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Exponential stability of the solutions of the initial-value problem for systems with impulse effect *

P.S. SIMEONOV and D.D. BAINOV
Plovdiv University, P.O. Box 45, 1504 Sofia, Bulgaria

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Abstract: In the present paper conditions have been found under which the exponential stability of a given solution of a system with impulse effect follows from the exponential stability of the respective system in variations.

Keywords: Exponential stability, systems with impulse effect.

1. Introduction

It is characteristic for the development of many processes that in certain moments they change their state by jumps. Adequate mathematical models of such processes are provided by the systems with impulses. Their study began with the work by Mil'man and Myshkis [1] after which the number of the publications dedicated to these theme grows rapidly.

In some of the works on systems with impulses the apparatus of the generalized functions is used [2]–[7].

In the works [8]–[16] the apparatus of the classical theory of the ordinary differential equations is used. In them systems with impulse effect of the following three types are investigated:

$$\mathrm{d}x/\mathrm{d}t = f(t, x), \quad t \neq \tau_k, \qquad \Delta x|_{t=\tau_k} = I_k(x), \quad k = 1, 2, \ldots; \tag{1}$$

$$\mathrm{d}x/\mathrm{d}t = (t, x), \quad t \neq \tau_k(x), \qquad \Delta x|_{t=\tau_k(x)} = I_k(x) \tag{2}$$

and

$$\mathrm{d}x/\mathrm{d}t = g(x), \quad x \overline{\in} M, \qquad \Delta x|_{x \in M} = I(x), \tag{3}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$.

Systems (1), (2), (3) are defined by a system of ordinary differential equations and by conditions which determine the moments and the magnitude of the impulse effect.

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The moments of impulse effect of system (1) are $t=\tau_k$, $k=1, 2, \ldots$ and are fixed.

The moments of impulse effect of system (2) occur when the mapping point $(t, x(t))$ from the extended phase space meets some of the hypersurfaces σ_k given by the equations $t=\tau_k(x)$, $k=1, 2, \ldots$.

The moments of impulse effect of system (3) occur when the point $x(t)$ from the phase space X meets the set $M \subset X$.

The solutions $x(t)$ of systems (1), (2), (3) are piecewise continuous functions. At the moment $t=\tau_k$ of impulse effect the solution $x(t)$ has a discontinuity of first type and the following equalities are assumed fulfilled

$$x(\tau_k-0)=x(\tau_k), \qquad x(\tau_k+0)=x(\tau_k)+\Delta x(\tau_k).$$

Between two successive moments of impulse effect ($t \in (\tau_k, \tau_{k+1}]$) the solution $x(t)$ of system (1) coincides with the solution of the initial value problem

$$\mathrm{d}\xi/\mathrm{d}t=f(t, \xi), \quad \xi(\tau_k)=x(\tau_k+0).$$

The solutions of systems (2) and (3) for $t \in (\tau_k, \tau_{k+1}]$ are determined analogously.

We shall note that the moments of impulse effect of the different solutions of systems (2) and (3) are different. That is why by their study we meet some additional difficulties not met by the study of system (1).

In the present paper conditions have been found under which the exponential stability of the solution $p(t)$ of one of the systems (1), (2) or (3) follows from the exponential stability of the system in variations of the respective system with respect of the solution $p(t)$.

2. Preliminary notes

Let $\mathbb{R}_+[0, \infty)$ and $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$ of the element $x=\mathrm{col}(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Further on we shall use the following notations:

$B_h(x_0)=\{x \in \mathbb{R}^n\colon |x-x_0|<h\}$: h-neighbourhood of the point $x_0 \in \mathbb{R}^n$;
$\bar{G}$: the closure of the set $G \subset \mathbb{R}^n$;
$\rho(x, L)=\inf_{y \in L}|x-y|$: the distance from the point $x \in \mathbb{R}^n$ to the set $L \subset \mathbb{R}^n$;
$|A|=\sup_{|x|=1}|Ax|$: the norm of the $(n \times n)$-matrix A;
E_n: the unit $(n \times n)$-matrix; $\partial f/\partial x=(\partial f_i/\partial x_j)$, $i=1, \ldots, m$, $j=1, \ldots, n$: the matrix of Jacobi for the function $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $x \to f(x)$;
$[a; b]$: the interval $[a, b]$ if $a \leqslant b$ or $[b, a]$ if $b<a$;
$x(t; t_0, x_0)$: the solution of system (1), (2) or (3) which satisfies the initial condition $x(t_0+0; t_0, x_0)=x_0$;
$\mathscr{J}^+(t_0, x_0)$: the maximal interval of the form (t_0, ω) in which the solution $x(t; t_0, x_0)$ is continuable to the right.

