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Some basic solutions for nonlinear creep

V. Kobelev *

University of Siegen, Paul-Bonatz-Str. 9-11, D-57076 Siegen, Germany

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ABSTRACT

The aim of the paper is to derive the exact analytical expressions for torsion and bending creep of rods that obey the Norton–Bailey, Prandtl–Garofalo and Naumenko–Altenbach–Gorash constitutive models. The common secondary creep constitutive model is the Norton–Bailey law which gives a power law relationship between creep rate and stress. The closed form solutions for fractional Norton–Bailey creep law are derived. The analytical formulas express the torque and bending moment as functions of the time for the period of relaxation. Other formulas express the twist rate and curvature as functions of the time for the duration of engineering creep experiment. The derived formulas are suitable for the practically important problems of machinery. Namely, the formulas are relevant for calculation of hereditary effects for helical, leaf and disk springs and twisted shafts.

1. Introduction

Stress analysis for creep has a long history in engineering mechanics driven by the requests of design for elevated temperature. The permanent high stress leads to creep of these structural elements. The modeling of creep under multi-axial stress states is the key step in the adequate prediction of the long-term structural behavior. Such a modeling requires the introduction of tensors of stress, strain, strain rate and corresponding inelastic parts.

The basic mechanical elements, such as bolts, shafts, torsion members, helical and leaf springs, are usually exposed for a long time to a constant or variable high stress. Even if the stress tensor of these elements is predominantly uniaxial, the stress fields are principally inhomogeneous. The only exception of the homogeneous stress is the uniaxial stress field in the rod of a constant cross-section in tension or compression. This is the argument for choice of the constant cross-section rod as the preferred object for creep experiments. In this case the strain rate and the stress could be immediately measured. The acquired dependence of creep strain rate over the constant uniaxial stress provides the uniaxial creep law.

In contrast, the stress fields in the majority of mechanical elements are inhomogeneous. In contrast to uniaxial tension tests, the stresses are not distributed uniformly over the cross section of structural members. Firstly, the shear strain in the twisted circular rods depends linearly upon the radius. Torsional loading has a significant influence on the initial choice of section for maximum structural efficiency. The elastic solution covers by StVenant theory of torsion. Secondly, the axial elongation of bent beams is the linear function of coordinate according to the Euler–Bernoulli theory.

However, the closed form solutions were not derived – to the author's knowledge – up to now, despite the fact that the classical theories of StVenant for torsion and Euler–Bernoulli for bending are applicable for creep problems. The problem consists in the nonlinear dependence of strain rate upon stress and calculation of resulting integrals over the rod's cross-section.

The nonlinear dependence of strain rate upon stress leads to the nonlinear differential equations that describe creep effect. Usually these nonlinear differential equations are being solved numerically using finite-element codes. The application of finite-element codes is unpractical for the structural members with such a primitive geometry, like rectangular beam or circular rods.

Another argument for closed analytical solution is the following. The closed form solution allows inversely derive the uniaxial creep law from torsion or bending experiments. The shear stress–strain response of materials can be extremely important in the design, analysis and manufacture of a wide variety of products and components which are loaded primarily in shear or torsion. When the applied loadings are primarily shear in nature, the shear creep laws must be known in order to apply the usual closed form equations commonly used in engineering design and analysis. The same argument could be applied for experiments on beam-like elements under flexure.

Consequently, the essential task is the derivation of the exact closed form expressions for torsion and bending creep for isotropic materials, which obey the commonly accepted constitutive laws. It is shown in this Article, that the laws of Norton–Bailey,

* Tel.: +49 (2722)629503.
E-mail address: kobelev@imr.mb.uni-siegen.de

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Prandtl–Garofalo and Naumenko–Altenbach–Gorash allow the closed form solution. These basic constitutive models were based on the time- and strain-hardening constitutive equations for time-varying stress. The models adequately describe the secondary creep stage from constant load/stress uniaxial tests where creep rate is nearly constant. Among others, the most widespread secondary creep constitutive model has been the Norton–Bailey law. The closed form solution is found for the fractional strain rate generalization of the Norton–Bailey law as well. This creep law provides a power law relationship between fractional strain rate and equivalent stress.

The constitutive models and the solution methods for creep problems are discussed in Kassner (2008). A summary of creep laws for common engineering materials is provided in Naumenko and Altenbach (2007) and Yao et al. (2007). The results of creep simulation are applied to practically important problem of engineering, namely for simulation of creep and relaxation of helical and disk springs.

There are two kinds of hereditary effects in structural members, namely relaxation and creep (Findley et al., 1976). The two terms are sometimes used interchangeably, although they are really different. Stress relaxation is a decrease in stress under constant strain. The deformation of body during relaxation does not alter, but the stress gradually reduces. The stress relaxation occurs when the deformation is held constant such as in bolt in flange where the constraint is that the total length of the system is fixed. Creep is an increase in strain under constant stress. For terminology clarity, the hereditary effects under constant stress will be referred to as "engineering creep".

2. Constitutive equations for creep

2.1. Tensorial generalization of creep laws

The groundwork of the engineering creep theory is the introduction of the inelastic strain, the creep potential, the flow rule, the equivalent stress and internal state variables. The creep component of strain rate is defined by material specific creep law. In this article we adopt, following the common procedure (Betten, 2008), an isotropic stress function

$$\dot{\varepsilon}'_{ij} = \frac{3\sigma'_{ij}}{2\sigma_{eff}} F(\sigma_{eff}, t). \tag{1}$$

The equations must be presented in the direct tensor notation. This notation assures the invariance with respect to the choice of the coordinate system and has the advantage of clear and compact representation of constitutive hypothesis. The special case of incompressible behavior of material ($\dot{\varepsilon}_{kk} = 0$) is assumed hereafter.

In Eq. (1) the following notations are used:

$\dot{\varepsilon}'_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3}\dot{\varepsilon}_{kk}\delta_{ij}$ is the deviatoric component of creep strain rate,

$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$ is the deviatoric component of stress,

$\sigma_{eff} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sqrt{3J'_2}$ is the Mises equivalent stress,

$J'_2 = \frac{1}{2}\sigma'_{ij}\sigma'_{ij}$ is the second invariant of the stress tensor.

The expressions for strain rate in uniaxial and shear stress states for the definite representations of stress function are next derived. The effects of hardening or softening and damage process are not accounted.

2.2. Norton–Bailey law

Firstly, consider Norton–Bailey law (Odquist and Hult, 1962). The isotropic stress function reads in this case

$$F(\sigma_{eff}, t) = \bar{\varepsilon}\left(\frac{t}{\bar{t}}\right)^{k-1}\left(\frac{\sigma_{eff}}{\bar{\sigma}}\right)^{m+1}, \tag{2}$$

where $\bar{\varepsilon}$, $\bar{\sigma}$, $\bar{t}$, m and k are the experimental constants. The effect of time softening is accounted, if the time exponent is k less that one. If the time exponent k is greater than one, the time hardening occurs.

