Pergamon

Topology Vol. 35, No. 4, pp. 1069–1083, 1996

0040-9383/96/$15.00 + 0.00

0040-9383(95)00059-3

THE POISSON STRUCTURE ON THE MODULI SPACE OF FLAT CONNECTIONS AND CHORD DIAGRAMS

JØRGEN ELLEGAARD ANDERSEN,† JOSEF MATTES and NICOLAI RESHETIKHIN‡

(*Received in revised form* 13 *February* 1995)

WE INTRODUCE the notion of chord diagrams on arbitrary compact (possibly punctured) oriented surfaces. In the case of the 2-spheres these are just the usual chord diagrams used in test study of Vassiliev invariants of links. We consider the algebra of chord diagrams on a surface and prove that this algebra has a natural Poisson structure. Suppose now that G is a Lie group with an invariant bilinear form on $\mathfrak{g} = Lie(G)$. We can associate to each chord diagram (coloured by representations of G) a function on the moduli space of flat G-connections on the surface. Our main result states that this map is a Poisson algebra homomorphism. Moreover, for most classical groups we prove that any algebraic function on moduli space can be obtained this way and we conjecture that this holds for all simple groups. In this way we obtain a universal description of the Poisson algebra of the moduli space, decoupling the Lie group in question. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

The Poisson structure on the algebra of functions on the moduli space of flat connections on an oriented genus g surface has been studied in recent years in a number of papers, for example [16, 19, 4, 12, 11, 13, 7, 9]. We start with a very brief summary of these works.

Let Σ be a closed oriented genus g surface and let G be a (real) Lie group with a bi-invariant inner product on its Lie algebra $\mathfrak{g}$. Consider the moduli space $\mathcal{M}^G$ of flat G-connections on Σ. The symplectic structure on the space $\mathcal{M}^G$ was first introduced by Narasimhan and Seshadri in [16, 19], by using the correspondence with semistable holomorphic bundles. Atiyah and Bott [4] then gave a construction of the symplectic structure on $\mathcal{M}^G$ using symplectic reduction from the infinite dimensional space of G-connections on Σ. Subsequently, Goldman studied this symplectic structure from the purely representation theoretical point of view ([12, 11], compare also [13] where Karshon completed the purely representation theoretical point of view by showing the closedness of Goldman's 2-form using only group cohomology). He studied the Hamiltonian flows of traces of holonomy functions of simply closed curves and gave a geometric formula for the Poisson bracket of two such functions. Later Biswas and Guruprasad in [7] extended Goldman's results to punctured surfaces. We shall make use of this extension.

One of the motivations for introducing chord diagrams on surfaces and the corresponding functions on $\mathcal{M}^G$ was to construct a set of functions which linearly spans the space of algebraic functions on $\mathcal{M}^G$, and to give a geometric formula for the Poisson bracket of any two such functions, hence generalizing Goldman's paper [12] and decoupling the geometric content of the Poisson structure. Recently, the Poisson structure on the moduli space of graph connections has been studied by Fock and Rosly [9].

† Supported in part by NSF grant DMS-93-09653.
‡ Supported by the A. Sloan Foundation and by NSF grant DMS-9015821.

In the present paper we describe the Poisson structure on (the smooth parts of) moduli spaces of flat connections on a punctured oriented surface from a universal point of view: we show that there exists a natural Poisson algebra $ch(\Sigma)$ associated to any surface Σ which is spanned by chord diagrams on that surface. In the case of the sphere these diagrams appeared before in the context of Vassiliev invariants of links [6, 5, 14]. Colouring the chord diagrams by representations of G we get Poisson algebras $ch(\Sigma)^G$. This construction of a Poisson algebra is universal in the sense that there exists a homomorphism of Poisson algebras $f: ch(\Sigma)^G \to C(\mathcal{M}^G)$ for any Lie group G (with a G-invariant bilinear form on $\mathfrak{g} = Lie(G)$. Furthermore, if $\phi: G \to H$ is a group homomorphism, then the diagram