Let $p(t)$ $(t \in \mathbb{R}_+)$ be a solution of one of systems (1)–(3) with moments of impulse effect $\{\tau_k\}_1^\infty$:

$$\tau_0=0<\tau_1<\tau_2<\cdots, \quad \lim_{k \to \infty}\tau_k=\infty.$$

Introduce the following sets:

$$\Gamma_k = \{x \in \mathbb{R}^n:\ x = p(t),\ \tau_{k-1} < t \leqslant \tau_k\} \cup p(\tau_{k-1}+0),$$
$$\Gamma_k^* = \{(t,\ x) \in \mathbb{R}^{n+1}:\ x = p(t),\ \tau_{k-1} < t \leqslant \tau_k\} \cup (\tau_{k-1},\ p(\tau_{k-1}+0)),$$
$$D_k(h) = \{x \in \mathbb{R}^n:\ \rho(x,\ \Gamma_k) < h\},$$
$$G_k(h) = \{(t,\ x) \in \mathbb{R}^{n+1}:\ |t-\tau| < h,\ |x-y| < h,\ (\tau,\ y) \in \Gamma_k^*\}.$$

Definition 1. The solution $p(t)$ of system (1), ((2) or (3)) is called exponentially stable if

$$(\exists B > 0,\ \alpha > 0)(\forall \eta > 0)(\exists \delta > 0)(\forall t_0 \in \mathbb{R}_+,\ |t_0 - \tau_k| > \eta)$$
$$\times (\forall x_0 \in \mathbb{R}:\ |x_0 - p(t_0)| < \delta)$$
$$(\forall t \geqslant t_0,\ |t-\tau_k| > \eta) \quad |x(t;\ t_0,\ x_0) - p(t)| \leqslant B|x_0 - p(t_0)|\ \mathrm{e}^{-\alpha(t-t_0)}.$$

3. Main results

First we shall consider the system with impulse effect (3). Assume that the set M is $(n-1)$-dimensional manifold contained in $\mathbb{R}^n$.

We shall say that conditions (A) are satisfied if the following conditions hold:

(A1) System (3) has a solution $p(t)$, $(t \in \mathbb{R}_+)$ with moments of impulse effect $\{\tau_k\}_1^\infty$:

$$\tau_0 = 0 < \tau_1 < \tau_2 < \cdots, \qquad \tau_k - \tau_{k-1} \geqslant \theta > 0, \quad k = 1, 2, \ldots .$$

(A2) There exists a number $H > 0$ such that:

(i) The function g: $\Omega \to \mathbb{R}^n$ is differentiable in the set $D = \bigcup_{k=1}^\infty D_k(H) \subset \Omega$ and is continuous in $\bar{D}$.

(ii) The function I: $\Omega \to \mathbb{R}^n$ is differentiable in the set $\bigcup_{k=1}^\infty B_H(p(\tau_k)) \subset \Omega$.

(A3) The set $M \cap B_H(p(\tau_k))$ coincides with the set of solutions of the system $\Phi_k(x) = 0$ where the function Φ_k: $B_H(p(\tau_k)) \to \mathbb{R}$ is differentiable in $B_H(p(\tau_k))$.

(A4) There exist constants $C > 0$, $\beta > 0$ and $h \in (0,\ H)$ such that for any $k = 1, 2, \ldots$ and $x \in B_h(p(\tau_k))$

$$\left|\frac{\partial \Phi_k}{\partial x}(x)\right| \leqslant C, \qquad |g(x)| \leqslant C, \qquad \left|\frac{\partial g}{\partial x}(x)\right| \leqslant C,$$
$$\left|\frac{\partial \Phi_k}{\partial x}(x) g(x)\right| \geqslant \beta.$$

(A5) For each $h \in (0,\ H)$ there exists $\gamma > 0$ such that for any $k = 1, 2, \ldots$ and $x \in \Gamma_k \setminus B_h(p(\tau_k))$

$$\rho(x,\ M) \geqslant \gamma.$$

(A6) For any $\mu \in (0,\ H)$ the functions

$$\alpha(t,\ y) = g(p(t)+y) - g(p(t)) - \frac{\partial g}{\partial x}(p(t))y$$

and

$$\beta_k(y) = I(p(\tau_k) + y) - I(p(\tau_k)) - \frac{\partial I}{\partial x}(p(\tau_k))y$$

satisfy the inequalities

$$|\alpha(t, y) - \alpha(t, v)| \leqslant L(\mu)|y - v|,$$
$$|\beta_k(y) - \beta_k(v)| \leqslant L(\mu)|y - v|$$

for $|y| \leqslant \mu$, $|v| \leqslant \mu$, $t \in \mathbb{R}_+$, $k = 1, 2, \ldots$, where

$$\lim_{\mu \to 0_+} L(\mu) = 0.$$

(A7) The Cauchy matrix $W(t, s)$ of the linear system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial g}{\partial x}(p(t))y, \quad t \neq \tau_k$$
$$\Delta y|_{t=\tau_k} = N_k y, \quad k = 1, 2, \ldots, \tag{4}$$
$$N_k = \frac{\partial I}{\partial x}(p(\tau_k)) + \left[g(p(\tau_k + 0)) - g(p(\tau_k)) - \frac{\partial I}{\partial x}(p(\tau_k))g(p(\tau_k))\right] \times \frac{\frac{\partial \Phi_k}{\partial x}(p(\tau_k))}{\frac{\partial \Phi_k}{\partial x}(p(\tau_k))g(p(\tau_k))}$$

satisfies an estimate of the form

$$|W(t, s)| \leqslant K\,\mathrm{e}^{-2\alpha(t-s)} \quad \text{for } 0 \leqslant s \leqslant t < \infty,$$

where $K \geqslant 1$ and $\alpha > 0$ are constants.