There is only non-vanishing component of stress tensor ($\sigma_{11} = \sigma$) for the uniaxial stress state. Correspondingly, the non-vanishing components of strain rate are

$$\dot{\varepsilon} \equiv \dot{\varepsilon}_{11} = -2\dot{\varepsilon}_{22} = -2\dot{\varepsilon}_{33},$$

where

$$\dot{\varepsilon} = \bar{\varepsilon}\left(\frac{t}{\bar{t}}\right)^{k-1}\left(\frac{\sigma}{\bar{\sigma}}\right)^{m+1}.$$

For brevity of equations, we introduce the material constant

$$c_\sigma = \frac{\bar{\varepsilon}}{\bar{t}^{k-1}\bar{\sigma}^{m+1}}.$$

With this constant the dependence of uniaxial strain rate upon stress reads

$$\dot{\varepsilon} = c_\sigma t^{k+1}\sigma^{m+1}. \tag{3}$$

For pure shear stress state ($\sigma_{12} = \sigma_{21} = \tau$) the non-vanishing components of deformation rate are

$$\dot{\varepsilon}_{12} = \dot{\varepsilon}_{21} \equiv \frac{\dot{\gamma}}{2} = \frac{1}{2}\bar{\gamma}\left(\frac{t}{\bar{t}}\right)^{k-1}\left(\frac{\tau}{\bar{\tau}}\right)^{m+1},$$

where

$$\bar{\gamma} = \sqrt{3}\bar{\varepsilon}.$$

With the creep constant for shear strain

$$c_\tau = \frac{\bar{\gamma}}{\bar{t}^{k-1}\bar{\tau}^{m+1}}$$

and equivalent shear stress

$$\bar{\tau} = \frac{\bar{\sigma}}{\sqrt{3}}$$

the Norton–Bailey creep law for pure shear deformation reduces to

$$\dot{\gamma} = c_\tau t^{k-1}\tau^{m+1}. \tag{4}$$

There is a simple relation between the constants in Eqs. (3) and (4):

$$c_\tau = 3^{m/2+1}c_\sigma.$$

2.3. Prandtl–Garofalo creep law

Secondly, consider the creep law with the hyperbolic sine function. This kind of creep law was originally suggested by Prandtl (1928) and employed by Nadai (1937) to describe the stress dependence of the steady creep rate. The isotropic stress function of Garofalo creep law (Garofalo, 1963; Abu-Haiba et al., 2002) could be represented as

$$F(\sigma_{eff}, t) = \bar{\varepsilon}\left(\frac{t}{\bar{t}}\right)^{k-1}\sinh^P\left(\frac{\sigma_{eff}}{\bar{\sigma}}\right). \tag{5}$$

Once again, the case $k \neq 1$ corresponds to the time hardening or softening. The Garofalo creep law describes creep in the breakdown and power-law range. Hereafter the case of Garofalo creep law with $P = 1$ is considered and is referred to as Prandtl–Garofalo creep law.

For the uniaxial stress state the deformation rate reads

$$\dot{\varepsilon} = \bar{\varepsilon}\left(\frac{t}{\bar{t}}\right)^{k-1} \sinh\left(\frac{\sigma}{\bar{\sigma}}\right)$$

or

$$\dot{\varepsilon} \equiv \dot{\varepsilon}_{11} = c_\sigma t^{k-1} \sinh\left(\frac{\sigma}{\bar{\sigma}}\right) \quad (6)$$

Here

$$c_\sigma = \frac{\bar{\varepsilon}}{\bar{t}^{k-1}}$$

is the creep constant for uniaxial strain.

For pure shear stress state the deformation rate reads

$$\dot{\varepsilon}_{12} = \dot{\varepsilon}_{21} = \frac{1}{2}\bar{\gamma}\left(\frac{t}{\bar{t}}\right)^{k-1} \sinh\left(\frac{\sqrt{3}\tau}{\bar{\sigma}}\right).$$

Finally, the shear strain rate according to Prandtl–Garofalo creep law is

$$\dot{\gamma} \equiv 2\dot{\varepsilon}_{12} = c_\tau t^{k-1} \sinh\left(\frac{\tau}{\bar{\tau}}\right) \quad (7)$$

with the corresponding constant

$$c_\tau = \frac{\sqrt{3}\bar{\varepsilon}}{\bar{t}^{k-1}} \equiv \frac{\bar{\gamma}}{\bar{t}^{k-1}} \equiv \sqrt{3}c_\sigma.$$

2.4. Naumenko–Altenbach–Gorash law

Thirdly, the isotropic stress function for Naumenko–Altenbach–Gorash creep law (Naumenko et al., 2009) is

$$F(\sigma_{eff}, t) = \bar{\varepsilon} \cdot \left[\frac{\sigma_{eff}}{\bar{\sigma}} + \left(\frac{\sigma_{eff}}{\bar{\sigma}}\right)^{m+1}\right]. \quad (8)$$

Eq. (8) adequately describes creep in diffusion range and power-law range.

For the uniaxial stress state the strain rate reads

$$\dot{\varepsilon} \equiv \dot{\varepsilon}_{11} = \bar{\varepsilon} \cdot \left[\frac{\sigma}{\bar{\sigma}} + \left(\frac{\sigma}{\bar{\sigma}}\right)^{m+1}\right] = -2\dot{\varepsilon}_{22} = -2\dot{\varepsilon}_{33}. \quad (9)$$

For pure shear stress state the shear deformation rate reduces to

$$\dot{\gamma} \equiv 2\dot{\varepsilon}_{12} = \bar{\gamma} \cdot \left[\frac{\tau}{\bar{\tau}} + \left(\frac{\tau}{\bar{\tau}}\right)^{m+1}\right]. \quad (10)$$

For the creep laws (1)–(10) the closed form solutions of basic creep problems are derived. Needless to say, that the numerical values for the creep constants $\bar{\varepsilon}$, $\bar{\gamma}$, $\bar{\sigma}$, $\bar{\tau}$ are apparently different for diverse creep laws.

3. Creep and relaxation of twisted rods

3.1. Basic constitutive equations for relaxation in torsion

Torsion is twisting of a structural member, when it is loaded by couples that produce rotation about its longitudinal axis. Consider the relaxation problem for a rod with circular cross-section under the constant twist. Let $\tau(r,t)$ is shear stress in the cross-section of rod. The total shear strain in any instant of the time is $\gamma(r,t)$, is the sum of the elastic and the creep components of shear strain:

$$\gamma = \gamma_e + \gamma_c. \quad (11)$$

The creep component of shear strain is $\gamma_c(r,t)$. The elastic component of shear strain is

$$\gamma_e = \tau/G, \quad (12)$$

where G is the shear modulus.

In this Article the creep for the total deformation that remains constant in time is investigated. Thus, the total strain $\gamma_0(r)$ is function of radius only but constant in time. However, the elastic and the creep components of strain are the functions as well of radius and of time, such that:

$$\gamma(r,t) = \gamma_e(r,t) + \gamma_c(r,t) \equiv \gamma_0(r). \quad (13)$$

The time differentiation of (13) leads to the differential equation for elastic and creep strain rates:

$$\dot{\gamma}(r,t) = \dot{\gamma}_e(r,t) + \dot{\gamma}_c(r,t) \equiv 0, \quad (14)$$

where dot denotes the time derivative.