$$\begin{array}{ccc} ch(\Sigma) & \xrightarrow{id} & ch(\Sigma) \\ \Big| & & \Big| \\ ch(\Sigma)^H & \xrightarrow{\phi^*} & ch(\Sigma)^G \\ \Big|{\scriptstyle f} & & \Big|{\scriptstyle f} \\ C(\mathcal{M}^H) & \xrightarrow{(\phi_*)^*} & C(\mathcal{M}^G) \end{array}$$

is commutative. We will show that the algebra $ch(\Sigma)$ has a quotient algebra which corresponds to Goldman's algebra of loops [12].

The question of surjectivity of the map f is reduced to a question about the structure of the space of G-invariants in $U\mathfrak{g}^{\otimes n}$. For most classical Lie groups the answer is positive and we conjecture that it is positive for all simple Lie groups.

One of the main motivations for this paper was an attempt to understand the relation between the gauge theoretic and combinatorial approaches to the Chern–Simons topological field theory [3, 18, 21, 17] and to relate these to the perturbative version. The quantization of the Poisson algebra of functions on the moduli space $\mathcal{M}^G$ will be studied in a forthcoming paper [2]. For a description of a quantization of $C(\mathcal{M}^G)$ based on the paper [9] see [1, 8].

The structure of the paper is as follows. In the next section we describe the notion of a chord diagram. The Poisson algebra structure on the space of chord diagrams is described in Section 3. The following section describes the homomorphism from the algebra of chord diagrams to the algebra of functions on moduli space. Finally, Section 5 contains an analysis of the question of surjectivity of the map introduced in Section 3.

2. DIAGRAMS

Definition 1. A *chord diagram* is a graph consisting of disjoint oriented circles S_i, $i \in \{1, \dots, n\}$, disjoint arcs C_j, $j \in \{1, \dots, m\}$, such that

1. the endpoints of the arcs are distinct,
2. $\bigcup_j \partial C_j = (\bigcup_i S_i) \cap (\bigcup_j C_j)$.

The arcs are called *chords*, the circles S_i are called the *core components* of the diagram.

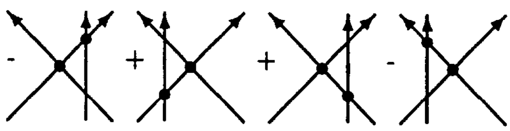
Definition 2. Given a closed oriented surface Σ, a *geometrical chord diagram* on Σ is a smooth map from a chord diagram D to Σ, mapping the chords to points. A *chord diagram on* Σ is a class of geometric chord diagrams modulo homotopy.

Note that diffeomorphism classes of chord diagrams correspond to chord diagrams on S^2. We will see this fact when we draw chord diagrams: we will draw chord diagrams on an arbitrary oriented genus g surface Σ in the plane, assuming some choice of identification of the plane with an open disk in Σ. Images of chords will be drawn as fat dots.

Definition 3. By a *generic chord diagram* (on Σ) we will mean a geometrical chord diagram on Σ such that all circles are immersed, and with all double points transverse.

Clearly every chord diagram on Σ contains generic chord diagrams.

Consider the complex vector space V with the basis given by the set of diffeomorphism classes of chord diagrams and the subspace W generated by the following linear combinations, called $4T$-*relations* (the diagrams are supposed to be identical outside the picture and the parts drawn are assumed to be immersed in S^2),



as well as all relations obtained from this by reversing orientations of strings using the following rule: for every chord that intersects a component whose orientation is reversed we get a factor of -1 for the diagram.

We define a (commutative) multiplication of chord diagram by defining their product to be their disjoint union. The subspace W is an ideal with respect to this multiplication, we denote the quotient ring by $ch := V/W$.

Definition 4. ch is called the *algebra of chord diagrams*.

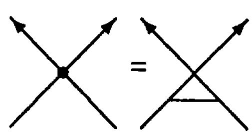
Analogously we can define an algebra of chord diagrams on Σ denoted by $ch(\Sigma)$.

