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ADJACENT VERTICES ON THE b-MATCHING POLYHEDRON

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Given a graph $G=(V,E)$ and an integer vector $b\in\mathbf{N}^V$, a b-matching is a set of edges $F\subset E$ such that any vertex $v\in V$ is incident to at most b_v edges in F. The adjacency on the convex hull of the incidence vectors of the b-matchings is characterized by a very general adjacency criterion, the coloring criterion, which is at least sufficient for all 0–1-polyhedra and which can be checked in the b-matching case by a linear algorithm.

1. Introduction

Let $G=(V,E)$ be an undirected graph without loops and multiple edges. For a vertex $v\in V$, let $\omega(v)$ denote the set of all edges incident to v. Now let $b\in\mathbf{N}^V$ be a tuple of positive integers assigned to the vertices of G. Then a subset of edges $F\subset E$ is called a *b-matching* (c.f. [1, p. 150], [4, p. 77]), if each vertex $v\in V$ is incident to at most b_v edges in F, i.e. if

$$(1)\qquad |\omega(v)F|\le b_v \quad \text{for any } v\in V.$$

(Throughout this paper we will use the notation AB for $A\cap B$ where A and B are any two sets.) A b-matching can be interpreted as a 0–1-vector $x\in\{0,1\}^E$ with the property

$$(2)\qquad \sum_{e\in\omega(v)} x_e\le b_v \quad \text{for any } v\in V.$$

Different but closely related is the concept of an *integer b-matching* which is a nonnegative integer vector $x\in\mathbf{N}^E$ satisfying (2). Some authors use the term b-matching for integer b-matching. A b-matching F where $b=(1,\ldots,1)$ is simply called a *matching*.

Due to the great importance of matchings and b-matchings for combinatorial optimization problems, these graph-theoretical concepts have been extensively studied in the literature. In particular it has been shown by Edmonds [3] that the

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matching problem (i.e. the determination of a maximum-weight-sum (b-) matching) can be solved by a polynomial algorithm. Moreover, Edmonds examined the *b-matching polyhedron*, i.e. the convex hull of the incidence vectors x^F of the b-matchings F, and gave a complete (however redundant) description of this polyhedron in terms of linear inequalities. The facets of a special kind of the b-matching polyhedron (G a complete graph and $b=(2,\dots,2)$) were determined in [6], the facets of the integer b-matching polyhedron were described in [11].

In this paper we will give another characterization of the b-matching polyhedron by developing a sufficient and necessary condition for the adjacency of (the incidence vectors of) two b-matchings on the b-matching polyhedron. The adjacency structure of the ordinary matching polyhedron was studied by Chvátal in [2] but his methods seem not to carry over to the general case. Therefore we characterize here the adjacency on the b-matching polyhedron by the so-called *coloring criterion*, a very general adjacency criterion developed in [9]. This criterion is sufficient for the adjacency on any 0–1-polyhedron and–as was shown in [5] and [7]–also necessary for many polyhedra belonging to important combinatorial optimization problems.

In Section 2 we will describe the coloring criterion for the b-matching case. Section 3 derives some properties of the "coarsest coloring" and using these, Section 4 proves the main result. Finally in Section 5 we show that the coloring criterion for the adjacency of two b-matchings in a graph with n vertices can be checked by an algorithm with time and space complexity $O(n)$. We will prove all results for an arbitrary integer vector $b\in\mathbf{N}^V$ but recommend that the reader concentrate on the case $b=(2,\dots,2)$ because in this special case the essential ideas are much easier to comprehend.

2. Coloring criterion

Let $G=(V,E)$ be an undirected graph, $b\in\mathbf{N}^V$ a positive integer vector, S the set of all b-matchings in G, and P the convex hull of the incidence vectors x^F of the b-matchings $F\in S$. Two distinct b-matchings $F_1,F_2\in S$ are called *adjacent* if their incidence vectors x^{F_1}, x^{F_2} lie on a common edge of the polyhedron P, i.e. if there are no b-matchings $F_3,\dots,F_r$ satisfying

$$\text{(3)}\qquad \sum_{i=1}^{2}\alpha_i x^{F_i}=\sum_{j=3}^{r}\alpha_j x^{F_j},\qquad \sum_{i=1}^{2}\alpha_i=\sum_{j=3}^{r}\alpha_j=1,$$
$$\alpha_j\geqslant 0,\quad 1\leqslant j\leqslant r,\qquad F_i\neq F_j,\quad i\leqslant 2,\ j\geqslant 3.$$

Obviously, (E,S) is an *independence system*, that is, a subset of a b-matching is also a b-matching. The *circuits* of (E,S), i.e. the minimal subsets of E not belonging to S, are the sets Z_v satisfying

$$\text{(4)}\qquad Z_v\subset\omega(v),\quad |Z_v|=b_v+1,\qquad v\in V.$$

A *coloring* of a set A is a partition of A into nonvoid subsets, i.e. a subset C of the power set of A such that $K \neq \emptyset$, $KL = \emptyset$ for any $K, L \in C$ and $\bigcup C = A$ ($\bigcup C$ denotes the union of all sets in C). The sets $K \in C$ are called color classes. The purpose of this paper is to prove that the following criterion is sufficient and necessary for the adjacency of $F_1, F_2 \in S$:

Coloring criterion. For any coloring C of the symmetric difference $F_1 \Delta F_2$ $(= F_1 \cup F_2 \backslash F_1 F_2)$ with $|C| \geq 2$ there exist two color classes $K, L \in C$ and a circuit Z_v such that

(5) $$Z_v \subset KF_1 \cup LF_2 \cup F_1 F_2.$$

As the number of all colorings of $F_1 \Delta F_2$ is extraordinarily high, it seems to be hard, if not impossible, to check the validity of this criterion. But in fact it is enough to check (5) for at most $|V|$ colorings using the concept of *feasible colorings*. To give a recursive definition of feasible colorings, we call a coloring C of $F_1 \Delta F_2$ feasible if either $|C'| = |F_1 \Delta F_2|$ (every edge in $F_1 \Delta F_2$ has a different color) or if there exists a feasible coloring C with $|C'| = |C| - 1$, a circuit Z_v, and two color classes $K, L \in C$ satisfying (5) and

(6) $$C' = C \backslash \{K, L\} \cup \{K \cup L\}.$$

Obviously the feasible colorings are just the colorings produced by the *coloring algorithm* that starts with the coloring $C = \{\{e\} \mid e \in F_1 \Delta F_2\}$ (every edge a distinct color) and that, whenever there exist color classes $K, L \in C$ and a circuit Z_v with (5), "combines" K and L to a common color class $K \cup L$.