The differentiation of the Eq. (12) over time delivers the elastic component of strain rate

$$\dot{\gamma}_e = \dot{\tau}/G. \quad (15)$$

3.2. Torque relaxation for the materials, that obeys Norton–Bailey law

Firstly, the Norton–Bailey law for the state of shear stress (Boyle, 2012) is assumed

$$\dot{\gamma}_c(r,t) = c_\tau t^{k-1}\tau^{m+1}, \quad (16)$$

The substitution of material law (16) in the Eq. (14) delivers the ordinary nonlinear differential equation of the first order for total shear stress $\tau(r,t)$:

$$\frac{\dot{\tau}}{G} + c_\tau t^{k-1}\tau^{m+1} = 0. \quad (17)$$

The initial condition for the Eq. (17) presumes the pure elastic shear stress in the initial moment $t = 0$:

$$\tau(r, t = 0) = \tau_0(r). \quad (18)$$

The shear stresses in the moment $t = 0$ for the rod with circular cross-section are

$$\tau_0(r) = G\theta r,$$

where θ is the twist angle per unit length. The torque at the moment $t = 0$ is

$$M_T^0 = \frac{1}{2}G\pi\theta R^4.$$

The solution of the ordinary differential equation (17) with initial condition (18) delivers the shear stress over the cross-section of the twisted rod as the function of time and radius:

$$\tau(r,t) = \left[\tau_0^{-m}(r) + \frac{c_\tau G m t^k}{k}\right]^{-1/m} \equiv G\theta\left[\frac{1}{r^m} + \frac{c_\tau G^{m+1}\theta^m m}{k}t^k\right]^{-1/m}. \quad (19)$$

The couple as the function of time is

$$M_T(t) = 2\pi\int_0^R r^2\tau(r,t)dr. \quad (20)$$

With the expression for total shear stress (19) we can calculate the couple

$$M_T(t) = 2\pi G\theta\int_0^R r^2\left[\frac{1}{r^m} + \frac{c_\tau G^{m+1}\theta^m m}{k}t^k\right]^{-1/m} dr$$

For evaluation of the integral (20) the formula (A4) for $J_p(a,m;X)$ is applied for the case $p = 2$. The integral could be expressed in terms of hypergeometric function (Lewin, 1981)

$$M_T(t) = 2\pi G\theta J_p\left(\frac{c_\tau\,\theta^m G^{m+1}\,m\,t^k}{k}, m; R\right) = {}_2F_1\left(\frac{4}{m}, \frac{1}{m}; \frac{4+m}{m}; -\frac{c_\tau\,\theta^m G^{m+1}\,m\,t^k}{k}R^m\right)M_T^0. \quad (21)$$

3.3. *Torque relaxation for the material, that obeys Prandtl–Garofalo law*

Secondly, the Prandtl–Garofalo law for shear stress is presumed

$$\dot{\gamma}_r(r,t) = c_\tau t^{k-1}\sinh\left(\frac{\tau}{\bar{\tau}}\right). \quad (22)$$

The solution of the differential equation (14) with initial condition (18) for the Garofalo creep law (22) reads

$$\tau(r,t) = \bar{\tau}\ln\left\{\tanh\left[\frac{Gc_\tau}{2k\bar{\tau}}t^k + \operatorname{arctanh}\left(\exp\left(\frac{Gr\theta}{\bar{\tau}}\right)\right)\right]\right\}. \quad (23)$$

The time dependent torque (20) could be expressed in terms of polylogarithms (Abramowitz and Stegun, 1972) using formula (A3) for $I_2(a,b;X)$,

$$M_T(t) = 2\pi\int_0^R r^2\tau(r,t)dr = 2\pi\bar{\tau}I_2\left(\frac{Gc_\tau}{2k\bar{\tau}}t^k, \frac{G\theta}{\bar{\tau}}; R\right). \quad (24)$$

3.4. *Torque relaxation for the material, that obeys Naumenko–Altenbach–Gorash*

Thirdly, the Naumenko–Altenbach–Gorash law for the state of shear stress is applied

$$\frac{\dot{\gamma}_r(r,t)}{\bar{\gamma}} = \frac{\tau}{\bar{\tau}} + \left(\frac{\tau}{\bar{\tau}}\right)^{m+1}, \quad (25)$$

The substitution of material law (25) in Eq. (14) leads to the nonlinear ordinary homogeneous differential equation for total shear stress

$$\frac{\dot{\tau}}{G} + \bar{\gamma}\left[\frac{\tau}{\bar{\tau}} + \left(\frac{\tau}{\bar{\tau}}\right)^{m+1}\right] = 0. \quad (26)$$

The solution of the differential equation (26) with initial condition (18) delivers the shear stress over the cross-section of the twisted rod

$$\tau(r,t) = \frac{r}{\left(r^m\frac{m\bar{\gamma}G\xi - \tau_c}{\bar{\tau}^{m+1}} + \frac{\xi}{G^m\theta^m}\right)^{1/m}} \quad (27)$$

with the auxiliary time function

$$\xi = \xi(t) = \exp\left(\frac{m\bar{\gamma}G}{\bar{\tau}}t\right).$$

The torque evaluates after the substitution of (27) in the expression (20). For evaluation the formula (A4) for $J_p(a,m;X)$ is applied for the case $p=2$. Finally, torque as function of time is expressed in terms of hypergeometric function:

$$M_T(t) = {}_2F_1\left(\frac{4}{m}, \frac{1}{m}; \frac{4+m}{m}; \frac{(\theta GR)^m}{\xi\bar{\tau}^{m+1}}(\bar{\tau} - \bar{\gamma}\xi Gm)\right)\xi^{-1/m}M_T^0. \quad (28)$$

The results of this section could be instantly applied to estimate the effects of creep and relaxation of twisted shafts. On the other hand, the wire of helical compression springs is also in twist state. The derived formulas express relaxation and creep of helical compression springs.

3.5. *"Engineering creep" phenomenon in torsion*

An "engineering creep" phenomenon is an increased tendency toward more strain and plastic deformation with no change in stress. Thus, the applied torque does not alter in time, but the twist angle of the rod continuously increases.

According to Norton–Bailey law (4) the shear stress due to creep is

$$\tau = \left(\frac{\dot{\gamma}}{c_\tau t^{k-1}}\right)^{\frac{1}{m+1}} = \left(\frac{\dot{\theta}r}{c_\tau t^{k-1}}\right)^{\frac{1}{m+1}}.$$

Performing the integration in (20) over the area of wire we get the torque due to creep

$$M_T^0(t) = 2\pi\int_0^R r^2\tau(r,t)dr = 2\pi\int_0^R r^2\left(\frac{r\dot{\theta}}{c_\tau t^{k-1}}\right)^{\frac{1}{m+1}}dr = \left(\frac{\dot{\theta}}{c_\tau t^{k-1}}\right)^{\frac{1}{m+1}}T_1,$$

where

$$T_1 = 2\pi\int_0^R r^{2+\frac{1}{m+1}}dr = 2\pi\frac{m+1}{3m+4}R^{\frac{3m+4}{m+1}}.$$

Assuming that torque M_T^0 remains constant, from the algebraic equation the twist rate resolves as:

$$\dot{\theta} = \eta_1 t^{k-1}$$

with $\eta_1 = c_\tau\left(\frac{M_T^0}{T_1}\right)^{m+1}$ providing the inhomogeneous ordinary differential equation of the first order with the initial condition

$$\theta(0) = 0.$$

The twist rate during the "engineering creep" is the following solution of this initial value problem:

$$\theta = \eta_1 t^k / k.$$

4. Creep and relaxation of beams in bending

4.1. *Basic constitutive equations for relaxation in bending*

Consider the problem of stress relaxation in the pure bending of a rectangular cross-section ($B \times H$) beam. In the applied Euler–Bernoulli theory of slender beams subjected to a bending moment M_B, a major assumption is that 'plane sections remain plane'. In other words, any deformation due to shear across the section is not accounted for (no shear deformation). During the relaxation experiment the curvature of the neutral axis of beam κ remains constant over time, such that the bending moment M_B continuously decreases. This case describes the relaxation of bending stress, assuming that the flexure deformation of beam does not alter in time.