Before we prove Theorem 1, we shall prove some auxiliary assertions.

Denote by $z(t; t_0, z_0)$ the solution of the system

$$\mathrm{d}z/\mathrm{d}t = g(z),$$

which satisfies the condition $z(t_0; t_0, z_0) = z_0 \in \mathbb{R}^n$.

Lemma 1. *Let conditions* (A1)–(A4) *hold.*

Then there exist numbers $h \in (0, H)$ and $\tau > 0$ such that for any $k = 1, 2, \ldots$ and $w \in B_h(p(\tau_k))$ there exists a unique differentiable function T_k: $B_h(p(\tau_k)) \to \mathbb{R}$, $w \to T_k(w)$ such that

$$T_k(p(\tau_k)) = \tau_k, \qquad \Phi_k(z(T_k(w); \tau_k, w)) = 0,$$
$$\frac{\partial \Phi_k}{\partial x}(z(T_k(w); \tau_k, w))\left[g(z(T_k(w); \tau_k, w))\frac{\partial T_k}{\partial w}(w) + \frac{\partial z}{\partial w}(T_k(w); \tau_k, w)\right] = 0,$$
$$|T_k(w) - T_k(u)| \leqslant \tau|w - u|$$

for w, $u \in B_h(p(\tau_k))$.

Proof. Lemma 1 follows immediately from the implicit function theorem applied to the function

$$\phi_k\colon\ (\tau_k - H,\ \tau_k + H) x B_H(p(\tau_k)) \to \mathbb{R}, \qquad (t,\ w) \to \phi_k(t,\ w) = \Phi_k(z(t;\ \tau_k,\ w))$$

and the equation $\phi_k(t, w) = 0$. □

Lemma 2. *Let the following conditions be satisfied*:

(1) *The sequences* $\{\gamma_k\}$ *and* $\{\tau_k\}$ *are such that*

$$\tau_1 > 0, \quad \tau_{k+1} - \tau_k \geqslant \theta > 0, \quad \gamma_k \geqslant 0, \qquad k = 1, 2, \ldots .$$

(2) *The function* $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is piecewise continuous.*

(3) *The following inequalities hold*

$$\gamma(s) \leqslant r\,\mathrm{e}^{-\alpha(s-t_0)} \quad \text{for } 0 \leqslant t_0 \leqslant s < \infty,$$

$$\gamma_k \leqslant r\,\mathrm{e}^{-\alpha(\tau_k - t_0)} \quad \text{for } 0 \leqslant t_0 \leqslant \tau_k < \infty, \tag{5}$$

where $r > 0$ *and* $\alpha > 0$ *are constants.*

Then for $t > t_0$

$$\int_{t_0}^{t} \mathrm{e}^{-2\alpha(t-s)} \gamma(s)\,\mathrm{d}s + \sum_{t_0 < \tau_k < t} \mathrm{e}^{-2\alpha(t-s)} \gamma_k \leqslant Qr\,\mathrm{e}^{-\alpha(t-t_0)}, \tag{6}$$

where $Q = 1/\alpha + 1/\alpha\theta + 1$.

Proof. Let $\gamma(s)$ and γ_k satisfy inequalities (5) and $\tau_i \leqslant t_0 < \tau_{i+1} < \tau_n < t \leqslant \tau_{n+1}$. Then from the estimates

$$\int_{t_0}^{t} \mathrm{e}^{-2\alpha(t-s)}\,\mathrm{e}^{-\alpha(s-t_0)}\,\mathrm{d}s = \frac{\mathrm{e}^{-2\alpha t}}{\alpha}\left[\mathrm{e}^{\alpha(t+t_0)} - \mathrm{e}^{2\alpha t_0}\right] < \frac{1}{\alpha}\mathrm{e}^{-\alpha(t-t_0)},$$