These algebras are bigraded by $(n, m) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ (number of circles and chords):

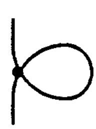
$$ch(\Sigma) = \bigoplus_{m,n \geq 0} ch^{m,n}(\Sigma)$$

$$ch(\Sigma) := \bigoplus_{n \geq 0} ch^{m,n}(\Sigma).$$

Remark 1. Our notion of chord diagrams in the case $\Sigma := S^2$ is identical to the one in the theory of Vassiliev invariants for *framed* links. One has to make the following identifications:



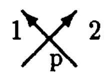
We will not go into details of this identification. Let us just notice that diagrams containing the following parts are non-zero in our cases:



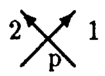
3. POISSON STRUCTURE ON THE ALGEBRA OF CHORD DIAGRAMS

Given two chord diagram A, B on Σ we can pick geometrical chord diagrams D_1, D_2 representing them such that $D_1 \cup D_2$ is a generic chord diagram. For $p \in D_1 \cap D_2$ we define the oriented intersection number as follows:

$\varepsilon_{12}(p) :=$ $+$ for p



$-$ for p



where 1 and 2 indicate components of the corresponding diagrams.

For each $p \in D_1 \cap D_2$ we define $D_1 \cup_p D_2$ to be the chord diagram on Σ given by joining $D_1^{-1}(p)$ and $D_2^{-1}(p)$ by a chord. Under the above assumptions, for chord diagrams D_1, D_2 we consider the following sum:

$$\sum_{p \in D_1 \cap D_2} \varepsilon_{12}(p)[D_1 \cup_p D_2]. \tag{1}$$

PROPOSITION 1. *The following hold:*

1. *The sum* (1) *depends only on the classes* $[D_1], [D_2] \in ch(\Sigma)$.

2. *The bracket* $\{[D_1], [D_2]\} := \sum_{p \in D_1 \cap D_2} \varepsilon_{12}(p)[D_1 \cup_p D_2]$ *determines the structure of a Poisson algebra on* $ch(\Sigma)$.

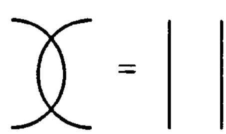
Remark 2. The proof shows that the Poisson bracket is well-defined even for unframed chord diagrams.

Proof. 1. Any two regular chord diagrams representing $[D_1 \cup D_2]$ can be connected by a sequence of applications of the following moves:

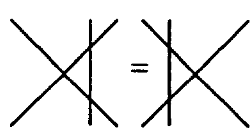
(a) isotopy

(b) first Reidemeister move

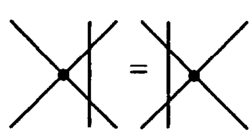
(c) second Reidemeister move



(d) third Reidemeister move

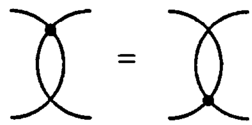


(e)



(f) 4T-relations

(g)



Invariance:

(a) clear

(b) new chords are introduced only between different components

(c) by the last relation

(d) by the relation immediately following

(e) by the 4T-relation (note that in our situation the given dot cannot connect components from different diagrams)

(f) clear

(g) clear.

2. Antisymmetry is clear by definition of ε.

3. Jacobi identity:

$$\{[D_1],\{[D_2],[D_3]\}] = \sum_{p\in D_1\cap D_2,\, q\in D_2\cap D_3} \varepsilon_{12}(p)\varepsilon_{23}(q)(D_1 \cup_p D_2 \cup_p D_3) + \sum_{p\in D_1\cap D_3,\, q\in D_2\cap D_3} \varepsilon_{13}(p)\varepsilon_{23}(q)(D_1 \cup_p D_2 \cup_p D_3)$$