Let $\sigma(z,t)$ is uniaxial stress in the beam in the direction of beam axis. The total strain in any instant of the time is $\varepsilon(z,t)$; is the sum of the elastic and the creep components of the strain:

$$\varepsilon = \varepsilon_e + \varepsilon_c. \quad (29)$$

The elastic component of shear strain is

$$\varepsilon_e = \sigma/E, \quad (30)$$

where E is the shear modulus and $\varepsilon_c(z,t)$ is the creep component of normal strain.

Consider creep under constant in time total strain

$$\varepsilon(z,t) = \varepsilon_e(z,t) + \varepsilon_c(z,t) \equiv \varepsilon_0(r). \quad (31)$$

The normal strain $\varepsilon_0(r) = \varepsilon(z, t=0)$ is the function of radius, but remains constant over time. The time differentiation of (31) leads to

$$\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_c \equiv 0. \tag{32}$$

The kinematic assumptions upon which the Euler–Bernoulli beam theory is founded allow it to be extended to more advanced analysis. Simple superposition allows for three-dimensional transverse loading. Euler–Bernoulli beam theory can also be extended to the analysis of curved beams, beam buckling, composite beams, and geometrically nonlinear beam deflection. The common constitutive models of creep in bending state are studied below.

4.2. Bending moment relaxation for the material, that obeys Norton–Bailey law

The Norton–Bailey law for a uniaxial state of stress reads

$$\dot{\varepsilon}_c(z,t) = c_\sigma t^{k-1}\sigma^{m+1}, \tag{33}$$

The substitution of material laws results in the ordinary differential equation for uniaxial stress

$$\frac{\dot{\sigma}}{E} + c_\sigma t^{k-1}\sigma^{m+1} = 0. \tag{34}$$

The initial condition for the Eq. (34) delivers the pure elastic shear stress in the initial moment

$$\sigma(z, t=0) = \sigma_0(z). \tag{35}$$

For pure elastic bending the following hypothesis of Euler–Bernoulli for initial distribution of stresses over the cross-section of the beam is valid:

$$\sigma_0(z) = E\kappa z,$$

where κ is the bending curvature, which presumed to be constant over time and z is the perpendicular distance to the neutral axis. The hypothesis of Euler–Bernoulli beam does not account for the effects of transverse shear strain.

The solution of the ordinary differential equation (34) with initial condition (35) is

$$\sigma(z,t) = \left[\sigma_0^{-m}(z) + \frac{c_\sigma E m t^k}{k}\right]^{-1/m} = E\kappa \cdot \left[\frac{1}{z^m} + \frac{c_\sigma \kappa^m E^{m+1} m}{k} t^k\right]^{-1/m}. \tag{36}$$

The bending moment for the rectangular cross-section of width B and height H is the function of time.

$$M_B(t) = B\int_{-H/2}^{H/2} z\sigma(z,t)dz. \tag{37}$$

With the expression (36) we can calculate the moment in (37) as

$$M_B(t) = 2BE\kappa \int_0^{H/2} z \cdot \left[\frac{1}{z^m} + \frac{c_\sigma \kappa^m E^{m+1} m}{k} t^k\right]^{-1/m} dz. \tag{38}$$

Using the Eq. (A4) for $J_p(a,m;X)$ for case $p=1$, the integral in (38) could be expressed in terms of hypergeometric function

$$M_B(t) = 2BE\kappa \cdot J_1\left(\frac{c_\sigma \kappa^m E^{m+1} m t^k}{k}, m; \frac{H}{2}\right) = {}_2F_1\left(\frac{3}{m}, \frac{1}{m}; \frac{3+m}{m}; -\frac{c_\sigma \kappa^m E^{m+1} m t^k}{k}\left(\frac{H}{2}\right)^m\right) M_B^0, \tag{39}$$

where

$$M_B^0 = EBH^3\kappa/12$$

is the elastic bending moment at time $t=0$.

4.3. Bending moment relaxation for the material, that obeys Prandtl–Garofalo law

The Prandtl–Garofalo law for uniaxial state of stress reads

$$\dot{\varepsilon}_c = c_\sigma t^{k-1}\sinh\left(\frac{\sigma}{\bar{\sigma}}\right). \tag{40}$$

The solution of the ordinary differential equation (32) with initial condition (35) for the Prandtl–Garofalo creep law (40) leads to the expression of normal stress as function of coordinate z and time t:

$$\sigma(z,t) = \bar{\sigma}\ln\left\{\tanh\left[\frac{Ec_\sigma}{2k\bar{\sigma}}t^k + \operatorname{arctanh}\left(\exp\left(\frac{E\kappa}{\bar{\sigma}}z\right)\right)\right]\right\}. \tag{41}$$

For evaluation the formula (A2) for $I_1(a,b;X)$ is applied. With this formula the integral in (37) could be expressed in terms of polylogarithm

$$M_B(t) = 2B\bar{\sigma}I_1\left(\frac{Ec_\sigma}{2k\bar{\sigma}}t^k, \frac{E\kappa}{\bar{\sigma}}; \frac{H}{2}\right). \tag{42}$$

4.4. Bending moment relaxation for the material, that obeys Naumenko–Altenbach–Gorash law

In this section the problem of the pure bending of a rectangular cross-section beam with a modified power law (stress range-dependent constitutive model) subjected to a bending moment is solved. The description of creep is based on Naumenko–Altenbach–Gorash law.

The substitution of modified power material law (9) results in the ordinary differential equation for uni-axial stress

$$\frac{\dot{\sigma}}{E} + \bar{\varepsilon}\cdot\left[\frac{\sigma}{\bar{\sigma}} + \left(\frac{\sigma}{\bar{\sigma}}\right)^{m+1}\right] = 0. \tag{43}$$

When loaded by a bending moment, the beam bends so that the inner surface is in compression and the outer surface is in tension. The neutral plane is the surface within the beam between these zones, where the material of the beam is not under stress, either compression or tension. The solution of the ordinary differential equation (43) with initial condition (35) delivers the stress over the cross-section of the beam as the function of time and distance z to neutral plane

$$\sigma(z,t) = \frac{z}{\left[z^m \frac{m\bar{\varepsilon}E\varsigma - \bar{\sigma}}{\bar{\sigma}^{m+1}} + \frac{\varsigma}{E^m\kappa^m}\right]^{1/m}}, \tag{44}$$

where

$$\varsigma = \varsigma(t) = \exp\left(\frac{m\bar{\varepsilon}E}{\bar{\sigma}}t\right).$$

For calculation of the bending moment (37) the formula (A4) for $J_p(a,m;X)$ used for the case $p=1$. With this formula the bending moment in the cross-section could be expressed in terms of hypergeometric function:

$$M_B(t) = {}_2F_1\left(\frac{3}{m}, \frac{1}{m}; \frac{3+m}{m}; \frac{(\kappa EH)^m}{2^m\bar{\sigma}^{m+1}}(\bar{\sigma} - m\bar{\varepsilon}E\varsigma)\right)\varsigma^{-1/m}M_B^0. \tag{45}$$

The curvature κ of the beam remains constant in time. In the expressions (39), (42), and (45) the bending moment $M_B(t)$ is the function of time and continuously relaxes with time.