$$\begin{aligned}
\sum_{t_0 < \tau_k < t} \mathrm{e}^{-2\alpha(t-\tau_k)}\,\mathrm{e}^{-\alpha(\tau_k - t_0)} &= \sum_{k=i+1}^{n} \mathrm{e}^{-2\alpha t}\,\mathrm{e}^{\alpha(\tau_k + t_0)} \\
&\leqslant \mathrm{e}^{-2\alpha t}\left[\frac{1}{\theta}\sum_{k=i+1}^{n-1} (\tau_{k+1} - \tau_k)\,\mathrm{e}^{\alpha(\tau_k + t_0)} + \mathrm{e}^{\alpha(\tau_n + t_0)}\right] \\
&\leqslant \mathrm{e}^{-2\alpha t}\left[\frac{1}{\theta}\int_{\tau_{i+1}}^{\tau_n} \mathrm{e}^{\alpha(s+t_0)}\,\mathrm{d}s + \mathrm{e}^{\alpha(\tau_n + t_0)}\right] \\
&= \mathrm{e}^{-2\alpha t}\left[\frac{1}{\alpha\theta}\left(\mathrm{e}^{\alpha(\tau_n + t_0)} - \mathrm{e}^{\alpha(\tau_{i+1} + t_0)}\right) + \mathrm{e}^{\alpha(\tau_n + t_0)}\right] \\
&\leqslant \left(\frac{1}{\alpha\theta} + 1\right)\mathrm{e}^{-\alpha(t-t_0)}
\end{aligned}$$

there follows estimate (6).

This completes the proof of Lemma 2. □

Theorem 1. *Let conditions* (A) *hold. Then the solution* $p(t)$ *of system* (3) *is exponentially stable.*

Proof. Choose $h > 0$ and $\tau > 0$ by Lemma 1.

Let $\eta > 0$ be given. We shall prove that there exists a constant $\delta > 0$ such that for any $t_0 \in \mathbb{R}_+$, $|t_0 - \tau_k| > \eta$ and $x_0 \in \mathbb{R}^n$, $|x_0 - p(t_0)| < \delta$ there exists a solution $x(t) = x(t;\ t_0,\ x_0)$ $(t \geqslant t_0)$ of system (3) with moments of impulse effect t_k and the following estimates hold:

$$|t_k - \tau_k| \leqslant B|x_0 - p(t_0)|\ \mathrm{e}^{-\alpha(\tau_k - t_0)} \equiv \eta_k < \eta \quad \text{for } \tau_k \geqslant t_0, \tag{7}$$

$$|x(t) - p(t)| \leqslant B|x_0 - p(t_0)|\mathrm{e}^{-\alpha(t-t_0)} \quad \text{for } t \geqslant t_0, \quad |t - \tau_k| > \eta_k. \tag{8}$$

For this purpose we form the sequences $y_n(t)$ and t_k^n, $n = 0, 1, 2, \ldots$, setting for $t \geqslant t_0$ and k: $\tau_k > t_0$

$$y_0(t) = p(t), \quad t_k^0 = \tau_k$$

after which we define successively

$$y_{n+1}(t) = p(t) + W(t,\ t_0)(x_0 - p(t_0)) + \int_{t_0}^{t} W(t,\ s) f_n(s)\ \mathrm{d}s + \sum_{t_0 < \tau_k < t} W(t,\ \tau_k + 0) b_k^n, \tag{9}$$

where

$$f_n(t) = F(t,\ y_n(t)) = g(y_n(t)) - g(p(t)) - \frac{\partial g}{\partial x}(p(t))(y_n(t) - p(t)), \tag{10}$$

$$b_k^n = \beta_k(y_n(\tau_k)) = I(z(t_k^n;\ \tau_k,\ y_n(\tau_k))) - I(p(\tau_k)) + \int_{t_k^n}^{\tau_k} [g(z(s;\ \tau_k,\ y_n(\tau_k + 0))) - g(z(s;\ \tau_k,\ y_n(\tau_k)))]\ \mathrm{d}s - N_k(y_n(\tau_k) - p(\tau_k)); \tag{11}$$

t_k^{n+1} is determined as a solution with respect to t of the equation

$$\Phi_k(z(t;\ \tau_k,\ y_{n+1}(\tau_k))) = 0, \tag{12}$$

i.e. $t_k^{n+1} = T_k(y_{n+1}(\tau_k))$.

Let $K|x_0 - p(t_0)| < h$. Then for $n = 0$ from (9), (10) and (11) and condition (A7) it follows that

$$|y_1(t) - p(t)| \leqslant K|x_0 - p(t_0)|\ \mathrm{e}^{-2\alpha(t-t_0)} \quad \text{for } t \geqslant t_0. \tag{13}$$

Moreover, since

$$|y_1(\tau_k) - p(\tau_k)| \leqslant K|x_0 - p(t_0)| < h$$

then by Lemma 1 there exists a unique solution $t_k^1 = T_k(y_1(\tau_k))$ of the equation $\Phi_k(z(t;\ \tau_k,\ y_1(\tau_k))) = 0$ and the following estimate holds

$$|t_k^1 - t_k^0| = |t_k^1 - \tau_k| = |T_k(y_1(\tau_k)) - T_k(p(\tau_k))| \leqslant \tau|y_1(\tau_k) - p(\tau_k)| \leqslant \tau K|x_0 - p(t_0)|\ \mathrm{e}^{-2\alpha(\tau_k - t_0)}. \tag{14}$$

Let $\mu \in (0,\ h)$. In view of condition (A6) and Lemma 1 we conclude that the functions $F(t,\ y)$ and $\beta_k(y)$ satisfy inequalities of the form

$$|F(t,\ y) - F(t,\ v)| \leqslant L(\mu)|y - v| \tag{15}$$

for $|y-p(t)| \leqslant \mu$, $|v-p(t)| \leqslant \mu$, $t \in \mathbb{R}_+$ and

$$|\beta_k(y)-\beta_k(v)| \leqslant L(\mu)|y-v| \tag{16}$$

for $|y-p(\tau_k)| \leqslant \mu$, $|v-p(\tau_k)| \leqslant \mu$, $k=1, 2, \ldots$, where $\lim_{\mu \to 0_+} L(\mu)=0$.