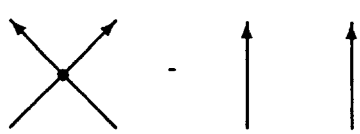
which equals

$$\begin{aligned}\{\{[D_1],[D_2]\},[D_3]\} &+ \{[D_2],\{[D_1],[D_3]\}]\\ = &\sum_{p\in D_1\cap D_2,\, q\in D_1\cap D_3} \varepsilon_{12}(p)\varepsilon_{13}(q)(D_1 \cup_p D_2 \cup_p D_3)\\ &+ \sum_{p\in D_1\cap D_2,\, q\in D_2\cap D_3} \varepsilon_{12}(p)\varepsilon_{23}(q)(D_1 \cup_p D_2 \cup_p D_3)\\ &+ \sum_{p\in D_2\cap D_1,\, q\in D_1\cap D_3} \varepsilon_{21}(p)\varepsilon_{13}(q)(D_2 \cup_p D_1 \cup_p D_3)\\ &+ \sum_{p\in D_2\cap D_3,\, q\in D_1\cap D_3} \varepsilon_{23}(p)\varepsilon_{13}(q)(D_2 \cup_p D_1 \cup_p D_3).\end{aligned}$$

The derivation property is obvious. □

Clearly we can colour components of chord diagrams with representations of some Lie group G. The above operations will define a Poisson algebra $ch(\Sigma)^G$ of coloured chord diagrams on Σ and we have the forgetful map $ch(\Sigma)^G \to ch(\Sigma)$.

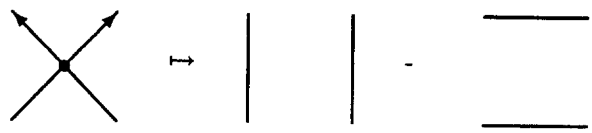
Now we will define two algebra homomorphisms $\pi_{1,2}$, one mapping to a quotient algebra, the other mapping to a subalgebra of $ch(\Sigma)$. Consider the idea I in $ch(\Sigma)$ generated by relations of the form



Proposition 2. *The ideal I is a Poisson idea and the quotient algebra $l(\Sigma)_1$ is Poisson algebra.*

Hence the projection $\pi_1 : ch(\Sigma) \to l(\Sigma)_1$ is a Poisson algebra homomorphism.

Next we define a map $\pi_2 : ch(\Sigma) \to l(\Sigma)_2 \subset ch(\Sigma)$. It will factor through the algebra of non-oriented links (definition obvious), $\pi_2 = \Phi_2 \circ \Phi_1$, where Φ_1 is given by the rule



and $\Phi_2 : D \mapsto (1/2^{\#D}) \sum_\varepsilon D^\varepsilon$, the sum going over all the $2^{\#D}$ possible orientations of D.

The Poisson algebra $l(\Sigma)_1$ was constructed in [12].

4. FUNCTIONS ON MODULI SPACE OBTAINED FROM DIAGRAMS

We introduce the following notations. Throughout the rest of the paper we let Σ denote a (possibly punctured) compact, oriented genus g surface with non-abelian fundamental group $\pi := \pi_1(\Sigma)$ and we let G be a connected real or complex Lie group with a non-degenerate Ad-invariant symmetric bilinear form $\langle , \rangle$ on its Lie algebra $\mathfrak{g}$.

Let D be a chord diagram on Σ. We write $S_i \backslash (\bigcup_j C_j) = \{S_{i,j}\}_{j=0}^{n_i}$, ordered according to the cyclic order of the $S_{i,j}$ given by the orientation of S_i. Next choose a representation (ρ_i, V_i) of G for every core component (i.e. we turn D into an element $(D, \{\rho_i\})$ of $ch(\Sigma)^G$). We will denote the representation of $\mathfrak{g}$ (or its universal enveloping algebra) corresponding to a representation of ρ_i of G again by ρ_i. Define $c(i, j)$ by $C_{c(i,j)} = \partial_+(S_{i,j})$.