A coloring C' is called *coarser* than a coloring C if any color class of C is contained in a color class of C'. For example equality (6) implies that C' is coarser than C. The following theorem which relates the coloring criterion and the concept of feasible colorings was proved in [7] and [9].

Theorem 1. *Each performance of the coloring algorithm leads to the same final coloring C which is the coarsest feasible coloring of $F_1 \Delta F_2$. The coloring criterion is fulfilled iff C consists of only one color class (i.e. all edges have the same color).*

Let us illustrate these concepts by an example where the graph $G = (V, E)$ and the vector $b \in \mathbb{N}^V$ are given by Fig. 1. For each vertex v the number on the left denotes the number of the vertex, the number on the right the b_v attached to v. Let F_1 and F_2 be the set of edges drawn as straight and wavy lines, respectively. Clearly they are b-matchings. Let us now apply the coloring algorithm. First of all we recognize that the vertices $v = 1, 2, 4, 5, 7, 8$ are incident to exactly one edge in $F_1 \backslash F_2$ and to one in $F_2 \backslash F_1$ and moreover to $b_v - 1$ edges in $F_1 F_2$. Therefore the corresponding sets $\omega(v)$ are circuits which permit to combine the two edges in $F_1 \Delta F_2$ to a common color class. Thus after six steps of the coloring algorithm, we

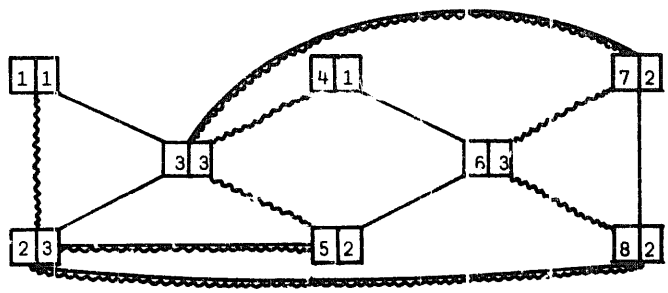

Fig. 1.

get the following feasible coloring

$$C=\{K=\{12,13,23\},\ L=\{34,46\},\ M=\{35,56\},\ N=\{67,68,78\}\}.$$

The circuits $Z_3=\{13,23,34,37\}$ and $Z_3'=\{13,23,35,37\}$ satisfy

$$Z_3\subset F_1K\cup F_2L\cup F_1F_2,\qquad Z_3'\subset F_1K\cup F_2M\cup F_1F_2.$$

Hence the three color classes K, L, M can be combined in two steps to a new class $K\cup L\cup M$. Finally $Z_6=\{46,56,67,68\}$ is a circuit with

$$Z_6\subset F_1(K\cup L\cup M)\cup F_2N\cup F_1F_2,$$

thus also the two remaining color classes $K\cup L\cup M$ and N can be combined. By Theorem 1, $C=\{F_1\Delta F_2\}$ is the coarsest feasible coloring and by our main results in Section 4, F_1 and F_2 are adjacent.

A closer look on the example above reveals a remarkable phenomenon: In any feasible coloring produced during the coloring algorithm, each color class K possesses at most two vertices which "contact" another color class L. For example, the only contact vertex for color class K is vertex 3; the contact vertices for color class $K\cup L$ are vertices 3 and 4. As we will show in the next section, this property holds for all b-matchings and has great importance for the proof of the necessity of the coloring criterion as well as for an efficient implementation of the coloring algorithm.

3. Properties of the feasible colorings

Let $G=(V,E)$ be a graph, $b\in\mathbf{N}^V$ a positive integer vector, F_1, F_2 two distinct b-matchings in G, C a feasible coloring, and $K\in C$ a color class. A vertex $v\in V$ is called a *contact vertex* of K if v is incident to an edge in K and to an edge in another color class $L\in C\setminus\{K\}$. The following lemma will be the main tool for the proof of our adjacency criterion:

Lemma 1. *Let K be a color class of a feasible coloring C. Then:*

(a) *K has at most two contact vertices.*

(b) *For each contact vertex v we have:*
$|\omega(v)K| = 1$,
or $|\omega(v)K| = 2$ *and v is the only contact vertex of K,*
or $|\omega(v)K| = 3$ *and* $|\omega(v)F_1F_2| = b_v - 2$,
or $|\omega(v)K| = 4$, $|\omega(v)KF_1| = |\omega(v)KF_2| = 2$ *and v is the only contact vertex of K.*

Proof. We prove the lemma by an induction on $|C|$. If $|C| = |F_1 \Delta F_2|$, every color class of C consists of a single edge and the lemma is clearly true. Now let C' be a feasible coloring with $|C'| < |F_1 \Delta F_2|$. By the definition of feasible colorings, there exists a feasible coloring C with $|C'| = |C| - 1$, a circuit Z_w, and $K, L \in C$ satisfying (5) and (6). By the induction hypothesis, the assertion holds for C and has only to be shown for the new color class $K \cup L$. The following two relations are obvious:

(7) $\quad |Z_wKF_1| + |Z_wLF_2| + |Z_wF_1F_2| = b_w + 1,$

(8) $\quad |Z_wF_i| \leq |\omega(w)F_i| \leq b_w \quad$ for $i \in \{1, 2\}$.