4.5. "Engineering creep" phenomenon members subjected to bending

As mentioned above, the "engineering creep" phenomenon is an increased tendency toward more strain under constant load. The beam is stressed with a invariable bending moment. Over time, the moment and stress do not change, although the curvature of the beam $\kappa = \kappa(t)$ continuously increases. The elongation rate of the strip, which locates on the perpendicular distance z to the neutral axis, is

$$\dot{\varepsilon}(t,z) = \dot{\kappa}z \qquad \text{for} \ -H/2 < z < H/2.$$

According to Norton–Bailey law (3) the shear stress due to creep is

$$\sigma = \left(\frac{\dot{\varepsilon}}{c_\sigma t^{k-1}}\right)^{\frac{1}{m+1}} = \left(\frac{\dot{\kappa}z}{c_\sigma t^{k-1}}\right)^{\frac{1}{m+1}}.$$

Performing the integration over the area of wire we get the bending moment due to creep

$$M_B^0 = B\int_{-H/2}^{H/2} z\sigma(z,t)dz = \frac{BH^2}{2}\frac{m+1}{3+2m}\left(\frac{H\dot{\kappa}}{2t^{k-1}c_\sigma}\right)^{\frac{1}{1+m}}.$$

Assuming that bending moment M_B^0 remains constant over time, from this equation the curvature time rate resolves as

$$\dot{\kappa} = \beta_1 t^{k-1} \quad \text{with} \quad \kappa(0) = 0 \tag{46}$$

and

$$\beta_1 = \frac{2c_\sigma}{H}\left(\frac{2M_B^0}{BH^2}\frac{3+2m}{m+1}\right)^{m+1}$$

Thus, the flexure

$$\kappa = \beta_1 t^k / k$$

increases over time as t^k.

The results of this section are applicable for springs, that overwhelmingly stressed by bending loads, like the leaf springs and torsion springs. Torsion springs may be of helix or spiral type.

5. Fractional creep models

5.1. Fractional generalization of creep laws

The motivation for the introduction of fractional models is as follows. Scott-Blair (1947) revealed a framework that enabled the power-law equation proposed by Nutting (1921) through the use of fractional calculus. Its model was the analogy to the classical the Hookean spring element, in which the stress in the spring is proportional to fractional strain rate. This element interpolates between the constitutive responses of a spring and a dashpot. In the work (Koeller, 1984) this canonical modal leaded to a mechanical element called the spring-pot, known also as the Scott-Blair element. Similar to the Maxwell element in hereditary mechanics, the fractional Scott-Blair element could be considered as the fundamental building block from which more complex constitutive models could be constructed. The creep and relaxation power laws of the Scott–Blair model are interpreted sometimes in terms of a continuous spectrum of retardation and relaxation times, respectively (Mainardi, 2010).

In the paper (Jaishankar and McKinley, 2013) was demonstrated, that fractional stress–strain relationships are also applicable to viscoelastic interfaces, and result in simple constitutive models that may be used to quantitatively describe the power-law rheological behavior exhibited by such interfaces.

The use of fractional calculus leads to generalizations of the classical mechanical models: the basic Newton element of order one is substituted by the more general Scott–Blair element of order α. The straightforward way to introduce fractional derivatives in creep (Mainardi and Spada, 2011) is the replacing in the constitutive equation the creep model the first derivative with a fractional derivative D^α of order $0 < \alpha \leqslant 1$,

$$D^\alpha \varepsilon'_{ij} = \frac{3\sigma'_{ij}}{2\sigma_{eff}} F(\sigma_{eff}, t), \tag{47}$$

where

$$D^\alpha \varepsilon'_{ij} = D^\alpha \varepsilon_{ij} - \frac{1}{3}\delta_{ij} D^\alpha \varepsilon_{kk},$$

is the deviatoric component of fractional creep strain rate.

The function $F(\sigma_{eff}, t)$ describes the material properties, and D^α is the fractional derivative operator (Miller and Ross, 1993). The basic definitions of fractional calculus in a form most useful for applications in rheology are briefly outlined in Mainardi and Gorenflo (2007), Atangana1 and Secer (2013). The actual definition of the fractional derivative operator is in Appendix D.

5.2. Fractional Norton–Bailey law

Firstly, consider fractional generalization of Norton–Bailey law. The isotropic stress function reads in this case

$$F(\sigma_{eff}, t) = \bar{\varepsilon}\left(\frac{t}{\bar{t}}\right)^{K-1}\left(\frac{\sigma_{eff}}{\bar{\sigma}}\right)^{M+1}, \tag{48}$$

where $\bar{\varepsilon}$, $\bar{\sigma}$, $\bar{t}$, M and K are the experimental constants.

There is only non-vanishing component of stress tensor ($\sigma_{11} = \sigma$) for the uniaxial stress state. Correspondingly, the non-vanishing components of fractional strain rate are

$$D^\alpha \varepsilon \equiv D^\alpha \varepsilon_{11} = -2D^\alpha \varepsilon_{22} = -2D^\alpha \varepsilon_{33},$$

where

$$D^\alpha \varepsilon = \bar{\varepsilon}\left(\frac{t}{\bar{t}}\right)^{K-1}\left(\frac{\sigma}{\bar{\sigma}}\right)^{M+1}.$$

For brevity of equations, we introduce the material constant

$$C_\sigma = \frac{\bar{\varepsilon}}{\bar{t}^{K-1}\bar{\sigma}^{M+1}}.$$

With this constant the dependence of uniaxial fractional strain rate upon stress reads

$$D^\alpha \varepsilon = C_\sigma t^{K+1}\sigma^{M+1}. \tag{49}$$

For pure shear stress state ($\sigma_{12} = \sigma_{21} = \tau$ the non-vanishing components of deformation rate are

$$D^\alpha \varepsilon_{12} = D^\alpha \varepsilon_{21} \equiv \frac{1}{2}D^\alpha \gamma = \frac{1}{2}\bar{\gamma}\left(\frac{t}{\bar{t}}\right)^{K-1}\left(\frac{\tau}{\bar{\tau}}\right)^{M+1},$$

with the creep constant for shear strain

$$C_\tau = \frac{\bar{\gamma}}{\bar{t}^{K-1}\bar{\tau}^{M+1}}.$$

The fractional Norton–Bailey creep law for pure shear deformation reduces to

$$D^a \gamma = C_\tau t^{K-1}\tau^{M+1}. \tag{50}$$

There is a relation between the constants in Eqs. (49) and (50):

$$C_\tau = 3^{M/2+1} C_\sigma.$$

5.3. Basic constitutive equations for relaxation in torsion

The creep for the total deformation that remains constant in time is studied. The elastic and the creep components of strain are the functions as well of radius and of time, such that:

$$\gamma(r,t) = \gamma_e(r,t) + \gamma_c(r,t) \equiv \gamma_0(r). \tag{51}$$

The fractional time differentiation of (52) leads to the fractional differential equation for elastic and creep strain rates:

$$D^{a}\gamma(r,t)=D^{a}\gamma_{e}(r,t)+D^{a}\gamma_{c}(r,t)\equiv 0,\quad 0<\alpha\leqslant 1. \tag{52}$$