Choose $\gamma > 0$ according to condition (A5) and the numbers $\mu \in (0, h/K)$ and $\delta > 0$ so that

$$L(\mu)Q = L(\mu)\frac{K}{\alpha\theta}(\alpha\theta+\theta+1) < \tfrac{1}{2}, \quad K\delta < \mu. \tag{17}$$

By induction with respect to n we shall prove that the members of the sequences $y_n(t)$, t_k^n can be determined successively and the following estimates hold

$$|y_n(t)-y_{n-1}(t)| \leqslant K|x_0-p(t_0)|2^{-n}\,\mathrm{e}^{-\alpha(t-t_0)} \tag{18}$$

$$|y_n(t)-p(t)| \leqslant K|x_0-p(t_0)|\mathrm{e}^{-\alpha(t-t_0)}, \tag{19}$$

$$|t_k^n-t_k^{n-1}| \leqslant \tau K|x_0-p(t_0)|2^{-n}\,\mathrm{e}^{-\alpha(\tau_k-t_0)}, \tag{20}$$

$$|t_k^n-\tau_k| \leqslant \tau K|x_0-p(t_0)|\,\mathrm{e}^{-\alpha(\tau_k-t_0)} \tag{21}$$

for $n=1, 2, \ldots$, $t \geqslant t_0$, $\tau_k \geqslant t_0$.

In fact, from (13) and (14) it follows that estimates (18)–(21) holds for $n=1$. Let these estimates hold for $n=1, \ldots, m$. Then $|y_m(t)-p(t)| < \mu$ and from (15), (16) and (18) we find

$$\begin{aligned}|f_m(t)-f_{m-1}(t)| &= |F(t, y_m(t))-F(t, y_{m-1}(t))|\\ &\leqslant L(\mu)|y_m(t)-y_{m-1}(t)| \leqslant L(\mu)K|x_0-p(t_0)|2^{-m}\,\mathrm{e}^{-\alpha(t-t_0)},\end{aligned} \tag{22}$$

$$|b_k^m-b_k^{m-1}| \leqslant L(\mu)K|x_0-p(t_0)|2^{-m}\,\mathrm{e}^{-\alpha(\tau_k-t_0)}. \tag{23}$$

Afterwards, by Lemma 2, in view of (9), (22), (23), (17) we obtain

$$\begin{aligned}&|y_{m+1}(t)-y_m(t)|\\ &\quad=\left|\int_{t_0}^t W(t, s)[f_m(s)-f_{m-1}(s)]\,\mathrm{d}s+\sum_{t_0<\tau_k<t} W(t, \tau_k+0)[b_k^m-b_k^{m-1}]\right|\\ &\quad\leqslant \frac{K}{\alpha\theta}(\alpha\theta+\theta+1)L(\mu)K|x_0-p(t_0)|2^{-m}\,\mathrm{e}^{-\alpha(t-t_0)}\\ &\quad< K|x_0-p(t_0)|2^{-m-1}\,\mathrm{e}^{-\alpha(t-t_0)}.\end{aligned} \tag{24}$$

From (24) it follows immediately that

$$\begin{aligned}|y_{m+1}(t)-p(t)| &\leqslant \sum_{j=1}^{m}|y_j(t)-y_{j-1}(t)|\\ &\leqslant K|x_0-p(t_0)|\left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\cdots\right)\mathrm{e}^{-\alpha(t-t_0)}\\ &= K|x_0-p(t_0)|\,\mathrm{e}^{-\alpha(t-t_0)}.\end{aligned}$$

In particular, $|y_{m+1}(\tau_k)-p(\tau_k)| \leqslant K|x_0-p(t_0)| \leqslant K\delta < h$ and by Lemma 1 for $n=m+1$ equation (12) has a unique solution $t_k^{m+1}=T_k(y_{m+1}(\tau_k))$ for which we obtain

$$\begin{aligned}|t_k^{m+1}-t_k^m| &= |T_k(y_{m+1}(\tau_k))-T_k(y_m(\tau_k))| \leqslant \tau|y_{m+1}(\tau_k)-y_m(\tau_k)|\\ &\leqslant \tau K|x_0-p(t_0)|2^{-m-1}\,\mathrm{e}^{-\alpha(\tau_k-t_0)}.\end{aligned}$$

Then

$$|t_k^{m+1}-\tau_k| \leqslant \sum_{j=1}^{m+1}|t_k^j-t_k^{j-1}| \leqslant \tau K|x_0-p(t_0)|\,\mathrm{e}^{-\alpha(\tau_k-t_0)}.$$

Thus estimates (18)–(21) hold for $n=m+1$, hence for each $n=1, 2,\ldots$.