(a) Clearly any contact vertex of $K \cup L$ is also a contact vertex of K or of L. Vertex w is a common contact vertex of K and L and by part (a) of the lemma for C, each of these sets has at most one contact vertex apart from w. Hence $K \cup L$ can have at most three contact vertices. However we will show now that one of these three candidates is in fact not a contact vertex of $K \cup L$ which yields (a) for $K \cup L$. By part (b) of the lemma for C, one of the four stated cases holds for K and L. It is easy to see that in each of these cases we have

(9) $\quad |Z_wKF_1| \leq 2, \qquad |Z_wLF_2| \leq 2.$

In view of (7), this implies

(10) $\quad |Z_wF_1F_2| \geq b_w - 3.$

This leads to the following four cases for $|Z_wF_1F_2|$:

Case 1. $|Z_wF_1F_2| \geq b_w - 1$. By (7), (8), we have then: $|Z_wKF_1| = |Z_wLF_2| = 1$ and $Z_wMF_i = \emptyset$ for any $M \in C \setminus \{K, L\}$ and $i \in \{1, 2\}$. Hence w is not a contact vertex of $K \cup L$.

Case 2. $|Z_wF_1F_2| = b_w - 2$. By (7) and (8) we can without loss of generality assume that $|Z_wKF_1| = 2$ and $|Z_wLF_2| = 1$. If moreover $\omega(w)KF_2 \neq \emptyset$ then as in case 1, w is not a contact vertex of $K \cup L$. Otherwise we have $|\omega(w)K| = 2$ and part (b) of the lemma implies that K has no contact vertex different from w.

Case 3. $|Z_wF_1F_2| = b_w - 3$. By (7) and (9) we have then $|Z_wKF_1| = |Z_wLF_2| = 2$. If moreover $\omega(w)KF_2 \neq \emptyset \neq \omega(w)LF_1$, then $|Z_w(K \cup L \cup F_1F_2)F_i| = b_w$ for $i \in \{1, 2\}$, hence w is no contact vertex of $K \cup L$. Otherwise we have without loss of generality $|\omega(w)K| = 2$, and by part (b), w is the only contact vertex of K. In all these cases there are at most two vertices which are contact vertices of $K \cup L$.

(b) Let v be a contact vertex of $K \cup L$. Without loss of generality assume $|\omega(v)K| \geq |\omega(v)L|$.

Case 1. $|\omega(v)(K\cup L)|=2$. If v were identical to w, then (7) and (8) would imply

$$|\omega(v)K|=|\omega(v)L|=1, \qquad |\omega(v)F_1F_2|=b_v-1$$

and v could not be a contact vertex of $K\cup L$, a contradiction. Thus $v\neq w$. If $\omega(v)L=\emptyset$, then v would be a contact vertex of K with $|\omega(v)K|=2$; hence v would be the only contact vertex of K, thus $v=w$, a contradiction. Hence $\omega(v)L\neq\emptyset$ and also $\omega(w)K\neq\emptyset$, therefore v is a common contact vertex of K and L. Since K as well as L has at most one contact vertex apart from w, v is the only contact vertex of $K\cup L$.

Case 2. $|\omega(v)(K\cup L)|=3$. If $|\omega(v)K|=3$, the assertion $|\omega(v)F_1F_2|=b_v-2$ follows immediately from the lemma for C. Hence we can assume that $|\omega(v)K|=2$, $|\omega(v)L|=1$. Therefore v is the only contact vertex of K, thus $v=w$ and

$$|\omega(v)F_1F_2|\geqslant|Z_wF_1F_2|\geqslant b_v-3.$$

$|\omega(v)F_1F_2|=b_v-3$ would imply the contradiction

$$\begin{aligned}|Z_w|&=|Z_w(K\cup L)|+|Z_wF_1F_2|\\&\leqslant|\omega(v)(K\cup L)|+|\omega(v)F_1F_2|=b_v=b_w.\end{aligned}$$

And if $|\omega(v)F_1F_2|\geqslant b_v-1$, then $|\omega(v)(K\cup L)|=3$ would imply $|\omega(v)F_i|>b_v$ for some $i\in\{1,2\}$, a contradiction. Thus $|\omega(v)F_1F_2|=b_v-2$.

Case 3. $|\omega(v)(K\cup L)|=4$. If $|\omega(v)K|=4$, the assertion for this case follows immediately. If $|\omega(v)K|=3$ and thus $|\omega(v)L|=1$, the lemma for C would imply $|\omega(v)F_1F_2|=b_v-2$. Then clearly

$$|\omega(v)(K\cup L\cup F_1F_2)F_i|=b \quad\text{for}\quad i\in\{1,2\},$$

and v could be no contact vertex of $K\cup L$, a contradiction. Therefore

$$(11)\qquad |\omega(v)K|=|\omega(v)L|=2.$$

By part (b) of the lemma, v is the only contact vertex of K as well as of L, thus $v=w$ and v is the only contact vertex of $K\cup L$. Let us suppose now that $|\omega(v)KF_1|\leqslant 1$. Then, by (11), $|\omega(v)KF_2|\geqslant 1$ and from (7) and (8):

$$\begin{aligned}|\omega(v)(LF_2\cup F_1F_2)|&\geqslant|Z_w(LF_2\cup F_1F_2)|\\&=b_w+1-|Z_wKF_1|\\&\geqslant b_w+1-|\omega(v)KF_1|\geqslant b_w.\end{aligned}$$

But this yields the contradiction

$$\begin{aligned}|\omega(v)F_2|&\geqslant|\omega(v)KF_2|+|\omega(v)(LF_2\cup F_1F_2)\\&\geqslant 1+b_v.\end{aligned}$$

