On best rational approximation of analytic functions

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Abstract

This paper contains some theorems related to the best approximation $\rho_n(f; E)$ to a function $f$ in the uniform metric on a compact set $E \subset \mathbb{C}$ by rational functions of degree at most $n$. We obtain results characterizing the relationship between $\rho_n(f; K)$ and $\rho_n(f; E)$ in the case when complements of compact sets $K$ and $E$ are connected, $K$ is a subset of the interior $\Omega$ of $E$, and $f$ is analytic in $\Omega$ and continuous on $E$.

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1. Meromorphic approximation and Hankel operators

1.1. Notation

Let $G \subset \mathbb{C}$ be a bounded domain and let $\Gamma$ be the boundary of $G$. We assume that $\Gamma$ consists of a finite number of closed analytic Jordan curves. Denote by $L_p(\Gamma)$, $1 \leq p < \infty$, the Lebesgue space of measurable functions $\varphi$ on $\Gamma$ with the norm given by
the formula
\[ ||\varphi||_p = \left( \int_G |\varphi(\xi)|^p |d\xi| \right)^{1/p}. \]

We use the following notation for the inner product in the Hilbert space \( L_2(\Gamma) \):
\[ (\varphi, \psi) = \int_G \varphi(\xi) \overline{\psi}(\xi) |d\xi|, \quad \varphi, \psi \in L_2(\Gamma). \]

Let \( L_\infty(\Gamma) \) be the space of essentially bounded on \( \Gamma \) functions with the norm
\[ ||\varphi||_\infty = \text{ess sup}_G |\varphi(\xi)|, \quad \varphi \in L_\infty(\Gamma). \]

Denote by \( E_p(G) \), \( 1 \leq p \leq \infty \), the Smirnov class of analytic functions on \( G \). For \( 1 \leq p < \infty \) the class \( E_p(G) \) consists of the functions \( \varphi \) for which there is a sequence of domains \( G_k \) with rectifiable boundaries having the following properties:
\[ G_{k+1} \subset G_k, \quad \overline{G}_k \subset G, \quad \bigcup_k G_k = G \]
and
\[ \sup_k \int_{\partial G_k} |\varphi(\xi)|^p |d\xi| < \infty. \]

\( E_\infty(G) \) is the class of bounded analytic functions on \( G \). The condition
\[ \int_\Gamma \frac{\varphi(\xi)}{\xi - z} d\xi = 0 \quad \text{for all} \quad z \in \overline{C} \setminus \overline{G} \]
is necessary and sufficient for a function \( \varphi \in L_1(\Gamma) \) to be the boundary value of a function in the Smirnov class \( E_1(G) \) (see [4,9] for more details about the classes \( E_p(G) \)).

Let \( \sigma \) be a positive Borel measure with support \( \text{supp} \sigma = F \subset G \). Let \( L_q(\sigma, F) \), \( 1 \leq q < \infty \), be the Lebesgue space of measurable functions \( \varphi \) on \( F \) with the norm
\[ ||\varphi||_{q,\sigma} = \left( \int_F |\varphi(\xi)|^q d\sigma(\xi) \right)^{1/q}. \]

Denote by \( J : E_2(G) \to L_2(\sigma, F) \) the embedding operator. The operator \( J \) is given by restricting an element \( \varphi \in E_2(G) \) to \( F \): \( J \varphi = \varphi|_F \). It is not hard to see that \( J \) is a compact operator.

1.2. Auxiliary results from the theory of Hankel operators

Consider a function \( f \) continuous on \( \Gamma \). We define the Hankel operator \( A_f = A_{f,G} : E_2(G) \to E_2^\perp(G) = L_2(\Gamma) \ominus E_2(G) \) by
\[ A_f(\varphi) = P_- (\varphi f), \quad \varphi \in E_2(G), \]
where \( P_- \) is the orthogonal projection from \( L_2(\Gamma) \) onto \( E_2^\perp(G) \). The function \( f \) is called a symbol of the Hankel operator \( A_f \). We remark that \( A_f \) is a compact operator. Let \( \{s_n(A_f)\}, n = 0, 1, 2, \ldots \), be the sequence of singular numbers of the operator \( A_f \) (the
sequence of eigenvalues of the operator \((A^*_f A_f)^{1/2}\), where \(A^*_f : E^1_2(G) \to E_2(G)\) is the adjoint of \(A_f\). We assume that \(s_0(A_f) \geq s_1(A_f) \geq \cdots \geq s_n(A_f) \geq \cdots\).