From (18), (20) it follows that the sequence $y_n(t)$ and t_k^n are convergent uniformly with respect to $t \geqslant t_0$ and $k=1, 2,\ldots$. Let $y(t)$ and t_k be their limits.

Then

$$|y(t)-p(t)| \leqslant K|x_0-p(t_0)|\,\mathrm{e}^{-\alpha(t-t_0)}, \quad t \geqslant t_0, \tag{25}$$

$$|t_k-\tau_k| \leqslant \tau K|x_0-p(t_0)|\,\mathrm{e}^{-\alpha(\tau_k-t_0)}, \quad \tau_k > t_0 \tag{26}$$

and

$$\begin{aligned}y(t) &= p(t)+W(t, t_0)(x_0-p(t_0))\\ &\quad+\int_{t_0}^{t}W(t, s)f(s)\,\mathrm{d}s+\sum_{t_0<\tau_k<t}W(t, \tau_k+0)b_k,\end{aligned} \tag{27}$$

where

$$\begin{aligned}f(t) &= g(y(t))-g(p(t))-\frac{\partial g}{\partial x}(p(t))(y(t)-p(t)),\\ b_k &= I(z(t_k;\ \tau_k,\ y(\tau_k)))-I(p(\tau_k))\\ &\quad+\int_{t_k}^{\tau_k}[g(z(s;\ \tau_k,\ y(\tau_k+0)))-g(z(s;\ \tau_k,\ y(\tau_k)))]\,\mathrm{d}s\\ &\quad-N_k(y(\tau_k)-p(\tau_k)).\end{aligned}$$

By an immediate verification from (27) and (4) we obtain that $y(t)$ is a solution of the system with impulse effect

$$\begin{aligned}&\mathrm{d}y/\mathrm{d}t = g(y), \quad t \neq \tau_k,\\ &\Delta y|_{t=\tau_k} = I(z(t_k;\ \tau_k,\ y(\tau_k)))+\int_{t_k}^{\tau_k}[g(z(s;\ \tau_k,\ y(\tau_k+0)))-g(z(s;\ \tau_k,\ y(\tau_k)))]\,\mathrm{d}s.\end{aligned} \tag{28}$$

For $t \geqslant t_0$ we define the function $x(t)$ by the formula

$$x(t)=\begin{cases}y(t) & \text{if } t \overline{\in} [\tau_k;\ t_k],\\ z(t;\ \tau_k,\ y(\tau_k+0)) & \text{if } t_k < t \leqslant \tau_k,\\ z(t;\ \tau_k,\ y(\tau_k)) & \text{if } \tau_k \leqslant t < t_k,\end{cases} \tag{29}$$

$$x(t_k)=x(t_k-0). \tag{30}$$

From (28)–(30) it follows that the function $x(t)$ is differentiable for $t \neq t_k$ and satisfies the equalities

$$\mathrm{d}x(t)/\mathrm{d}t = g(x(t)) \quad \text{for } t \geqslant t_0, \quad t \neq t_k$$

and $x(t_0) = x_0$.

Moreover,

$$\Delta x(t_k) = \begin{cases} z(t_k;\ \tau_k,\ y(\tau_k + 0)) - y(t_k) & \text{if } t_k < \tau_k, \\ y(t_k) - z(t_k;\ \tau_k,\ y(\tau_k)) & \text{if } t_k > \tau_k, \\ \Delta y(\tau_k) & \text{if } t_k = \tau_k. \end{cases}$$

Let $t_k < \tau_k$. Then

$$\begin{aligned} \Delta x(t_k) &= z(t_k;\ \tau_k,\ y(\tau_k + 0)) - y(t_k) \\ &= \Delta y(\tau_k) + z(t_k;\ \tau_k,\ y(\tau_k + 0)) - y(\tau_k + 0) + y(\tau_k) - y(t_k) \\ &= I(z(t_k;\ \tau_k,\ y(\tau_k))) + \int_{t_k}^{\tau_k} [g(z(s;\ \tau_k,\ y(\tau_k + 0))) - g(z(s;\ \tau_k,\ y(\tau_k)))]\ \mathrm{d}s \\ &\quad + \int_{\tau_k}^{t_k} g(z(s;\ \tau_k,\ y(\tau_k + 0)))\ \mathrm{d}s - \int_{\tau_k}^{t_k} g(z(s;\ \tau_k,\ y(\tau_k)))\ \mathrm{d}s \\ &= I(y(t_k)) = I(x(t_k)). \end{aligned}$$

By analogous calculations it is proved that also in the cases when $t_k > \tau_k$ or $t_k = \tau_k$ the following equality holds

$$\Delta x(t_k) = I(x(t_k)).$$

Taking into account $z(t_k;\ \tau_k,\ y(\tau_k)) = x(t_k)$, passing to the limit in the equality $\Delta_k(z(t_k^n;\ \tau_k,\ y_n(\tau_k))) = 0$, we obtain that $\Phi_k(x(t_k)) = 0$, i.e. t_k are moments of impulse effect for $x(t)$. Moreover, in view of Lemma 1 and condition (A5) from (25) and (29) it follows that $x(t)$ has no other moments of impulse effect.