Therefore $|\omega(v)KF_1|\geqslant 2$; in view of (11) this implies

$$|\omega(v)(K\cup L)F_1|=|\omega(v)KF_1|=2,$$

and analogously one gets

$$|\omega(v)(K\cup L)F_2|=|\omega(v)LF_2|=2.$$

Case 4. $|\omega(v)(K\cup L)|\geqslant 5$. The lemma for C implies that $|\omega(v)K|\leqslant 4$. If $|\omega(v)K|=3$, hence $|\omega(v)L|\geqslant 2$, it would follow $|\omega(v)F_1F_2|=b_v-2$ yielding a contradiction to $|\omega(v)F_i|\leqslant b_v$, $i\in\{1,2\}$. Hence $|\omega(v)K|=4$, $|\omega(v)L|\geqslant 1$. Then we have $v=w$, $|\omega(v)KF_i|=2$ for $i\in\{1,2\}$. But since

$$Z_w\subset\omega(w)KF_1\cup\omega(w)(LF_2\cup F_1F_2),$$

it follows

$$|\omega(w)(LF_2\cup F_1F_2)|\geqslant|Z_w|-|\omega(w)KF_1|=b_w+1-2=b_w-1,$$
$$|\omega(w)F_2|\geqslant|\omega(w)KF_2|+|\omega(w)(LF_2\cup F_1F_2)|\geqslant 2+b_w-1=b_w+1,$$

a contradiction.

4. Main result

Using the lemma just proved, we can now show that the coloring criterion is necessary and sufficient for the adjacency of b-matchings.

Theorem 2. *Let $G=(V,E)$ be a graph, $b\in\mathbf{N}^V$ and F_1, F_2 two distinct b-matchings. Then the following statements are equivalent:*

(i) *F_1, F_2 are adjacent b-matchings.*

(ii) *There are no b-matchings F_3, F_4 different from F_1, F_2 with the property $F_1\cup F_2=F_3\cup F_4$, $F_1F_2=F_3F_4$.*

(iii) *The coloring criterion (with respect to F_1, F_2) is satisfied.*

Proof. The two implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) hold for arbitrary 0–1-polyhedra and have been proven in [7] and [9]. Thus we will give only a brief sketch of the proofs.

"(i) $\Rightarrow$ (ii)". If there were b-matchings F_3, F_4 as in (ii), then $\frac{1}{2}(x^{F_1}+x^{F_2})=\frac{1}{2}(x^{F_3}+x^{F_4})$ would be a representation of the form (3).

"(iii) $\Rightarrow$ (i)". Suppose there is a representation (3). Then it can be shown by induction that for any feasible coloring C the following property holds

(12) $\quad KF_j=KF_1 \quad$ or $\quad KF_j=KF_2 \quad$ for any $\ j\geqslant 3 \ $ and $\ K\in C.$

By Theorem 1, (iii) implies that $C=\{F_1\Delta F_2\}$ is a feasible coloring and (12) for $K=F_1\Delta F_2$ yields $F_j=F_1$ or $F_j=F_2$, a contradiction to (3). The induction mentioned above uses the recursive definition of feasible colorings and the fact that a b-matching F_j and two color classes K and L with (12) which can be combined to $K\cup L$ according to a circuit Z_v with (5) cannot fulfil $KF_j=KF_1$ and $LF_j=LF_2$.

"(ii) $\Rightarrow$ (iii)". This is the most important result of this paper. Let (ii) be satisfied and let C be the coarsest feasible coloring. Then there is no "partial coloring" C_1 with $\emptyset \neq C_1 \subsetneqq C$ such that, where $C_2 = C \setminus C_1$, each contact vertex v incident to an edge in $\bigcup C_1$ and to an edge in $\bigcup C_2$ satisfies

$$(13) \qquad |\omega(v)((F_i \cap \bigcup C_1) \cup (F_{-i} \cap \bigcup C_2) \cup F_1F_2)| \leqslant b_v \quad \text{for } i \in \{1,2\}.$$

(F_{-i} is to denote F_1 if $i = 2$ and F_2 if $i = 1$.) For if there existed such a C_1, then

$$F_3 = (F_1 \cap \bigcup C_1) \cup (F_2 \cap \bigcup C_2) \cup F_1F_2,$$
$$F_4 = (F_2 \cap \bigcup C_1) \cup (F_1 \cap \bigcup C_2) \cup F_1F_2$$

would yield a contradiction to (ii). We suppose now that C consists of more than one color class and get a contradiction by constructing a partial coloring C_1 satisfying (13). Then the assertion follows immediately from Theorem 1. We give a short sketch of the proof:

First we will show in Proposition 1 that out of the four cases which are permitted by the lemma in Section 3, only the first and, with further restrictions, the second can hold in our context.

This enables the construction of a sequence (K_j) of color classes analogous to a path in a graph. In Proposition 2 we will show that this "path" leads through all color classes of C. We then define "simple vertices", that are vertices which are–apart from "loops"–passed only once. In Proposition 3, we exclude the occurrence of most such simple vertices and then perform the main construction. The following argument is used several times in the proof. If a vertex v satisfies

$$(14) \qquad |\omega(v)F_1 \cap \bigcup C_1| = |\omega(v)F_2 \cap \bigcup C_1|,$$

then it follows for $i \in \{1,2\}$:

$$\begin{aligned}&|\omega(v)((F_i \cap \bigcup C_1) \cup (F_{-i} \cap \bigcup C_2) \cup F_1F_2)| = \\ &\quad = |\omega(v)((F_{-i} \cap \bigcup C_1) \cup (F_{-i} \cap \bigcup C_2) \cup F_1F_2)| \\ &\quad \leqslant |\omega(v)F_{-i}| \leqslant b_v\end{aligned}$$

and thus inequality (13) holds for v. Now we have

Proposition 1. *Any contact vertex v of a color class K of a coarsest feasible coloring C satisfies $|\omega(v)K| = 1$ or $= 2$; if $|\omega(v)K| = 2$, then $b_v \geqslant 3$. The inclusion $\omega(v)K \subset F_i$ holds for some $i \in \{1,2\}$. But there are no two color classes $K, L \in C$ such that $|\omega(v)K| = |\omega(v)L| = 2$, $\omega(v)K \subset F_1$, $\omega(v)L \subset F_2$.*

Proof. Let v be a contact vertex of a color class $K \in C$, it is incident with an edge $vw \in L \in C \setminus \{K\}$. By part (b) of the lemma we have the following cases:

Case 1. $|\omega(v)K| = 1$. In this case, nothing remains to be shown.