Let \(\mathcal{M}_n(G)\) be the following class of meromorphic in \(G\) functions with at most \(n\) poles (counted with multiplicities):

\[
\mathcal{M}_n(G) = \{h : h = p/q, \ p \in E_\infty(G), \ \deg q \leq n, \ q \not\equiv 0\}.
\]

Let \(\Lambda_n(f; G)\) be the least deviation of \(f\) in the space \(L_\infty(\Gamma)\) from the class \(\mathcal{M}_n(G)\):

\[
\Lambda_n(f; G) = \inf_{h \in \mathcal{M}_n(G)} \|f - h\|_\infty.
\]

The AAK theorem (see [1]) asserts that for \(G = \{z : |z| < 1\}\) and \(f \in C(\Gamma)\), we have

\[
s_n(A_f) = \Lambda_n(f; G), \ n = 0, 1, 2, \ldots.
\]

In the case when \(G\) is a bounded domain and \(\Gamma\) consists of \(N\) closed analytic Jordan curves the following generalization of the AAK theorem was proved by the author (see [5]):

**Let \(f\) be continuous on \(\Gamma\). Then**

\[
s_n(A_f) \leq \Lambda_n(f; G), \quad n = 0, 1, 2, \ldots, \tag{1}
\]

and,

\[
\Lambda_{n+N-1}(f; G) \leq s_n(A_f), \quad \text{for} \quad n \geq N - 1. \tag{2}
\]

There exist (see [6]) orthonormal systems \(\{q_n\}, \ \{z_n\}, \ n = 0, 1, 2, \ldots\) of eigenfunctions of the operator \((A^*_f A_f)^{1/2}\) corresponding to the sequence of singular numbers \(\{s_n(A_f)\}, \ n = 0, 1, 2, \ldots\), such that

\[
(q_n f - p_n)(\xi) \ d\xi = s_n(A_f)(z_n(\xi)) \ d\xi \quad \text{a.e. on} \quad \Gamma,
\]

\[
(z_n f - \beta_n)(\xi) \ d\xi = s_n(A_f)(q_n(\xi)) \ d\xi \quad \text{a.e. on} \quad \Gamma,
\]

where \(p_n, \ \beta_n \in E_2(G)\). Clearly,

\[
\int_{\Gamma} (q_i z_j)(\xi) f(\xi) \ d\xi = s_i(A_f) \delta_{i,j}, \quad i, j = 0, 1, 2, \ldots, \tag{3}
\]

where \(\delta_{i,j}\) is the Kronecker symbol.

We will need the following theorem (see [7]):

**Theorem.** Let \(G\) be a bounded domain whose boundary consists of a finite number of closed analytic Jordan curves. Let \(f\) be continuous on \(\Gamma\) and let \(\phi_0, \ldots, \phi_n \in E_2(G)\) and \(\psi_0, \ldots, \psi_n \in E_2(G)\). Then the following estimate of the absolute value of a Hadamard-type determinant of order \(n + 1\) is valid:

\[
\left|\int_{\Gamma} (\phi_i \psi_j f)(\xi) \ d\xi\right|^n_{i,j=0} \leq \prod_{k=0}^n s_k(A_f) \left((\phi_i, \phi_j)_{n,j=0}^{n}\right)^{1/2} \left((\psi_i, \psi_j)_{n,j=0}^{n}\right)^{1/2} \tag{4}
\]

(with the Gram determinants of order \(n + 1\) on the right).
1.3. Estimates of errors in best meromorphic approximation

