$\begin{aligned} |Q| &\le \sum_{y \in P} |Q(y)| \le \sum_{y \in P} [\omega_x(G)-1] = |P|[\omega_x(G)-1] \\ &\le \omega(G \setminus \{x\})[\omega_x(G)-1]. \end{aligned}$$

If R is the set of vertices of D (other than x) having common prey with x in D, then by a similar argument

$$|R| \le \omega(G \setminus \{x\})[\omega_x(G)-1].$$

Though R and Q need not be disjoint, still $d(x) \le |Q| + |R|$ so the proof is complete. □

Theorem 3.1 is a refinement of a result from [1] which can be stated as follows:

> If G is a graph with finite niche number, then for every vertex x of G $d(x) \le 2\omega(G)[\omega(G)-1]$.

Corollary 3.2 exploits the benefits of this refinement.

Let W_m represent the "wheel" on $m+1$ vertices, i.e.,

$$V(W_m) = \{x_0, x_1, \ldots, x_m\}$$

and

$$E(W_m) = \{[x_0, x_i],\ [x_i, x_{i+1}]:\ i = 1, \ldots, m-1,\ [x_0, x_m],\ [x_1, x_m]\}.$$

Corollary 3.2. *If $m \ge 9$, then $n(W_m) = \infty$.*

Proof. If x_0 is the center of W_m, then $\omega_x(W_m) = 3$ and $\omega(W_m \setminus \{x_0\}) = 2$, violating the conclusion of Theorem 3.1, which means that $n(W_m) = \infty$ for $m = d(x_0) > 8$. □

Comment. A careful analysis of the proof (rather than the result) of Theorem 3.1 can be used to show that $n(W_8)=\infty$. The (quite tedious) details are omitted here. $W_3=K_4$ so (as mentioned in the introduction) $n(W_3)=1$ and for $m=4,5,6,7$ digraphs have been found to demonstrate that $n(W_m)=0$. As an example, we present a niche-minimal digraph for W_7 in Fig. 5.

The final result gives a bound on (finite) niche number in terms of the order of the original graph. The proof we give requires the following.

Lemma 3.3. *In a niche-minimal acyclic digraph D with niche graph $G\cup I_k$, every $b\in I_k$ is adjacent in D to at least one vertex which is not adjacent to any vertex in $I_k-\{b\}$.*

Proof. Let D be a niche-minimal acyclic diagraph and suppose $b\in V(D)$ is isolated. Assume, without loss of generality, that b is a source and suppose (contrary to the conclusion of this lemma) that every prey of b is a predator of an isolated vertex. We note that this means that each prey of b has exactly one prey. Let $\tilde{b}$ be any fixed (isolated) prey of a prey of b and let D_1 be the subgraph of D generated by b, $\tilde{b}$, all prey of b, and all predators of $\tilde{b}$. If every prey of b is a predator of $\tilde{b}$, then b is superfluous, i.e., the digraph obtained from D by removing vertex b and all arcs originating at b has niche graph $G\cup(I_K-\{b\})$. This contradicts the assumed niche minimality of D. Thus b has at least one prey, say y, which is not a predator of $\tilde{b}$. Likewise there is a predator $\tilde{y}$ of $\tilde{b}$ which is not a prey of b.

Now let D_1' be the digraph on $V(D_1)$ obtained by replacing each arc in D_1 of the form $[z,\tilde{b}]$ by $[z,y]$ and each arc in D_1 of the form $[b,z]$ by $[\tilde{y},z]$ (see Fig. 6 for D_1 and D_1'), and let D' be the digraph obtained from D by replacing D_1 with D_1'. It is straightforward that D and D' have the same niche graph. Now a cycle in D' must, since D is acyclic, pass through a vertex of D_1', and must therefore pass through a prey (in D) of b and so must pass through an isolated vertex, denoted b' (due to the assumption that the prey of every prey of b is isolated). Clearly b' is outside

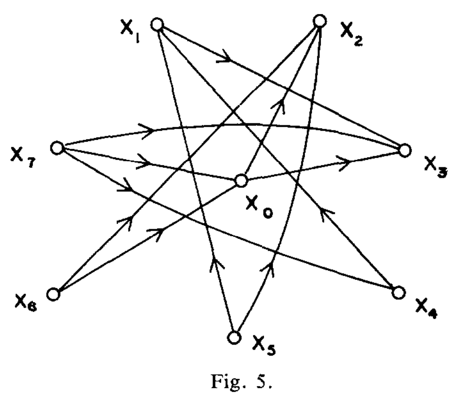

Fig. 5.