Case 2. $|\omega(v)K| = 2$. If $\omega(v)K$ consisted of an edge in F_1 and of one in F_2, then $C_1 = \{K\}$ would be a partial coloring with (14) and thus with (13), a contradiction. Therefore $\omega(v)K \subset F_i$ for some $i \in \{1,2\}$ Clearly $b_v \geqslant 2$. If $b_v = 2$, then $vw \in F_{-i}L$

and $Z_v = \omega(v)K \cup \{vw\}$ is a circuit which shows that $C \setminus \{K, L\} \cup \{K \cup L\}$ is a coarser feasible coloring, a contradiction. Therefore $b_v \geq 3$. Now suppose that also $|\omega(v)L| = 2$ and $\omega(v)L \subset F_{-i}$. Then $C_1 = \{K, L\}$ satisfies (14) and thus (13). Hence if $C \neq C_1$, this C_1 would lead to a contradiction. Therefore $C = \{K, L\}$. But if $b_v = 3$, then $\omega(v)$ is a circuit because of which $C \setminus \{K, L\} \cup \{K \cup L\}$ is a coarser feasible coloring, a contradiction, and if $b_v \geq 4$, then $C_1 = \{K\}$ is a partial coloring with (13). Thus in any case we get a contradiction.

Case 3. $|\omega(v)K| = 3$. By part (b) of the lemma, we have $|\omega(v)F_1F_2| = b_v - 2$. Hence there is an $i \in \{1, 2\}$ with $|\omega(v)F_iK| = 2$, $|\omega(v)F_{-i}K| = 1$, and $vw \in F_{-i}L$. Because of the circuit $Z = \omega(v)(F_iK \cup F_1F_2) \cup \{vw\}$, $C \setminus \{K, L\} \cup \{K \cup L\}$ is a coarser feasible coloring, contradiction.

Case 4. $|\omega(v)K| = 4$. Then by part (b) of the lemma, $C_1 := \{K\}$ satisfies (14) and thus (13).

Now we construct a sequence $(v_s, K_{s+1}, v_{s+1}, \ldots, v_{t-1}, K_t, v_t)$ which has properties similar to those of an alternating path in a graph, namely

(15) The K_j's, $s < j \leq t$, are distinct color classes in C.

(16) Each v_j, $s < j < t$, is a common contact vertex of K_j and K_{j+1}.

(17) If $v_j = v_{j+1}$, then this vertex is incident to exactly two edges of K_{j+1}, hence (by the lemma) it is the only contact vertex of K_{j+1}.

(18) For any j, $s < j < t$, there is an $i \in \{1, 2\}$ such that $\omega(v_j)K_j \subset F_i$, $\omega(v_j)K_{j+1} \subset F_{-i}$.

(19) The sequence is maximal, i.e. it cannot be extended on either side without violating one of the properties (15) to (18) above.

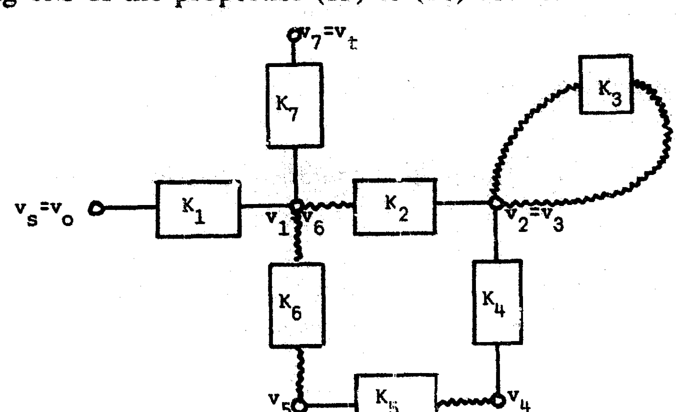

Fig. 2.

The existence of such a (nonvoid) sequence is evident: We start with a sequence consisting of a single color class and extend it on either side as long as possible without violating (15) to (18). A great deal of the technical difficulties in the following argument is due to the possibility of "loops" like at vertex v_2 in Fig. 2;

note that, by Proposition 1, in the case $b_v = 2$ for all $v \in V$, such "loops" do not exist.

Proposition 2. $\{K_{s+1}, \ldots, K_t\} = C$.

Proof. Let $C_1 = \{K_{s+1}, \ldots, K_t\}$. By (18) and Proposition 1, equation (14) and thus (13) hold for $C_1 = \{K_{s+1}, \ldots, K_t\}$ and each vertex v_j with $v_s \neq v_j \neq v_t$. Moreover, $\omega(v_t)K_t \subset F_i$ for some $i \in \{1, 2\}$ and by (18)

$$|\omega(v_t)F_{-i} \cap \bigcup C_1| \leq |\omega(v_t)F_i \cap \bigcup C_1|.$$

By (19) and Proposition 1, $C_2 = C \setminus C_1$ satisfies

$$\omega(v_t)F_{-i} \cap \bigcup C_2 = \emptyset.$$

Therefore

$$\begin{aligned} &|\omega(v_t)((F_i \cap \bigcup C_1) \cup (F_{-i} \cap \bigcup C_2) \cup F_1F_2| \leq |\omega(v_t)F_i| \leq b_{v_t}, \\ &|\omega(v_t)((F_{-i} \cap \bigcup C_1) \cup (F_i \cap \bigcup C_2) \cup F_1F_2)| \\ &\quad = |\omega(v_t)F_{-i} \cap \bigcup C_1| + |\omega(v_t)((F_i \cap \bigcup C_2) \cup F_1F_2)| \\ &\quad \leq |\omega(v_t)F_i \cap \bigcup C_1| + |\omega(v_t)((F_i \cap \bigcup C_2) \cup F_1F_2)| \\ &\quad \leq |\omega(v_t)F_i| \leq b_{v_t}. \end{aligned}$$

Analogous inequalities hold also at vertex v_s. Hence $C_1 = \{K_{s+1}, \ldots, K_t\}$ fulfills (13) and so $C_1 = C$.