Let $G$ be a bounded domain with boundary $\Gamma$ consisting of $N$ closed analytic Jordan curves. Consider a function $f$ continuous on $\partial G$. We assume that $f$ can be extended analytically on $G \setminus F$, where $F$ is a compact subset of $G$. Let $G_1, \overline{G}_1 \subset G$, be a domain bounded by a finite number of closed analytic Jordan curves which contains the compact set $F$. Denote by $\Gamma_1$ the boundary of $G_1$. We assume that $\Gamma$ and $\Gamma_1$ are positively oriented with respect to $G$ and $G_1$, respectively. Let $J : E_2(G) \to L_2(|d\tau|, \Gamma)$ be the corresponding embedding operator.

**Theorem 1.** We have

$$\prod_{k=0}^{n} s_k(A_{f,G}) \leq \prod_{k=0}^{n} s_k(A_{f,G_1}) \prod_{k=0}^{n} s_k^2(J).$$

We single out a result that follows directly from Theorem 1 (see (1) and (2)).

**Corollary 2.** Let $N \geq 2$ and $n \geq N - 1$. We have

$$\prod_{k=0}^{N-2} s_k(A_{f,G}) \prod_{k=0}^{n} \Delta_{k+N-1}(f; G) \leq \prod_{k=0}^{n} \Delta_k(f; G_1) \prod_{k=0}^{n} s_k^2(J).$$

In the case when $G$ is a simply connected domain we obtain the following:

**Corollary 3.** Let $G$ be a simply connected domain. Then

$$\prod_{k=0}^{n} \Delta_k(f; G) \leq \prod_{k=0}^{n} \Delta_k(f; G_1) \prod_{k=0}^{n} s_k^2(J).$$

**Proof of Theorem 1.** Let $\{q_n\}, \{\zeta_n\}, n = 0, 1, 2, \ldots$ be the orthonormal systems of eigenfunctions of the operator $(A_{f,G}^* A_{f,G})^{1/2}$ corresponding to the sequence of singular numbers $\{s_n(A_{f,G})\}, n = 0, 1, 2, \ldots$, and satisfying the following equations (see (3)):

$$\int_{\Gamma} (q_i(z_j) f(\zeta)) \, d\zeta = s_i(A_{f,G}) \delta_{i,j}, \quad i, j = 0, 1, 2, \ldots .$$

(5)

It follows immediately from (5) that the product of singular numbers $s_0(A_{f,G})s_1(A_{f,G}) \ldots s_n(A_{f,G})$ can be written as a determinant of order $n + 1$:

$$\prod_{k=0}^{n} s_k(A_{f,G}) = \left| \int_{\Gamma} (q_i(z_j) f(\zeta)) \, d\zeta \right|_{i,j=0}^{n}.$$
Since the functions $q_i, \alpha_j, i, j = 0, 1, 2, \ldots$, belong to $E_2(G)$ and $f$ is analytic on $G \setminus F$, the formula
\[
\prod_{k=0}^{n} s_k(A_{f,G}) = \left| \int_{\Gamma_1} (q_i \alpha_j)(t) f(t) \, dt \right|_{i,j=0}^{n}
\] (6)
can be written for the product of singular numbers.

Let $A_{f,G_1} : E_2(G_1) \rightarrow E_2^+(G_1)$ be the Hankel operator constructed from $f(t), t \in \Gamma_1$. Denote by $(q_i, q_j)_{2,|dt|}$ and $(\alpha_i, \alpha_j)$ the inner products of $q_i$ and $q_j$ in the spaces $L_2(|dt|, \Gamma_1)$ and $L_2(\Gamma)$, respectively. From (6), by (4), we obtain that
\[
\prod_{k=0}^{n} s_k(A_{f,G}) \leq \prod_{k=0}^{n} s_n(A_{f,G_1}) \left( \left| (q_i, q_j)_{2,|dt|} \right|_{i,j=0}^{n} \right)^{1/2} \left( \left| (\alpha_i, \alpha_j)_{2,|dt|} \right|_{i,j=0}^{n} \right)^{1/2}.
\]