Fractional derivation of (12) shows, that the elastic component of fractional strain rate is

$$D^{\alpha}\dot{\gamma}_{e}=\frac{1}{G}D^{\alpha}\dot{\tau}. \tag{53}$$

5.4. Torque relaxation for the materials, that obeys fractional Norton–Bailey law

The fractional Norton–Bailey law (50) for the state of shear stress reads

$$D^{\alpha}\gamma_{c}(r,t)=GC_{\tau}t^{K-1}\tau^{M+1}, \tag{54}$$

where the order of differentiation is $0<\alpha<1$.

The substitution of material laws (54) and (55) in Eq. (53) results in the nonlinear fractional differential equation of order α for total shear stress $\tau(r,t)$:

$$D^{\alpha}\tau+GC_{\tau}t^{K-1}\tau^{M+1}=0. \tag{55}$$

Eq. (56) is a nonlinear fractional differential equation of the order α of type (A10). The nonlinear fractional differential equation (56) with the initial condition (8) could be solved in closed from using the method of variable separation. The solution is given by Eq. (A12). The proper choice of constants makes possible, that the same formula expresses the solutions of both nonlinear fractional differential equation (56) and nonlinear ordinary differential equation (17). Namely, let the material constant C_{τ} in the Eq. (56) relates to the constant c_{τ}, which appears in the ordinary Norton–Bailey law with the formula

$$C_{\tau}=\frac{m\,\Gamma(1-m-\alpha)\,\Gamma(1+k)}{k\,\Gamma(1+k-\alpha)\,\Gamma(1-m)}c_{\tau}=\frac{(M-\alpha)\,\Gamma(1-M)\,\Gamma(\alpha+K)}{(K-1+\alpha)\,\Gamma(1+K-1)\,\Gamma(1-M-\alpha)}c_{\tau},$$

with

$$K=k+1-\alpha,$$

$$M=m+\alpha.$$

In this case the solution of the fractional differential equation (56) with initial condition (8) delivers the shear stress over the cross-section of the twisted rod as the function of time and radius by means of formula (19). This simplifies the calculation of torque, because the resulting torque for fractional Norton–Bailey law is given by the same formula (20).

5.5. Fractional "engineering creep" of rods subjected to torsion

The rod is twisted now with an applied torque M_T^0, which remains constant over time. Over time, the torque do not change, although the twist angle of the rod continuously increases.

According to Norton–Bailey law (50) the shear stress due to creep is

$$\tau=\left(\frac{D^{\alpha}\gamma}{C_{\tau}t^{K-1}}\right)^{\frac{1}{M+1}}=\left(\frac{rD^{\alpha}\theta}{C_{\tau}t^{K-1}}\right)^{\frac{1}{M+1}}.$$

Performing the integration in (20) over the area of wire we get the torque due to creep

$$M_T^0(t)=2\pi\int_0^R r^2\tau(r,t)dr=2\pi\int_0^R r^2\left(\frac{rD^{\alpha}\theta}{C_{\tau}t^{K-1}}\right)^{\frac{1}{M+1}}dr=\left(\frac{D^{\alpha}\theta}{C_{\tau}t^{K-1}}\right)^{\frac{1}{M+1}}T_{\alpha},$$

where

$$T_{\alpha}=2\pi\int_0^R r^{2+\frac{1}{M+1}}dr=2\pi\frac{M+1}{3M+4}R^{\frac{3M+4}{M+1}}.$$

Assuming that torque M_T^0 remains constant over time, from this equation the twist rate resolves as

$$D^{\alpha}\theta=t^{K-1}\eta_{\alpha}\qquad\text{with}\qquad\theta(0)=0,$$

where

$$\eta_{\alpha}=C_{\tau}\left(\frac{M_T^0}{T_{\alpha}}\right)^{M+1}.$$

The solution of this fractional inhomogeneous equation of type (A8) delivers the formula (A9). The twist rate during the "engineering creep" in fractional case reads

$$\theta(t)=\eta_{\alpha}\frac{\Gamma(K)}{\Gamma(K+\alpha)}t^{K-1+\alpha}.$$

5.6. Bending moment relaxation for the material, that obeys fractional Norton–Bailey law

Consider the problem of stress relaxation in the pure bending of a rectangular cross-section ($B\times H$) beam. The fractional time differentiation of (31) leads to

$$D^{\alpha}\varepsilon=D^{\alpha}\varepsilon_{e}+D^{\alpha}\varepsilon_{c}\equiv 0 \tag{56}$$

and Hook's law – likewise to Eq. (54) – reads

$$D^{\alpha}\varepsilon_{e}=\frac{1}{E}D^{\alpha}\sigma. \tag{57}$$

The fractional Norton–Bailey law for a uniaxial state of stress reads

$$D^{\alpha}\varepsilon_{c}(z,t)=C_{\sigma}t^{K-1}\sigma^{M+1}, \tag{58}$$

The substitution of the material law (58) and the Hook's law in the Eq. (57) results in the fractional differential equation for uniaxial stress

$$D^{\alpha}\sigma+EC_{\sigma}t^{K-1}\sigma^{M+1}=0. \tag{59}$$

This is a nonlinear fractional differential equation of the order α of type (A10). The initial condition (35) for the Eq. (59) delivers the pure elastic shear stress in the initial moment.

The Euler–Bernoulli hypothesis for initial distribution of stresses over the cross-section of the beam is valid at initial moment of time, as the deformation of beam starts from the pure elastic bending.