Now we call a vertex $v \in \{v_s, \ldots, v_t\}$ a *simple vertex* if $\{j \mid v_j = v\} = \{j', j'+1, \ldots, j''\}$ with $j' \leq j''$. Because of (17), (18), and Proposition 1, this definition is equivalent to $\{j \mid v_j = v\} = \{j'\}$ or $= \{j', j'+1\}$. Any other vertex from $\{v_s, \ldots, v_t\}$ is called a *multiple vertex*. For example in Fig. 2, $v_1 = v_6$ is a multiple vertex, any other vertex is a simple vertex.

Proposition 3. *After an appropriate renumbering, each v_j with $v_s \neq v_j \neq v_t$ is a multiple vertex.*

Proof. Suppose v with $v_s \neq v \neq v_t$ is a simple vertex. Suppose further that there exists a second simple vertex w. Without loss of generality we can assume

$$s < \min\{j \mid v_j = v\} = j' < j'' = \max\{j \mid v_j = w\} \leq t.$$

(In Fig. 2, consider the simple vertices $v = v_2$ and $w = v_4$ with $j' = 2, j'' = 4$.) Let $C_1 = \{K_{j'+1}, \ldots, K_{j''}\}$ and $C_2 = C \setminus C_1$. By (18) there is an $i \in \{1, 2\}$ with $\omega(v)K_{j'} \subset F_{-i}$, $\omega(v)K_{j'+1} \subset F_i$. Hence

$$\begin{aligned} &\omega(v)((F_i \cap \bigcup C_1) \cup (F_{-i} \cap \bigcup C_2) \cup F_1F_2) \\ &\quad = \omega(v)(F_iK_{j'+1} \cup F_{-i}K_{j'} \cup F_1F_2). \end{aligned}$$

If the cardinality of this set exceeded b_v, it would contain a circuit combining $K_{j'}$ and $K_{j''+1}$, a contradiction. Therefore (13) is satisfied for this index i, and for the other one it follows from

$$\begin{aligned}&|\omega(v)((F_{-i}\cap \bigcup C_1)\cup(F_i\cap \bigcup C_2)\cup F_1F_2)|\\ &\quad\leq|\omega(v)(F_iK_{j'+2}\cup F_1F_2|\leq b_v.\end{aligned}$$

Analogously (13) follows for w, and all other vertices $v_{j'+2},\ldots,v_{j''}$ satisfy (14) and thus (13). Moreover, because of $s<j'\leq j''$ we have $\emptyset\neq C_1\neq C$. This yields a contradiction.

Thus $v=v_{j'}$ is the only simple vertex. Suppose now that $v_s\neq v_t$. Then, for similar reasons as above, $C_1=\{K_{j'+1},\ldots,K_t\}$ satisfies inequality (13) at all vertices $v_{j'},\ldots,v_{t-1}$ and–as can easily be deduced from $v_s\neq v_t$ and Proposition 2–also at v_t. Moreover $s<j'<t$ implies $\emptyset\neq C_1\neq C$. Thus this assumption, too, yields a contradiction. Hence $v_s=v_t$ (the sequence $(v_s,K_{s+1},\ldots K_t,v_t)$ corresponds to a cycle). But then we can renumber the sequence such that $v=v_s=v_t$. Since v is the only simple vertex, the proposition is proved.

Now we come to the main argument of the proof: As C has more than one color class, Proposition 2 implies $|\{K_{s+1},\ldots,K_t\}|=|C|\geq 2$; hence there is a vertex v with $v_s\neq v\neq v_t$. By Proposition 3, v is a multiple vertex. Then there are indices j,k,l such that $s\leq j<k<l$ and $v=v_j=v_l\neq v_k$ (cf. Fig. 3).

Let v be chosen such that under the conditions above, the index l is minimal. Now v_k is a multiple vertex, for if it were a simple vertex, the definition of j,k,l would imply $v_s\neq v_k\neq v_t$ and by Proposition 3, v_k would nevertheless be a multiple vertex. Hence there are indices $m',m\neq k$ such that $k<m'<m$ and $v_k=v_m\neq v_{m'}$. By the minimality of l we have $k<l\leq m$ and $v_m=v_k\neq v_l$ implies even $l<m$. By Proposition 1 there is an $i\in\{1,2\}$ such that $\omega(v)K_{j+1}\subset F_i$ and either $\omega(v)K_l\subset F_{-i}$ or $\omega(v)K_l\subset F_i$. Let us first assume $\omega(v)K_l\subset F_{-i}$. Then $C_1=\{K_{j+1},\ldots,K_l\}$ satisfies

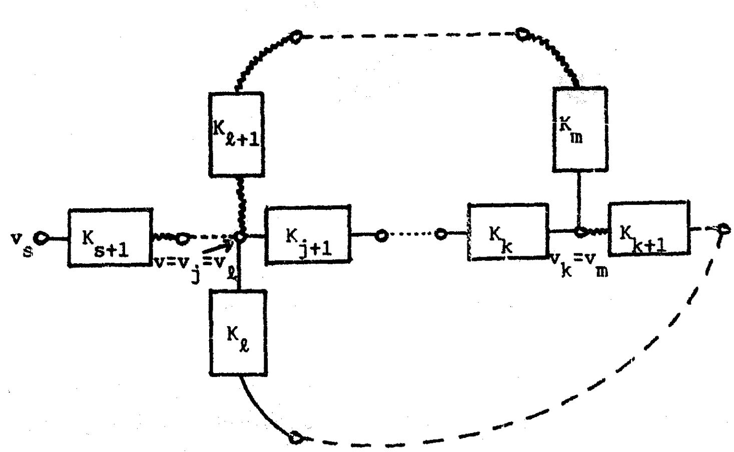

Fig. 3.