We also fix orthonormal bases $\{I_{\mu_j}\}_{\mu_j=1}^{\dim(\mathfrak{g})}$ in $\mathfrak{g}$ for every chord C_j as well as a flat connection Φ on some principal G-bundle over Σ which defines a gauge class of flat connections $[\Phi] \in \mathcal{M}^G \cong \mathrm{Hom}(\pi, G)/G$.

Choose a point $p_i \in S_i$ if S_i does not intersect any chord, and denote by T^{Φ}_{i,j,p_i} the parallel transport for Φ along $S_{i,j}$ (with respect to the basepoint p_i if $S_{i,j}$ is closed).

With the data above we associate a function on $\mathcal{M}^G$ in the following way:

Definition 5.

$$f_{D,\rho}([\Phi]) := \sum_{\mu_1, \dots, \mu_m} \prod_i tr_{V_i}\left(\prod_{j=0}^{n_i} \rho_i(I_{\mu_{c(i,j)}}) \rho_i(T^{\Phi}_{i,j,p_i}) \right). \tag{2}$$

LEMMA 3. *$f_{D,\rho}([\Phi])$ is well-defined.*

Proof. 1. Invariance under the choice of bases $I_{\mu_k} \mapsto A^{\nu}_{\mu_k} I_\nu$, $A^{\nu}_{\mu_k}$ orthogonal. Let $\tilde{\mu}_{c(i,j)}$ be equal to

$$\begin{cases} \mu_{c(i,j)} & \text{if } c(i,j) \neq k \\ \nu & \text{for some pair } (i,j) \text{ such that } c(i,j) = k \\ \lambda & \text{for the other pair } (i,j) \text{ such that } c(i,j) = k. \end{cases}$$

Then we get

$$f_{D,\rho} \mapsto \sum_{\mu_1, \dots, \mu_m, \nu, \lambda} A^{\nu}_{\mu_k} A^{\lambda}_{\mu_k} \prod_i tr_{V_i}\left(\prod_{j=0}^{n_i} \rho_i(I_{\tilde{\mu}_{c(i,j)}}) \rho_i(T^{\Phi}_{i,j,p_i}) \right)$$

$$= \sum_{\mu_1,\ldots,\hat{\mu}_k,\ldots,\mu_m,\nu,\lambda} \left(\sum_{\mu_k} A^{\nu}_{\mu_k} A^{\lambda}_{\mu_k}\right) \prod_i tr_{V_i}\left(\prod_{j=0}^{n_i} \rho_i(I_{\tilde{\mu}_{c(i,j)}})\rho_i(T^{\Phi}_{i,j,p_i})\right)$$

$$= \sum_{\mu_1,\ldots,\hat{\mu}_k,\ldots,\mu_m,\nu,\lambda} \delta_{\nu,\lambda} \prod_i tr_{V_i}\left(\prod_{j=0}^{n_i} \rho_i(I_{\tilde{\mu}_{c(i,j)}})\rho_i(T^{\Phi}_{i,j,p_i})\right)$$

$$= f_{D,\rho}. \tag{3}$$

2. Gauge invariance:

$$T^{g\cdot\Phi}_{i,j,p_i} = g_{\partial_+(S_{i,j})} T^{\Phi}_{i,j,p_i} g^{-1}_{\partial_+(S_{i,j})}$$

and $\partial_+(S_{i,j}) = \partial_-(S_{i,j+1})$ imply that $f_{D,\rho}$ changes to

$$\sum_{\mu_1,\ldots,\mu_m} \prod_i tr_{V_i}\left(\prod_{j=0}^{n_i} \rho_i(g_{\partial_+(S_{i,j})} I_{\mu_{c(i,j)}} g^{-1}_{\partial_+(S_{i,j})})\rho_i(T^{\Phi}_{i,j,p_i})\right)$$

which is equal to $f_{D,\rho}$ by the above.

3. Invariance under homotopy: Any two homotopic chord diagrams on Σ are homotopic by a sequence of homotopies of two types: those fixing all the chords and those moving only small neighbourhoods of chords. Invariance under the first type is clear and the second type of homotopies results in a base change in the copy of $\mathfrak{g}$ associated to the chords, hence f is invariant by the above.