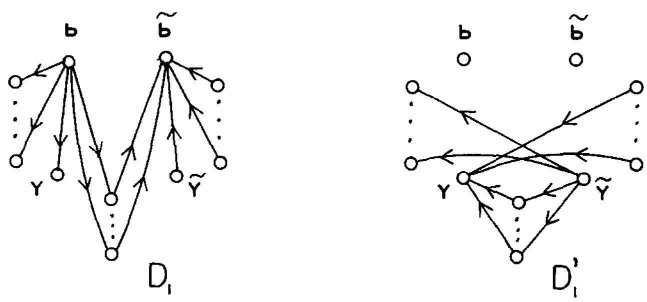

Fig. 6.

$V(D_1)=V(D_1')$ and so b' is an isolated vertex in D with positive in- and outdegree. This contradicts Corollary 2.2 since D is assumed niche minimal. So D' is acyclic. This, finally, contradicts the niche-minimal nature of D since b and $\tilde{b}$ are both superfluous in D'. Thus no such vertex b can exist. □

Theorem 3.4. *If $n(G)$ is finite, then $n(G) \le \frac{2}{3}|V(G)|$.*

Proof. Let D be a niche-minimal acyclic digraph with niche graph $G \cup I_K$ where $k=n(G)$. Let

$$V_i = \{y \in V(G)\colon y \text{ is adjacent in } D \text{ to exactly } i \text{ vertices of } I_k\}$$
$$\text{for } i=1,2,$$
$$A_2 = \{b \in V(I_k)\colon \exists\, y \in V(G) \text{ such that } y \text{ is adjacent in } D \text{ to both } b \text{ and another vertex of } I_k\},$$
$$A_1 = V(I_k) - A_2.$$

Note in the definition of A_2 that no such vertex y can be adjacent in D to more than two isolated vertices, i.e., vertices of I_K. We now define $f\colon V_1 \to V(I_k)$ by

$$f(y)=b \quad \text{iff} \quad b \in V(I_k) \text{ and } y \text{ is adjacent in } D \text{ to } b.$$

By construction of V_1, f is well defined and $V_1 = \bigcup_{b \in V(I_K)} f^{-}(b)$ is a partition of V_1, from which it follows that

$$|V_1| = \sum_{b \in V(I_K)} |f^{-1}(b)|. \tag{4}$$

By Lemma 3.3, $|f^{-1}(b)| \ge 1$ for all $b \in V(I_K)$. Every vertex in $V(I_K)$ must be adjacent in D to at least two nonisolated vertices since D is niche minimal. If $b \in A_1$, then these adjacent vertices are in V_1 and so $b \in A_1 \Rightarrow |f^{-1}(b)| \ge 2$. Finally, notice $|A_2| \le 2|V_2|$. Combining these observations with (4) one obtains

$$|V_1| = \sum_{b \in A_1} |f^{-1}(b)| + \sum_{b \in A_2} |f^{-1}(b)|$$
$$\ge 2|A_1| + |A_2|$$

and so

$$|V(G)| \geq |V_1| + |V_2| \geq 2|A_1| + |A_2| + \tfrac{1}{2}|A_2| \geq \tfrac{3}{2}(|A_1| + |A_2|)$$
$$= \tfrac{3}{2}|V(I_K)| = \tfrac{3}{2}n(G). \qquad \square$$

In closing we comment that Theorem 3.4 can probably be improved substantially. The use of Theorem 3.4 has already had a direct impact on the speed of an algorithm developed by one of the authors to compute niche numbers. Any further improvement would have an additional impact. The observation mae in [1] is, as far as we know, still true: no graph with finite niche number greater than two is known.

References

[1] C. Cable, K.F. Jones, J.R. Lundgren and S. Seager, Niche graphs, Discrete Appl. Math. 23 (1989) 231–241.