(14), thus (13) and as C_1 does not contain the color classes $K_{l+1}, \ldots, K_m$, we have also $\emptyset \neq C_1 \neq C$, a contradiction. Therefore $\omega(v)(K_{j+1} \cup K_l) \subset F_i$ and (by (18)) $\omega(v)K_{l+1} \subset F_{-i}$. Without loss of generality we can now assume that $\omega(v_k)K_k \subset F_1$, $\omega(v_k)K_{k+1} \subset F_2$. Now we define

$$C_1 = \begin{cases} \{K_{l+1}, \ldots, K_m\} \cup \{K_{k+1}, \ldots, K_l\} & \text{if } \omega(v_k)K_m \subset F_1, \\ \{K_{l+1}, \ldots, K_m\} \cup \{K_{j+1}, \ldots, K_k\} & \text{if } \omega(v_k)K_m \subset F_2 \end{cases}$$

in both cases, C_1 satisfies (14), thus (13) and $\emptyset \neq C_1 \neq C$. This contradiction terminates the proof.

5. Complexity of the coloring algorithm

Finally we show that Lemma 1 is of predominant importance, not only for the theoretical proof of necessity of the coloring criterion but also for the practical implementation of the coloring algorithm. In particular it implies that there are only three possibilities to combine two color classes $K, L \in C$ (see also [8]):

Lemma 2. *Two color classes K and L of a feasible coloring C and a circuit Z_w with $|Z_wK| \geq |Z_wL|$ satisfy*

(20) $\qquad Z_w \subset KF_i \cup LF_{-i} \cup F_1F_2$

iff one of the following cases holds for some $i \in \{1, 2\}$ (cf. Fig. 4 for $b_w = 4$):

(a) $|Z_wF_1F_2| = b_w - 1, |Z_wKF_i| = |Z_wLF_{-i}| = 1,$

(b) $|Z_wF_1F_2| = b_w - 2, |Z_wKF_i| = 2, |Z_wLF_{-i}| = 1,$

(c) $|Z_wF_1F_2| = b_w - 3, |Z_wKF_i| = |Z_wLF_{-i}| = 2.$

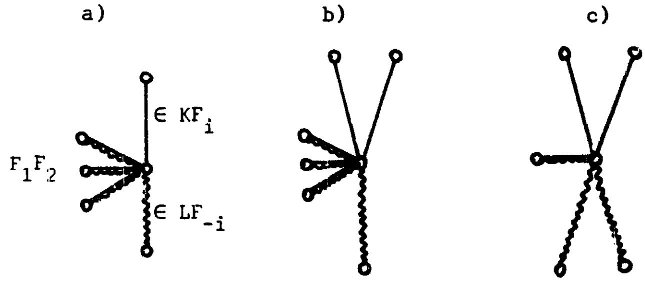

Fig. 4.

Proof. Obviously each of these cases is sufficient for (20). Now assume that (20) holds. Clearly $|Z_wF_1F_2| \leq b_w$.

Case 1. $|Z_wF_1F_2| = b_w$. Then as $|Z_w| = b_w + 1$, we have $|Z_wKF_i| = 1$ and $|\omega(w)F_i| \geq |Z_wF_i| > b_w$, a contradiction to the definition of b-matchings.

Case 2. $|Z_wF_1F_2| = b_w - 1$. Then $|Z_w(KF_i \cup LF_{-i})| = 2$ and since F_1, F_2 are b-matchings, $|Z_wKF_i| = |Z_wLF_{-i}| = 1$.

Case 3. $|Z_wF_1F_2| = b_w - 2$. Then $|Z_w(KF_i \cup LF_{-i})| = 3$ and $|Z_wK| \geq |Z_wL|$ imply $|Z_wKF_i| = 2$, $|Z_wLF_{-i}| = 1$.

Case 4. $|Z_wF_1F_2| = b_w - 3$. A similar argument as above yields $2 \leq |Z_wKF_i| \leq 3$. Suppose $|Z_wKF_i| = 3$. Then by Lemma 1 we have *either* $|\omega(w)K| = 3$ and $|\omega(w)F_1F_2| = b_w - 2$–but then $|\omega(w)F_i| \geq |Z_wKF_i| + |\omega(w)F_1F_2| > b_w$–*or* $|\omega(w)K| = 4$, $|\omega(w)KF_i| = 2$–a contradiction to $|Z_wKF_i| = 3$. Hence $|Z_wKF_i| = 2$ and also $|Z_wLF_{-i}| = 2$.

Case 5. $|Z_wF_1F_2| \leq b_w - 4$. Then

$$|\omega(w)KF_i| + |\omega(w)LF_{-i}| \geq |Z_w(KF_i \cup LF_{-i})| \geq 5.$$

But it follows from Lemma 1 that this inequality is impossible.