By the Weyl–Horn theorem (see, for example, [2, Lemma 3.1]),
\[
\left| (q_i, q_j)_{2,|dt|} \right|_{i,j=0}^{n} = \left| (J q_i, J q_j)_{2,|dt|} \right|_{i,j=0}^{n} \leq \prod_{k=0}^{n} s_k^2(J) \left| (q_i, q_j) \right|_{i,j=0}^{n}
\]
and
\[
\left| (\alpha_i, \alpha_j)_{2,|dt|} \right|_{i,j=0}^{n} = \left| (J \alpha_i, J \alpha_j)_{2,|dt|} \right|_{i,j=0}^{n} \leq \prod_{k=0}^{n} s_k^2(J) \left| (\alpha_i, \alpha_j) \right|_{i,j=0}^{n}.
\]

Taking into account now that $(\alpha_i, \alpha_j) = (q_i, q_j) = \delta_{i,j}$, we get
\[
\prod_{k=0}^{n} s_k(A_{f,G}) \leq \prod_{k=0}^{n} s_k(A_{f,G_1}) \prod_{k=0}^{n} s_k^2(J) \cdot \left( \left| (q_i, q_j) \right|_{i,j=0}^{n} \right)^{1/2} \left( \left| (\alpha_i, \alpha_j) \right|_{i,j=0}^{n} \right)^{1/2} \leq \prod_{k=0}^{n} s_k(A_{f,G_1}) \prod_{k=0}^{n} s_k^2(J). \quad \square
\]

2. Rational approximation

2.1. Estimates of errors in best rational approximation

Let $E$ be an arbitrary compact set in the extended complex plane $\mathbb{C}$. Consider a function $f$ continuous on $E$. For any nonnegative integer $n$ denote by $\rho_n(f; E)$ the best rational approximation of $f$ in the uniform metric on $E$ by rational function of order at most $n$. In other words,
\[
\rho_n(f; E) = \inf_{r \in \mathcal{R}_n} || f - r ||_E,
\]
where $\|\cdot\|_E$ is the supremum norm on $E$ and the infimum is taken in the class of all rational functions of order at most $n$:

$$R_n = \{ r : r = p/q, \deg p \leq n, \deg q \leq n, \, q \not\equiv 0 \}.$$ 

If $f$ is analytic on $\mathbb{C} \setminus F$, where $F$ is a compact set in the extended complex plane $\mathbb{C}$ such that $F \cap E = \emptyset$, then (see [6])

$$\limsup_{n \to \infty} n(f; E)^{1/n} \leq 1/\rho$$

and

$$\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leq 1/\rho,$$

where $\rho = \exp(1/C(E, F))$ and $C(E, F)$ denotes the condenser capacity associated with the condenser $(E, F)$ (see, for example, [3]).

Let $E \subset \mathbb{C}$ be a compact set with connected complement $U$, $U \neq \emptyset$. We assume that the interior $\Omega$ of $E$ is not empty. Denote by $\partial E$ the boundary of $E$. Let $f$ be a function analytic in $\Omega$ and continuous on $E$. We assume that $f$ is not a rational function. It follows easily from this that $n(f; E) \neq 0$ for all $n = 0, 1, 2, \ldots$.

Let $K \subset \mathbb{C}$ be a compact set and let $K$ belong to the interior $\Omega$ of $E$. We assume that the complement $G$ of $K$ is connected.

**Theorem 4.** We have

$$\limsup_{n \to \infty} \left( \frac{\prod_{k=0}^{n} \rho_k(f; K)}{\prod_{k=0}^{n} \rho_k(f; E)} \right)^{1/n^2} \leq \exp(-1/C(\partial E, K)).$$

As a consequence of Theorem 4 we obtain the following result characterizing the asymptotics behavior of $\rho_n(f; K)/\rho_n(f; E)$ as $n \to \infty$.