Thus the function $x(t)$ is a solution of system (3) which coincides with the solution $x(t;\ t_0,\ x_0)$ and has moments of impulse effect t_k. By (25), (26) and (29) the solution $x(t;\ t_0,\ x_0)$ and the moments t_k satisfy (7) and (8).

This completes the proof of Theorem 1. □

Consider the system with impulse effect (2). Assume that the hypersurfaces σ_k are given by the equations $t = \tau_k(x)$, $k = 1, 2, \ldots$, where the functions τ_k: $\Omega \to \mathbb{R}$ satisfy the conditions

$$0 < \tau_1(x) < \tau_2(x) < \cdots, \qquad \lim_{k \to \infty} \tau_k(x) = \infty$$

for any $x \in \Omega$.

We shall say that conditions (B) are satisfied if the following conditions hold:

(B1) System (2) has a solution $p(t)$ $(t \in \mathbb{R}_+)$ with moments of impulse effect $\{\tau_k\}_1^\infty$:

$$\tau_0 = 0 < \tau_1 < \tau_2 < \cdots, \qquad \tau_k - \tau_{k-1} \geqslant \theta > 0, \quad k = 1, 2, \ldots .$$

(B2) There exists a number $H > 0$ such that:

(i) The function $f\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is differentiable in the set $G = \bigcup_{k=1}^{\infty} G_k \subset \mathbb{R}_+ \times \Omega$ and is continuous in the set $\bar{G}$.

(ii) The functions $I_k\colon \Omega \to \mathbb{R}^n$, $\tau_k\colon \Omega \to \mathbb{R}$ are differentiable in the set $B_H(p(\tau_k)) \subset \Omega$, $k = 1, 2, \ldots$.

(B3) There exist constants $C > 0$, $\beta > 0$ and $h \in (0, H)$ such that for any $k = 1, 2, \ldots$ and $(t, x) \in [\tau_k - h, \tau_k + h] \times B_h(p(\tau_k))$

$$\left|\frac{\partial \tau_k}{\partial x}(x)\right| \leqslant C, \qquad |f(t, x)| \leqslant C, \qquad \left|\frac{\partial f}{\partial x}(t, x)\right| \leqslant C,$$

$$\left|1 - \frac{\partial \tau_k}{\partial x}(x) f(t, x)\right| \geqslant \beta.$$

(B4) For each $h \in (0, H)$ there exists $\gamma > 0$ such that for any $k = 1, 2, \ldots$ and $(t, x) \in \Gamma_k^* \setminus [\tau_k - h, \tau_k + h] \times B_h(p(\tau_k))$

$$\tau_{k-1}(x) < t < \tau_k(x).$$

(B5) For each $\mu \in (0, H)$ the functions

$$A(t, y) = f(t, p(t) + y) - f(t, p(t)) - \frac{\partial f}{\partial x}(t, p(t)) y,$$

$$\beta_k(y) = I_k(p(\tau_k) + y) - I_k(p(\tau_k)) - \frac{\partial I_k}{\partial x}(p(\tau_k)) y$$

satisfy the inequalities

$$|A(t, y) - A(t, v)| \leqslant L(\mu) |y - v|,$$
$$|\beta_k(y) - \beta_k(v)| \leqslant L(\mu) |y - v|$$

for $|y| \leqslant \mu$, $|v| \leqslant \mu$, $t \in \mathbb{R}_+$, $k = 1, 2, \ldots$ *where*

$$\lim_{\mu \to 0_+} L(\mu) = 0.$$

(B6) The Cauchy matrix $W(t, s)$ of the linear system

$$\frac{dy}{dt} = \frac{\partial f}{\partial x}(t, p(t)) y, \quad t \neq \tau_k,$$
$$\Delta y|_{t=\tau_k} = P_k y, \quad k = 1, 2, \ldots,$$
$$P_k = \frac{\partial I_k}{\partial x}(p(\tau_k)) + \left[\frac{\partial I_k}{\partial x}(p(\tau_k)) f(\tau_k, p(\tau_k)) + f(\tau_k, p(\tau_k)) - f(\tau_k, p(\tau_k + 0))\right] \times \frac{\frac{\partial \tau_k}{\partial x}(p(\tau_k))}{1 - \frac{\partial \tau_k}{\partial x}(p(\tau_k)) f(\tau_k, p(\tau_k))}$$

satisfies an estimate of the form

$$|W(t, s)| \leqslant K\, e^{-2\alpha(t-s)} \quad \text{for } 0 \leqslant s \leqslant t < \infty,$$

where $K \geqslant 1$ and $\alpha > 0$ are constants.