For the simplification of the mathematical formulas it is assumed, that the constants of fractional and ordinary differential equations relates to each other as following

$$C_{\sigma}=\frac{m\,\Gamma(1-m-\alpha)\,\Gamma(1+k)}{k\,\Gamma(1+k-\alpha)\,\Gamma(1-m)}c_{\sigma}=\frac{(M-\alpha)\,\Gamma(1-M)\,\Gamma(\alpha+K)}{(K-1+\alpha)\,\Gamma(1+K-1)\,\Gamma(1-M-\alpha)}c_{\sigma}.$$

In this case the solution of the fractional differential equation (59) with initial condition (35) and the solution of the ordinary differential equation (34) with initial condition (35) coincide. In other words, the fractional differential equation and the ordinary differential equation possess the same solution:

$$\sigma(z,t)=\left[\sigma_0^{-m}(z)+\frac{c_{\sigma}E\,m\,t^{k}}{k}\right]^{-1/m}=E\kappa\cdot\left[\frac{1}{z^{m}}+\frac{c_{\sigma}\kappa^{m}E^{m+1}m}{k}t^{k}\right]^{-1/m}. \tag{60}$$

For calculation of bending moment for the rectangular cross-section of width B and height H as the function of time the expression (37) is used. With the Eq. (37) we can calculate the time-depending bending moment

$$M_B(t) = 2BE\kappa \int_0^{H/2} z \cdot \left[\frac{1}{z^m} + \frac{c_\sigma \kappa^m E^{m+1}}{k} \frac{m}{} t^k\right]^{-1/m} dz. \tag{61}$$

This makes straightforward the calculation of bending moment, because the same formula delivers the resulting moment for fractional Norton–Bailey law also. Using the results of Appendix B ($J_p(a,m;X)$, case $p=1$), the integral in (61) is expressed in terms of hypergeometric function by formula (39).

5.7. Basic constitutive equations for creep phenomenon in structural members subjected to bending

The fractional elongation rate of the strip, which locates on the perpendicular distance z to the neutral axis, is

$$D^\alpha \varepsilon(t,z) = zD^\alpha \kappa \qquad \text{for} \qquad -H/2 < z < H/2.$$

According to Norton–Bailey law (58) the shear stress due to creep is

$$\sigma = \left(\frac{1}{C_\sigma t^{K-1}} D^\alpha \varepsilon\right)^{\frac{1}{M+1}} = \left(\frac{z}{C_\sigma t^{K-1}} D^\alpha \kappa\right)^{\frac{1}{M+1}}.$$

Performing the integration over the area of wire we get the bending moment due to creep

$$M_B^0 = B \int_{-H/2}^{H/2} z\sigma(z,t)dz = \frac{BH^2}{2} \frac{M+1}{3+2M} \left(\frac{H}{2t^{K-1}C_\sigma} D^\alpha \kappa\right)^{\frac{1}{1+M}}.$$

Assuming that bending moment M_B^0 remains constant over time, from this equation the curvature time rate resolves as

$$D^\alpha \kappa = \beta_\alpha t^{K-1}, \tag{62}$$

where

$$\beta_\alpha = \frac{2C_\sigma}{H} \left(\frac{2M_B^0}{BH^2} \frac{3+2M}{M+1}\right)^{M+1}.$$

Eq. (62) is a linear inhomogeneous fractional differential equation of type (A8). The solution of fractional differential equation (62) is given by Eq. (A9) and with the initial condition

$$\kappa(0) = 0$$

reads:

$$\kappa(t) = \beta_\alpha \frac{\Gamma(K)}{\Gamma(\alpha+K)} t^{K-1+\alpha}. \tag{63}$$

As shown above, the analogous time dependence of twist angle upon time appears in creep under constant torque.

6. Application of the derived formulas for design of structural elements

The application of the solutions allows accurate analytic description of creep and relaxation of practically important problems in mechanical engineering. As the bending and torsion frequently occur in structural members, the results are immediately applicable. The derived formulas are immediately relevant for creep calculation of torsion members with circular cross-sections, like shafts and twist beams.

Another example of high loaded elements of machinery deliver the springs made of steel. The springs store the elastic energy either by means of bending or torsion. Respectively, in material dominates either uniaxial or pure shear stress state. The physical phenomenon with metal springs is that at stress below the yield strength of the material a slow inelastic deformation take place. In the spring branch this is called creep when a spring under constant load loose length and it is called relaxation when a spring under constant compression lose load. The creep and relaxation rates depend on the temperature, the stress in the metal, the yield strength and the time. Increased temperature, stress and time also increase the creep and relaxation rates. Especially the temperature and stress have a major influence. The precise creep description is essentially important for correct dimensioning of springs.

7. Conclusion

The essential task of this Article is the derivation of the exact closed form expressions for torsion and bending creep for isotropic materials, which obey the commonly accepted constitutive laws. The laws of Norton–Bailey, Prandtl–Garofalo and Naumenko–Altenbach–Gorash allow the closed form solution. The fractional generalization of Norton–Bailey law is also solved in closed form. The relaxation of stresses was studied for structural elements subjected to torsion and bending moments.

The structures examined are elementary – a beam in bending, a rod in torsion and helical and disk springs – but demonstrate the basic characteristics of nonlinear creep. The closed form solutions with common creep models allow a deeper understanding of hereditary effects in structural members and make easier the design procedure.

Appendix A. Integrals with polylogarithm

The weighted integrals of the function

$$f(x) = \ln(\tanh(a + \text{arctanh}(exp(bx))))$$

are:

$$I_0(a,b;X) \equiv \int_0^X f(x)dx = X\ln(\tanh(a)) + \frac{1}{b}(\Lambda_2 - \mathrm{M}_2 + \mu_2 - \lambda_2), \tag{A1}$$

$$I_1(a,b;X) \equiv \int_0^X f(x)xdx = \tfrac{1}{6b^2}\left[\pi^2 \ln(\coth(a)) + \ln^3(\coth(a)) + 3b^2X^2\ln(\tanh(a))\right] + \tfrac{1}{b^2}(\mathrm{M}_3 - \Lambda_3) - \tfrac{X}{b}(\mathrm{M}_2 - \Lambda_2), \tag{A2}$$

$$I_2(a,b;X) \equiv \int_0^X f(x)x^2dx = \tfrac{X^3}{3}\ln(\tanh(a)) + \tfrac{2}{b^3}(\Lambda_4 - \mathrm{M}_4 + \mu_4 - \lambda_4) + \tfrac{2X}{b^2}(\mathrm{M}_3 - \Lambda_3) - \tfrac{X^2}{b}(\mathrm{M}_2 - \Lambda_2). \tag{A3}$$

The following abbreviations are used:

$$\mathrm{M}_k = \mathrm{Li}_k(-\coth(a)\exp(bX)),$$
$$\Lambda_k = \mathrm{Li}_k(-\tanh(a)\exp(bX)),$$
$$\mu_k = \mathrm{M}_k|_{X=0} \equiv \mathrm{Li}_k(-\coth(a)),$$
$$\lambda_k = \Lambda_k|_{X=0} \equiv \mathrm{Li}_k(-\tanh(a)).$$

In these expressions is $\mathrm{Li}_k(z)$ the polylogarithm of order k and argument z.

The expression of polylogarithm as the integral of the Bose–Einstein distribution is used:

$$\mathrm{Li}_k(z) = \frac{1}{\Gamma(k)} \int_0^\infty \frac{x^{k-1}}{z^{-1}\exp(x) - 1} dx$$

This integral converges for $\mathrm{Re}(k) > 0$ and all z except for z real and $\geqslant 1$.