**Corollary 5.** The following inequality is valid:

$$\liminf_{n \to \infty} \left( \frac{\rho_n(f; K)}{\rho_n(f; E)} \right)^{1/n} \leq \exp(-2/C(\partial E, K)).$$

**Proof of Theorem 4.** We first assume that $K$ and $E$ are bounded by finitely many disjoint closed analytic Jordan curves. Since quantities $\rho_n(f; K), \rho_n(f; E), n = 0, 1, \ldots$, and the condenser capacity $C(\partial E, K)$ are invariant under linear fractional transformations of the extended complex plane $\mathbb{C}$ we confine ourselves to the case when the complement of $K$ (the domain $G$) is bounded.

It is not hard to see that $\Omega = \mathbb{C} \setminus \overline{U}$. Moreover, since $U$ is connected, $\overline{U}$ is a continuum (a closed connected set with at least two points). Hence, $\Omega$ consists of a finite number of simply connected domains bounded by closed analytic Jordan curves.
Let \( w(z) \) be the solution of the Dirichlet problem constructed in the open set \( \Omega \setminus K \) with respect to boundary data equal 1 on \( \partial K \) and 0 on \( \partial \Omega \). It will be assumed that \( w(z) \) is extended by continuity to \( \overline{C} \): \( w(z) = 1 \) for \( z \in K \), and \( w(z) = 0 \) for \( z \in \overline{U} \). For arbitrary \( \varepsilon \), with \( 0 < \varepsilon < 1 \), let \( G(\varepsilon) = \{ z : w(z) < \varepsilon \} \) and \( \gamma(\varepsilon) = \{ z : w(z) = \varepsilon \} \).

Using the maximum principle for harmonic functions we can conclude that every connected component of the open set \( G(\varepsilon) \), \( 0 < \varepsilon < 1 \), contains at least one point of \( U \). Since \( U \) is a continuum, it follows from this that \( G(\varepsilon) \) is a domain. We assume that \( \gamma(\varepsilon) \), \( 0 < \varepsilon < 1 \), is positively oriented with respect to the domain \( G(\varepsilon) \). We distinguish components \( \gamma_i \) of \( \gamma(\varepsilon) \) such that \( \gamma_i \cap K \neq \emptyset \).

Let \( \gamma(\varepsilon_1) \) consist of \( N \) closed analytic Jordan curves. Denote by \( J : E_2(G(\varepsilon_1)) \to L_2(|dt|, \gamma(\varepsilon)) \) the corresponding embedding operator. Since \( f \) is analytic in \( C \setminus \overline{U} \), and since \( \overline{U} \subset G(\varepsilon) \subset G(\varepsilon_1) \), it follows from Corollary 2, that for \( n \geq N - 1 \)

\[
\prod_{k=N-1}^{n} \Delta_{k+N-1}(f; G(\varepsilon_1)) \leq C \prod_{k=0}^{n} \Delta_k(f; G(\varepsilon)) \prod_{k=0}^{n} s_k^2(J),
\]

where \( C \) is a positive quantity not depending on \( n \). Here and in what follows denote by \( C, C_1, \ldots, \) positive quantities not depending on \( n \).

Let us estimate

\[
\prod_{k=0}^{n} \Delta_k(f; G(\varepsilon)).
\]

It follows from the definitions of \( \Delta_k(f; G(\varepsilon)) \) and \( \rho_k(f; \gamma(\varepsilon)) \) that

\[
\Delta_k(f; G(\varepsilon)) \leq \rho_k(f; \gamma(\varepsilon)).
\]

Since \( \gamma(\varepsilon) \subseteq E \), we can write

\[
\rho_k(f; \gamma(\varepsilon)) \leq \rho_k(f; E).
\]
So,
\[ \prod_{k=0}^{n} \Delta_k(f; G(\varepsilon)) \leq \prod_{k=0}^{n} \rho_k(f; E), \]
and, by (10),
\[ \prod_{k=N-1}^{n} \Delta_k \prod_{k=0}^{n} \rho_k(f; E) \leq C \prod_{k=0}^{n} s_k^2(J). \] (11)