Theorem 2. *Let conditions* (B) *hold.*

Then the solution $p(t)$ of system (2) *is exponentially stable.*

The arguments in the proof of Theorem 2 are analogous to the arguments in the proof of Theorem 1, that is why we omit them.

Consider the system with impulse effect (1).

We shall say that conditions (C) are satisfied if the following conditions hold:

(C1) System (1) has a solution $p(t)$ $(t \in \mathbb{R}_+)$ with moments of impulse effect τ_k:

$$\tau_0 = 0 < \tau_1 < \tau_2 < \cdots, \qquad \tau_k - \tau_{k-1} \geqslant \theta > 0, \quad k = 1, 2, \ldots .$$

(C2) There exists a number $H > 0$ such that:

(i) the functions $f(t, x)$ and $\partial f(t, x)/\partial x$ are continuous in each of the sets

$$\{(t, x) \in \mathbb{R}^{n+1}: \tau_{k-1} < t \leqslant \tau_k, \ |x - p(t)| < H\}, \quad k = 1, 2, \ldots$$

and for any $k = 1, 2, \ldots$ and $x_0 \in B_H(p(\tau_k + 0))$ there exist the finite limits of the functions f and $\partial f/\partial x$ for $(t, x) \to (\tau_k, x_0)$, $t > \tau_k$.

(ii) The function I_k: $\Omega \to \mathbb{R}^n$ is differentiable in the set $B_H(p(\tau_k))$, $k = 1, 2, \ldots$.

(C3) For each $\mu \in (0, H)$ the functions

$$A(t, y) = f(t, p(t) + y) - f(t, p(t)) - \frac{\partial f}{\partial x}(t, p(t))y,$$

$$\beta_k(y) = I_k(p(\tau_k) + y) - I_k(p(\tau_k)) - \frac{\partial I_k}{\partial x}(p(\tau_k))y$$

satisfy the inequalities

$$|A(t, y) - A(t, v)| \leqslant L(\mu)|y - v|,$$

$$|\beta_k(y) - \beta_k(v)| \leqslant L(\mu)|y - v|$$

for $|y| \leqslant \mu$, $|v| \leqslant \mu$, $t \in \mathbb{R}_+$, $k = 1, 2, \ldots$, where

$$\lim_{\mu \to 0_+} L(\mu) = 0.$$

(C4) The Cauchy matrix $W(t, s)$ of the linear system

$$\frac{dy}{dt} = \frac{\partial f}{\partial x}(t, p(t))y, \quad t \neq \tau_k,$$

$$\Delta y|_{t=\tau_k} = \frac{\partial I_k}{\partial x}(p(\tau_k))y, \quad k = 1, 2, \ldots$$

satisfies an estimate of the form

$$|W(t, s)| \leqslant K\,e^{-2\alpha(t-s)} \quad \text{for } 0 \leqslant s \leqslant t < \infty,$$

where $K \geqslant 1$ and $\alpha > 0$ are constants.

For the system with impulse effect at fixed moments of time (1) we shall give the following definition of exponential stability.

Definition 2. The solution $p(t)$ of system (1) is called exponentially stable if

$$(\exists B > 0, \alpha > 0, \delta > 0)(\forall t_0 \in \mathbb{R}_+)(\forall x_0 \in \mathbb{R}^n: |x_0 - p(t_0)| < \delta)(\forall t > t_0)$$

$$|x(t; t_0, x_0) - p(t)| \leqslant B|x_0 - p(t_0)|\,e^{-\alpha(t-t_0)}.$$

We shall note that Definitions 1 and 2 of exponential stability of the solution $p(t)$ of system (1) are equivalent if the following condition (D) is satisfied.

(D) There exist constants $C>0$ and $H>0$ such that

$$|\partial f/\partial x(t, x)| \leqslant C$$

for $(t, x) \in [\tau_k - H, \tau_k + H] \times (B_H(p(\tau_k)) \cup B_H(p(\tau_k + 0)))$, $k = 1, 2, \ldots$

Theorem 3. *Let conditions* (C) *hold. Then the solution* $p(t)$ *of system* (1) *is exponentially stable in the sense of Definition* 2.

In the proof of Theorem 3 the arguments are as in the proof of Theorem 1 with the only difference that we construct only the sequence $y_n(t)$ by the formula

$$y_0(t) = p(t),$$
$$y_{n+1}(t) = p(t) + W(t, t_0+0)(x_0 - p(t_0+0)) + \int_{t_0}^{t} W(t, s) f_n(s)\,\mathrm{d}s + \sum_{t_0<\tau_k<t} W(t, \tau_k+0) b_k^n,$$

where

$$f_n(t) = f(t, y_n(t)) - f(t, p(t)) - \frac{\partial f}{\partial x}(t, p(t))(y_n(t) - p(t)),$$

$$b_k^n = I_k(y_n(\tau_k)) - I_k(p(\tau_k)) - \frac{\partial I_k}{\partial x}(p(\tau_k))(y_n(\tau_k) - p(\tau_k)).$$

We give Theorem 3 without proof.

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