4. Invariance under choice of base points and of total ordering of the $S_{i,j}$ is also clear by 1.

5. 4T-relations: We can assume that the connection is trivial on the part of the surface depicted in the 4T-relation, then for suitable $X_i \in End(V_i)$ and the strings enumerated at the bottom from left to right, the f's have the form (if the strings come from three different core components, the case where they come from two core components is entirely similar)

$$f_{D,\rho} = \sum tr_{V_1}(X_1 I_\nu I_\mu) tr_{V_2}(X_2 I_\nu) tr_{V_3}(X_3 I_\mu) \prod_{i>3} tr_{V_i}(X_i)$$

$$f_{D,\rho} = \sum tr_{V_1}(X_1 I_\mu I_\nu) tr_{V_2}(X_2 I_\nu) tr_{V_3}(X_3 I_\mu) \prod_{i>3} tr_{V_i}(X_i)$$

$$f_{D,\rho} = \sum tr_{V_1}(X_1 I_\mu) tr_{V_2}(X_2 I_\nu) tr_{V_3}(X_3 I_\nu I_\mu) \prod_{i>3} tr_{V_i}(X_i)$$

$$f_{D,\rho} = \sum tr_{V_1}(X_1 I_\mu) tr_{V_2}(X_2 I_\nu) tr_{V_3}(X_3 I_\mu I_\nu) \prod_{i>3} tr_{V_i}(X_i).$$

The 4T-relation gives

$$\sum\Bigg(tr_{V_1}(X_1 [I_\mu, I_\nu]) tr_{V_2}(X_2 I_\nu) tr_{V_3}(X_3 I_\mu) \prod_{i>3} tr_{V_i}(X_i)$$

$$+ tr_{V_1}(X_1 I_\mu) tr_{V_2}(X_2 I_\nu) tr_{V_3}(X_3 [I_\mu, I_\nu]) \prod_{i>3} tr_{V_i}(X_i)\Bigg). \tag{4}$$

Under changes of orientation this is compatible with the sign changes in the 4T-relations (up to an overall minus sign). It equals

$$\sum f^{\lambda}_{\mu,\nu} tr_{V_1}(X_1 I_\lambda) tr_{V_2}(X_2 I_\nu) tr_{V_3}(X_3 I_\mu) \prod_{i>3} tr_{V_i}(X_i)$$
$$+ \sum f^{\lambda}_{\mu,\nu} tr_{V_1}(X_1 I_\mu) tr_{V_2}(X_2 I_\nu) tr_{V_3}(X_3 I_\lambda) \prod_{i>3} tr_{V_i}(X_i) \tag{5}$$

hence, renaming indices, we end up with

$$\sum (f^{\lambda}_{\mu,\nu} + f^{\mu}_{\lambda,\nu}) \left(tr_{V_1}(X_1 I_\lambda) tr_{V_2}(X_2 I_\nu) tr_{V_3}(X_3 I_\mu) \prod_{i>3} tr_{V_i}(X_i) \right) \tag{6}$$

which is 0 since the structure constants satisfy $f^{\mu}_{\lambda,\nu} = -f^{\lambda}_{\mu,\nu}$. □

Remark 3. By invariance under homotopy of type two, if $D = D_1 \cup_p D_2$, $p \in S^1_{i,j} \cap S^2_{k,l}$, we can assume that both the parallel transports for Φ from p to $\partial_+(S^1_{i,j})$ and from p to $\partial_+(S^2_{k,l})$ are the identity.

Remark 4. Due to Lemma 3 (3) we can drop the base points p_i from our notation.

THEOREM 4.

$$\{f_{\rho_1, D_1}, f_{\rho_2, D_2}\} = f_{\rho_1 \cup \rho_2, \{D_1, D_2\}}.$$

Proof. Using $\mathcal{M}^G \cong Hom(\pi, G)/G$, we get a correspondence $[\Phi] \to [\rho]$. Choose a lift $\rho \in Hom(\pi, G)$ for $[\rho]$ (resp. a lift Φ for $[\Phi]$), then we have $T^{\Phi}_{\alpha} = \rho(\alpha)$, α a based loop.