Fix a nonnegative integer \( k \). For an arbitrary rational function \( r \in \mathcal{R}_k \) with poles outside \( \gamma(\varepsilon) \) and any function \( \varphi \in E_\infty(G(\varepsilon_1)) \) we have by the Cauchy formula
\[ (r' - f)(z) = \frac{1}{2\pi i} \int_{\gamma(\varepsilon_1)} \frac{(f - r - \varphi)(\xi)}{\xi - z} d\xi, \quad z \in K, \] (12)
where \( r' \) is the sum of the principal parts of \( r \) corresponding to poles of \( r \) lying in \( G(\varepsilon_1) \).

We remark that \( r' \in \mathcal{R}_k \). Estimating the integral in (12), we get
\[ \rho_k(f; K) \leq ||f - r'||_K \leq C_1 ||f - r - \varphi||_\infty. \]

Since \( r \) is an arbitrary function in \( \mathcal{R}_k \) with poles outside \( \gamma(\varepsilon_1) \) and \( \varphi \) is an arbitrary function in \( E_\infty(G(\varepsilon_1)) \),
\[ \rho_k(f; K) \leq C_1 \Delta_k(f; G(\varepsilon_1)). \]

From this, by (11), we can write
\[ \prod_{k=0}^{n} \rho_k(f; K) \leq C^n \prod_{k=0}^{n} \rho_k(f; E) \prod_{k=0}^{n} s_k^2(J). \] (13)

Using the result of Zaharjuta and Skiba (see [10]),
\[ \lim_{n \to \infty} s_n^{1/n}(J) = \exp(-1, C(\gamma(\varepsilon), \gamma(\varepsilon_1))) \]
from (13) we get
\[ \lim \sup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; K) / \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leq \exp(-1/C(\gamma(\varepsilon), \gamma(\varepsilon_1))). \]

Letting \( \varepsilon \to 0 \) and \( \varepsilon_1 \to 1 \), we obtain (see (9)) that
\[ \lim \sup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; K) / \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leq \exp(-1/C(\partial E, K)). \] (14)
We now get rid of the condition that \( K \) and \( E \) are bounded by finitely many closed analytic Jordan curves. Consider the general case when \( K \) and \( E \) are arbitrary compact sets satisfying the conditions:

(a) \( K \subset \Omega \);
(b) \( G = \overline{\mathbb{C}} \setminus K \) and \( U = \overline{\mathbb{C}} \setminus E \) are connected, \( U \neq \emptyset \).

Since \( U \) is the complement of \( E \), \( \partial E = \partial U \), and \( E = \partial E \cup \Omega \), it follows that \( \overline{U} = U \cup \partial E \) and \( \Omega \) coincides with the complement of a closed domain \( \overline{U} \). Using now the fact that \( U \) is the complement of \( E \), and \( \Omega \) is the complement of \( U \), we can write

\[
\partial \Omega = \partial \overline{U} \subseteq \partial U = \partial E. \tag{15}
\]

Since \( \overline{U} \) is a continuum, \( \Omega \) consists of an at most countable number of simply connected domains. We distinguish components \( \Omega_i \) of \( \Omega \) such that \( \Omega_i \cap K \neq \emptyset \). Since \( \Omega \) is an open cover of the compact set \( K \), it follows that there is only a finite number of such components \( \Omega_i \). Let \( \Omega' = \bigcup_i \Omega_i \). We remark that \( \Omega' \subset \Omega \). By the properties of the condenser capacity

\[
C(\partial \Omega', K) = C(\partial \Omega, K).
\]

So, by (15), we can write

\[
C(\partial \Omega', K) \leq C(\partial E, K). \tag{16}
\]