Appendix B. Integrals with hypergeometric function

The weighted integrals of the function

$g = (a + x^{-m})^{-1/m}$,

are

$$J_p(a,m;X) \equiv \int_0^X x^p g(x)dx \;=\; {}_2F_1\left(\frac{1}{m},\frac{2+p}{m};\frac{2+p+m}{m};-aX^m\right)\frac{X^{2+p}}{2+p}, \quad p \geqslant 0. \tag{A4}$$

Appendix C. Integrals with incomplete beta function

The weighted integrals of the function

$\left(\frac{|c-r|}{r}\right)^{1/n}$

are

$$\begin{aligned} K_n(a,b,c) &\equiv \frac{1}{c}\int_a^b \left(\frac{|c-r|}{r}\right)^{1/n} dx \\ &= \frac{1}{c}\left[\int_a^c \left(\frac{c-r}{r}\right)^{1/n} dx + \int_c^b \left(\frac{r-c}{r}\right)^{1/n} dx\right] \\ &= \frac{1}{c}\lim_{\varepsilon\to+0}\left[\int_a^c \left(\frac{c-r+\varepsilon}{r}\right)^{1/n} dx + \int_c^b \left(\frac{r-c+\varepsilon}{r}\right)^{1/n} dx\right] \\ &= \frac{(-1)^{-1/n}\pi}{n}\left(i+\cot\left(\frac{\pi}{2n}\right)\right) - \mathrm{B}\left(\frac{a}{c};\frac{n-1}{n},\frac{n+1}{n}\right) \\ &\quad - (-1)^{1/n}\mathrm{B}\left(\frac{b}{c};\frac{n-1}{n},\frac{n+1}{n}\right) \end{aligned} \tag{A5}$$

and

$$\begin{aligned} L_n(a,b,c) &\equiv \frac{1}{c^2}\left[\int_a^b \left(\frac{|c-r|}{r}\right)^{1/n}(c-r)dx\right] \\ &= \frac{1}{c^2}\left[\int_a^c \left(\frac{c-r}{r}\right)^{1/n}(c-r)dx + \int_c^b \left(\frac{r-c}{r}\right)^{1/n}(c-r)dx\right] \\ &= \frac{1}{c^2}\lim_{\varepsilon\to+0}\left[\int_a^c \left(\frac{c-r+\varepsilon}{r}\right)^{1/n}(c-r)dx + \int_c^b \left(\frac{r-c+\varepsilon}{r}\right)^{1/n}(c-r)dx\right] \\ &= \frac{i\pi}{\exp\left(\frac{i\pi}{n}\right)-1}\,\frac{1+n}{n^2} + \mathrm{B}\left(\frac{a}{c};\frac{2n-1}{n},\frac{n+1}{n}\right) - \mathrm{B}\left(\frac{a}{c};\frac{n-1}{n},\frac{n+1}{n}\right) \\ &\quad + (-1)^{-1/n}\left[\mathrm{B}\left(\frac{b}{c};\frac{2n-1}{n},\frac{n+1}{n}\right) - \mathrm{B}\left(\frac{b}{c};\frac{n-1}{n},\frac{n+1}{n}\right)\right]. \end{aligned} \tag{A6}$$

In these expressions

$$\mathrm{B}(x;p,q) = {}_2F_1(p,1-q;p+1;x)\frac{x^p}{p} = \int_0^x z^{p-1}(1-z)^{q-1}dz$$

is the incomplete beta function (Pearson, 1968).

Appendix D. Solutions of some fractional differential equations

The Davidson–Essex definition of derivative operator (Davison and Essex, 1998) is the following:

$$D^{\alpha,p,n}f(t) = \frac{d^{n+1-p}}{dt^{n+1-p}}\int_0^t \frac{(t-\varsigma)^{n-\alpha}}{\Gamma(1+n-\alpha)}\frac{d^p f(\varsigma)}{d\varsigma^p}d\varsigma$$

where $0 < p \leqslant n+1$.

If $n = 0$ and $p = 1$ the Davidson–Essex derivative operator turns to be Caputo derivative operator

$$D^\alpha f(t) = \int_0^t \frac{(t-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)}\frac{df(\varsigma)}{d\varsigma}d\varsigma \quad \text{for} \quad (0<\alpha<1. \tag{A7}$$

Table 1
Ordinary and fractional differential equations, that possess the same solution.

Ordinary equation	Fractional equation	Solution
$\dot{\tau} + G c_\tau t^{k-1}\tau^{m+1} = 0$	$D^\alpha\tau + G C_\tau t^{K-1}\tau^{M+1} = 0$	$\tau(r,t) = \left[\tau_0^{-m}(r) + \frac{c_\tau G m t^k}{k}\right]^{-1/m}$
c_τ	$C_\tau = -\frac{\Gamma(1-m-\alpha)\Gamma(1+k)}{\Gamma(1+k-\alpha)\Gamma(1-m)}c_\tau$	
$\dot{\sigma} + E c_\sigma t^{k-1}\sigma^{m+1} = 0$	$D_\alpha\sigma + E C_\sigma t^{K-1}\sigma^{M+1} = 0$	$\sigma(z,t) = \left[\sigma_0^{-m}(z) + \frac{c_\sigma E m t^k}{k}\right]^{-1/m}$
c_σ	$C_\sigma = \frac{m\Gamma(1-m-\alpha)\Gamma(1+k)}{k\Gamma(1+k-\alpha)\Gamma(1-m)}c_\sigma$	
k	$K = k+1-\alpha$	
m	$M = m+\alpha$	

The solution of linear fractional differential equation (Podlubny, 1999)

$$D^\alpha f = \beta t^{K-1} \tag{A8}$$

could be found using Laplace transformation. The Laplace transformation of the fractional differential equation reads

$$-f(0)q^{\alpha-1} + q^\alpha\hat{f}(q) = \beta\Gamma(K)q^{-K}$$

where $\hat{f}(q)$ is the Laplace transformation of the function $f(t)$.

The Laplace transformation is

$$\hat{f}(q) = \beta\Gamma(K)q^{-\alpha-K} + f(0)q^{-1}.$$

Its inversion delivers

$$f(t) = y(0) + \beta\frac{\Gamma(K)}{\Gamma(K+\alpha)}t^{K+\alpha-1}. \tag{A9}$$

The solution of nonlinear fractional differential equation

$$D^\alpha f + \beta t^{K-1}f^{M+1} = 0 \quad \text{with} \quad f(0) = f_0 \quad \text{for } (0<\alpha<1, M>0, K\geqslant 1. \tag{A10}$$

is found using separation of variables. The Eq. (A7) is equivalent to

$$\frac{D^\alpha f}{f^{M+1}} - \frac{D^\alpha f_0}{f_0^{M+1}} + \beta t^{K-1}D^\alpha t = 0.$$

Applying the fractional differential operator $D^{-\alpha}$, we get

$$\frac{\Gamma(1-M)}{\Gamma(1-M+\alpha)}(f^{\alpha-M} - f_0^{\alpha-M}) + \beta\frac{\Gamma(K)}{\Gamma(K+\alpha)}t^{K-1+\alpha} = 0. \tag{A11}$$

The resolution of the algebraic equation (A8) with respect to $f = f(t)$ provides the desired solution of fractional differential equation (A7) as

$$f^{\alpha-M} = \left(f_0^{\alpha-M} - \beta\frac{\Gamma(1-M+\alpha)\Gamma(K)}{\Gamma(K+\alpha)\Gamma(1-M)}t^{K-1+\alpha}\right)^{\frac{1}{\alpha-M}}. \tag{A12}$$

For the proper choice of parameters the solutions of ordinary and fractional equations coincide (see Table 1).

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