If we are now looking for an implementation of the coloring algorithm, we have first to decide upon an appropriate representation of the current coloring. A straightforward method is to store the single edges of each color class; this can be most conveniently be done in form of a tree represented by its "father" relation (cf. [10]). Although such a data structure would lead to an elegant method to combine color classes, an easy analysis shows that the resulting procedure has inevitably super-linear time complexity. A much better alternative is offered by our two lemmas above which demonstrate the great importance of the contact vertices for the coloring algorithm. Thus our implementation is based mainly on the contact vertices. The graph, the two b-matchings, and the current coloring C are represented by the following data structures of length $O(|V|)$:

CON is a $|C| \times 2$-matrix, each row CON(K, $*$) contains the two contact vertices of color class K; if K has only one color class, we set CON(K, 2) = 0.

A is a $|V| \times 4$-matrix whose rows correspond to the contact vertices which might be used to combine color classes. By Lemma 2 any such contact vertex v satisfies $|\omega(v)F_1F_2| \geq b_v - 3$ and meets at most four color classes. The numbers of these classes are stored in row $A(v, *)$ in the following form:

$$\begin{array}{ll} v \text{ meets} & A(v,*) = \\ \left.\begin{array}{l} KF_i,\ K'F_i,\ LF_{-i},\ L'F_{-i} \\ KF_i,\ K'F_i,\ LF_{-i} \\ KF_i,\ \qquad LF_{-i} \end{array}\right\} \Rightarrow & \left\{\begin{array}{l} (K, K', L, L') \\ (K, K'\,L, 0) \\ (K, 0, L, 0) \end{array}\right. \end{array}$$

Two more data structures are used in our implementation:

IS (for intersection) is a vector in which the cardinalities $|\omega(v)F_1F_2|$ for $v \in V$ are stored.

STACK contains for future use the vertices in which color classes can be combined. For each vertex w in STACK, one of the following three analoga to the cases in Lemma 2 holds:

(a) $|\omega(w)F_1F_2| = b_w - 1$ (these vertices are put on STACK from the very beginning)

(b) $|\omega(w)F_1F_2| = b_w - 2$, $A(w, 1) = A(w, 2)$ (then $K = A(w, 1)$ can be combined with $L = A(w, 3)$ and perhaps also with $A(w, 4)$)

(c) $|\omega(w)F_1F_2| = b_w - 3$, $A(w, 1) = A(w, 2)$, $A(w, 3) = A(w, 4)$ (then $K = A(w, 1)$ can be combined with $L = A(w, 3)$).

The combining of two color classes $K, L \in C$ at a vertex w from STACK can now be performed in three steps:

(1) The "free" (i.e. different from w) contact vertex v of L is determined in CON(L, $*$). It becomes the new contact vertex of $K := K \cup L$ and thus replaces w (more precisely: the first occurrence of w) in CON(K, $*$).

(2) The names of the color classes with contact vertices v are updated: each occurrence of L in $A(v, *)$ is replaced by K.

(3) Combining K and L, new possibilities to combine color classes can only arise at v. We check that by inspecting $A(v, *)$ and eventually push v onto STACK.

The coloring algorithm starts with initializing CON, A, and IS for the trivial coloring $C = \{\{e\} \mid e \in F_1 \Delta F_2\}$ (each edge a different color). Obviously this can be done in linear time starting with any resonable encoding of the graph G and the two b-matchings F_1, F_2. Simultaneously the vertices v satisfying (a) above are pushed onto STACK. Now the actual coloring algorithm starts. It is described in the following pseudo-ALGOL procedure.

begin
while (STACK is not empty **and** $|C| > 1$) **do**
begin pop w from STACK;

$K := A(w, 1)$; $L := A(w, 3)$;

comment check if there is one more possibility to combine color classes at w;

if $|\omega(w)F_1F_2| = b_w - 2$ **and** w meets a fourth color class $A(w, 4)$ which has not yet been combined with K

then begin $A(w, 3) := A(w, 4)$; $A(w, 4) := K$; add w again to STACK;
end;

comment in the following, the classes K and L are combined and get the common name K;

$|C| = |C| - 1$;

comment determine and update the "free" contact vertex v of L;
$v :=$ **if** CON(L, 1) $= w$ **then** CON(L, 2) **else** CON(L, 1);
if CON(K, 1) $= w$ **then** CON(K, 1) $:= v$; **else** CON(K, 2) $:= v$;
comment update the color classes met by v;
for $j := 1$ to 4 **do if** $A(v, j) = L$ **then** $A(v, j) := K$;
comment check if color classes can be combined at v;
if $v = w$ **or** $v = 0$ **then**;

else if $|\omega(v)F_1F_2| = b_v - 2$ **and** $A(v,1) = A(v,2)$
and $(A(v,3) \neq A(v,1)$ **or** $(A(v,4) \neq 0$
and $A(v,4) \neq A(v,1)))$
then add v to STACK;
else if $|\omega(v)F_1F_2| = b_v - 2$ **and** $A(v,3) = A(v,4)$
and $(A(v,1) \neq A(v,3)$ **or** $A(v,2) \neq A(v,3))$
then interchange $(A(v,3), A(v,4))$ with $(A(v,1), A(v,2))$
and add v to STACK;
else if $|\omega(v)F_1F_2| = b_v - 3$ **and** $A(v,1) = A(v,2)$
and $A(v,3) = A(v,4)$ **and** $A(v,1) \neq A(v,3)$
then add v to STACK;
end;
if $|C| = 1$ **then** output 'F_1, F_2 are adjacent';
else output 'F_1, F_2 are not adjacent';
end;

The length of the input for this algorithm is $O(n)$ where $n = |V|$. The initialization requires $O(n)$ steps. The time needed to process one vertex from STACK is bounded by a constant, and each vertex is added to STACK at most twice. Therefore the total number of steps required is $O(n)$. Moreover the space complexity is clearly $O(n)$. Thus we have proven:

Theorem 3. *Let $G = (V, E)$ be a graph with n vertices and $b \in \mathbb{N}^V$. Then the adjacency of two b-matchings of G on the b-matching polyhedron of G can be checked by the coloring algorithm with time and space complexity* $O(n)$.

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