Let \( B = \overline{\mathbb{C}} \setminus \Omega' \). Since \( \overline{U} \) is a continuum, we can conclude that \( B \) is a continuum. Moreover, since \( K \subset \Omega' \), \( B \cap K = \emptyset \). We construct a sequence of compacts \( \{K_m\} \) and \( \{B_m\}, m = 1, 2, \ldots \), bounded by finitely many closed analytic Jordan curves, that tends monotonically to \( K \) and \( B \), respectively:

\[
K \subset K_m \subset K_{m-1}, \quad \bigcap_{m=1}^{\infty} K_m = K,
\]

\[
B \subset B_m \subset B_{m-1}, \quad \bigcap_{m=1}^{\infty} B_m = B.
\]

We assume that for all \( m \), \( B_m \) is a continuum, the complement of \( K_m \) is connected, and \( B_m \cap K_m = \emptyset \).

Fix a positive integer \( m \). Let \( V_m \) be the closure of the complement of \( B_m \) in the extended complex plane \( \overline{\mathbb{C}} \). It is easy to see that \( V_m \subset \Omega' \subset E \). Since \( B_m \) is a continuum, the complement of \( V_m \) is connected. Using the relations \( K \subset K_m \) and \( V_m \subset E \), for all nonnegative integers \( n \) and \( m \) we can write

\[
\rho_n(f; K) \leq \rho_n(f; K_m) \tag{17}
\]

and

\[
\rho_n(f; V_m) \leq \rho_n(f; E). \tag{18}
\]
Since \( K_m \) and \( V_m \) are bounded by finitely many closed analytic Jordan curves, with the help of estimate (14) we get

\[
\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; K_m) / \prod_{k=0}^{n} \rho_k(f; V_m) \right)^{1/n^2} \leq \exp(-1/C(\partial V_m, K_m)).
\]

This implies (see (17) and (18))

\[
\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; K) / \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \leq \exp(-1/C(\partial V_m, K_m)).
\]

By properties of the condenser of the capacity we have

\[
\lim_{m \to \infty} C(\partial V_m, K_m) = C(\partial \Omega', K).
\]

So, we can pass to the limit on the right-hand side of (19), obtaining

\[
\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; K) / \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n} \leq \exp(-1/C(\partial \Omega', K)).
\]

Using now (16), we get (8). □

Let a function \( f \) be analytic in an open set \( D \) and let \( E \subset D \) be a compact set with connected complement. We assume that \( D \) consists of a finite number of domains \( D_i, i = 1, \ldots, \gamma \), and \( D_i \cap E \neq \emptyset \) for all \( i \). Denote by \( F \) the complement of \( D \) in the extended complex plane \( \overline{C} \). It is assumed that \( F \) is a continuum. It follows from this that the logarithmic capacity \( \text{cap} \) (\( F \)) (see [3,8]) of \( F \) is positive and \( F \) is a regular compact set in the sense of potential theory.

Let \( \rho = \exp(1/C(E, F)) \), where \( C(E, F) \) is the condenser capacity associated with the condenser \( (E, F) \). We assume that the logarithmic capacity \( \text{cap}(E) \) is positive. From this and the fact that \( \text{cap}(F) > 0 \) we can conclude that (see [3,8]) that \( C(E, F) > 0 \).

Denote by \( w(z) \) the solution of the generalized Dirichlet problem with the boundary function equal to 1 on \( \partial F \) and to 0 on \( \partial E \). For each \( i = 1, \ldots, \gamma \), the function \( w(z) \) is harmonic in the domain \( D_i \setminus E \). It is assumed that the compact set \( E \) is regular. Since \( E \) and \( F \) are regular compacts, \( w(z) \) is continuous on \( D \setminus \overline{E} \); \( w(z) = 1, z \in \partial D = \partial F \), and \( w(z) = 0, z \in \partial E \). It will be assumed that \( w(z) \) is extended by continuity to \( \overline{C} \); \( w(z) = 1 \) for \( z \in F \) and \( w(z) = 0 \) for \( z \in E \). For arbitrary \( r \), with \( 1 < r < \rho \), let \( E(r) = \{ z : w(z) \leq \ln r / \ln \rho \} \) and \( \gamma(r) = \{ z : w(z) = \ln r / \ln \rho \} \). We remark that, by properties of the condenser capacity (see, for example, [3,8]),

\[
C(E, \gamma(r)) = \frac{\ln \rho}{\ln r} C(E, F)
\]

and

\[
\exp(1/C(E, \gamma(r))) = r.
\]
Using (7), it is easy to obtain an upper estimate for \( \lim_{n \to \infty} \rho_n(f; E)^{1/n} \):

\[
\lim_{n \to \infty} \rho_n(f; E)^{1/n} \leq \frac{1}{\rho^2}.
\]