To simplify notation we write f_k for $tr_{V_k}(\prod_{j=0}^{n_k} \rho_k(I_{\mu_{c(k,j)}}) \rho_k(T^{\Phi}_{k,j}))$.

Let u be a 1-cocycle representing $[u] \in H^1_c(\Sigma, \mathfrak{g}_{[\Phi]}) \cong H^1(\pi, \mathfrak{g}_{Ad_\rho}) \cong T_{[\Phi]}\mathcal{M}^G$ (cf. [7], Lemma 2.2). Now we can compute

$$df_{\rho, D|\Phi}([u])$$
$$= \frac{d}{dt}\bigg|_{t=0} \sum_{\mu_1,\ldots,\mu_m} \prod_i tr_{V_i}\left(\prod_{j=0}^{n_i} \rho_i(I_{\mu_{c(i,j)}}) \rho_i(\exp(tu(S_{i,j}) + O(t^2)) T^{\Phi}_{i,j}) \right)$$
$$= \sum_{\mu_1,\ldots,\mu_m} \sum_{i=1}^{n} \left(\prod_{k<i} f_k \right) tr_{V_i}\left(\sum_{l_i=1}^{n_i} \left(\prod_{j=0}^{l_i-1} \rho_i(I_{\mu_{c(i,j)}}) \rho_i(T^{\Phi}_{i,j}) \right) \rho_i(I_{\mu_{c(i,j)}}) \right.$$
$$\left. \times \rho_i(u(S_{i,l_i})) \rho_i(T^{\Phi}_{i,l_i}) \left(\prod_{j=l_i+1}^{n_i} \rho_i(I_{\mu_{c(i,j)}}) \rho_i(T^{\Phi}_{i,j}) \right) \right) \left(\prod_{k>i} f_k \right)$$
$$= \sum_{\mu_1,\ldots,\mu_m} \sum_{i=1}^{n} \sum_{l_i=1}^{n_i} \left(\prod_{k<i} f_k \right) tr_{V_i}\left(\sum_{\beta} \left(\prod_{j=0}^{l_i-1} \rho_i(I_{\mu_{c(i,j)}}) \rho_i(T^{\Phi}_{i,j}) \right) \rho_i(I_{\mu_{c(i,l_i)}}) \right.$$
$$\left. \times \rho_i(I_\beta) \rho_i(T^{\Phi}_{i,l_i}) \left(\prod_{j=l_i+1}^{n_i} \rho_i(I_{\mu_{c(i,j)}}) \rho_i(T^{\Phi}_{i,j}) \right) \left(\prod_{k<i} f_k \right) \langle I_\beta, u(S_{i,l_i}) \rangle \right) \tag{7}$$
$$= \sum_{\mu_1,\ldots,\mu_m} \sum_{i=1}^{n} \sum_{l_i=1}^{n_i} \sum_{\beta} \left(\prod_{k<i} f_k \right) tr_{V_i}\left(\left(\prod_{j=0}^{l_i-1} \rho_i(I_{\mu_{c(i,j)}}) \rho_i(T^{\Phi}_{i,j}) \right) \right.$$
$$\left. \times \rho_i(I_{\mu_{c(i,l_i)}} I_\beta) \rho_i(T^{\Phi}_{i,l_i}) \left(\prod_{j=l_i+1}^{n_i} \rho_i(I_{\mu_{c(i,j)}}) \rho_i(T^{\Phi}_{i,j}) \right) \right) \left(\prod_{k>i} f_k \right) \langle I_\beta, u(S_{i,l_i}) \rangle. \tag{8}$$