We conclude this section with the result related to functions \( f \) having the following asymptotics of the errors in the best rational approximation:

\[
\lim_{n \to \infty} \rho_n(f; E)^{1/n} = \frac{1}{\rho^2}.
\]

**Theorem 6.** Let

\[
\lim_{n \to \infty} \rho_n(f; E)^{1/n} = \frac{1}{\rho^2},
\]

where \( \rho = \exp(1/C(E, F)) \). Then for any \( 1 < r < \rho \),

\[
\lim_{n \to \infty} \rho_n(f; E(r))^{1/n} = \exp(-2/C(E(r), F)) = \left( \frac{r}{\rho} \right)^2.
\]

**Proof.** Since

\[
\lim_{n \to \infty} \rho_n(f; E)^{1/n} = \frac{1}{\rho^2},
\]

we can write

\[
\lim_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} = \frac{1}{\rho}.
\]

Using (8) and (20), we get

\[
\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; E) \right)^{1/n^2} \prod_{k=0}^{n} \rho_k(f; E(r)) \leq \exp(-1/C(E, \gamma(r)) = \frac{1}{r}.
\]

From this, by (22), we obtain

\[
\liminf_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; E(r)) \right)^{1/n^2} \geq \frac{r}{\rho}.
\]

Since

\[
C(E(r), F) = C(E, F)/(1 - \ln r / \ln \rho)
\]

and

\[
\exp((1/C(E(r), F)) = \frac{\rho}{r},
\]
we have (see (7)),
\[
\limsup_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; E(r)) \right)^{1/n^2} \leq \exp(-1/C(E(r), F)) = \frac{r}{\rho}.
\]
So, by (24),
\[
\lim_{n \to \infty} \left( \prod_{k=0}^{n} \rho_k(f; E(r)) \right)^{1/n^2} = \frac{r}{\rho}. \tag{25}
\]
Fix an arbitrary \(0 < \theta < 1\). Choose a sequence of integers \(\{k_n\}, n = 1, 2, 3, \ldots\), such that \(0 \leq k_n \leq n, \lim_{n \to \infty} k_n/n = \theta\). Since the sequence \(\{\rho_n(f; E(r))\}, n = 1, 2, \ldots\) is nonincreasing,
\[
\left( \prod_{k=0}^{k_n} \rho_k(f; E(r)) \right) \rho_n^{n-k_n}(f; E(r)) \leq \prod_{k=0}^{n} \rho_k(f; E(r)). \tag{26}
\]
From (26), on account of (25), we obtain that
\[
\limsup_{n \to \infty} \rho_n(f; E(r))^{1/n} \leq \left( \frac{r}{\rho} \right)^{1+\theta}. \tag{27}
\]
Letting \(\theta \to 1\), we get
\[
\limsup_{n \to \infty} \rho_n(f; E(r))^{1/n} \leq \left( \frac{r}{\rho} \right)^{2}. \tag{27}
\]
Using now the inequality
\[
\left( \prod_{k=0}^{k_n} \rho_k(f; E(r)) \right) \leq \rho_n^{k_n-n}(f; E(r)) \prod_{k=0}^{n} \rho_k(f; E(r)),
\]
where \(k_n \geq n\), and the same arguments as above it is not hard to prove the following:
\[
\liminf_{n \to \infty} \rho_n(f; E(r))^{1/n} \geq \left( \frac{r}{\rho} \right)^{2},
\]
which with help of (27) implies the desired equality (21). \(\square\)

References