Therefore, in the notation of [7], Section 3, with $M = \Sigma$,

$$[Hf_{D,\rho}|_{[\Phi]}]\cap[\Sigma]$$
$$= ((B\circ l)^{-1}([df_{D,\rho}|_{\Phi}]))\cap[\Sigma]\times\eta^{-1}B^{-1}([df_{D,\rho}|_{\Phi}])$$
$$=\Bigg[\sum_{\mu_1,\dots,\mu_m}\sum_{i=1}^{n}\sum_{l_i=1}^{n_i}\sum_{\beta}[S_{i,l_i}]\otimes\Big(\prod_{k<i}f_k\Big)tr_{V_i}\Bigg(\Bigg(\prod_{j=0}^{l_i-1}\rho_i(I_{\mu_{c(i,j)}})\rho_i(T^{\Phi}_{i,j})\Bigg)$$
$$\times\rho_i(I_{\mu_{c(i,l_i)}}I_\beta)\rho_i(T^{\Phi}_{i,l_i})\Bigg(\prod_{j=l_i+1}^{n_i}\rho_i(I_{\mu_{c(i,j)}})\rho_i(T^{\Phi}_{i,j})\Bigg)\Bigg)\Big(\prod_{k>i}f_k\Big)I_\beta\Bigg]\in H_1(\Sigma,\partial\Sigma,\mathfrak{g}_{Ad\rho})\quad(9)$$

(as in the proof of [7], Prop. 3.1.), where $[S_{i,l_i}]$ denotes the 1-chain defined by the arc S_{i,l_i}. We denote by $S^1_{i_1}$ and $S^2_{i_2}$ components of D_1 and D_2 respectively. Now we can follow the proof of [7], Theorem 3.2 (compare [12] 2.3) to arrive at

$$\{f_{\rho_1,D_1},f_{\rho_2,D_2}\}([\Phi])$$
$$=\mathfrak{B}([Hf_{\rho_1,D_1}(\Phi)]\cap[\Sigma],[Hf_{\rho_2,D_2}(\Phi)]\cap[\Sigma])\quad(10)$$
$$=\Bigg\langle\Bigg(\sum_{\mu^1_1,\dots,\mu^1_{m_1}}\sum_{i_1=1}^{n_1}\sum_{l_1=1}^{n_{i_1}}\sum_{\beta^1}[S^1_{i_1,l_1}]\otimes\Big(\prod_{k<i_1}f_k\Big)$$
$$\times tr_{V^1_{i_1}}\Bigg(\Bigg(\prod_{j=0}^{l_1-1}\rho^1_{i_1}(I_{\mu^1_{c(i_1,j)}})\rho^1_{i_1}(T^{\Phi}_{i_1,j})\Bigg)\rho^1_{i_1}(I_{\mu^1_{c(i_1,l_1)}}I_{\beta^1})\rho^1_{i_1}(T^{\Phi}_{i_1,l_1})$$
$$\times\Bigg(\prod_{j=l_1+1}^{n_{i_1}}\rho^1_{i_1}(I_{\mu^1_{c(i_1,j)}})\rho^1_{i_1}(T^{\Phi_1}_{i_1,j})\Bigg)\Bigg)\Big(\prod_{k>i_1}f_k\Big)I_{\beta^1}\Bigg),$$
$$\Bigg(\sum_{\mu^2_1,\dots,\mu^2_{m_2}}\sum_{i_2=1}^{n_2}\sum_{l_2=1}^{n_{i_2}}\sum_{\beta^2}[S^2_{i_2,l_2}]\otimes\Big(\prod_{k<i_2}f_k\Big)$$
$$\times tr_{V^2_{i_2}}\Bigg(\Bigg(\prod_{j=0}^{l_2-1}\rho^2_{i_2}(I_{\mu^2_{c(i_2,j)}})\rho^2_{i_2}(T^{\Phi}_{i_2,j})\Bigg)\rho^2_{i_2}(I_{\mu^2_{c(i_2,l_2)}}I_{\beta^2})\rho^2_{i_2}(T^{\Phi}_{i_2,l_2})$$
$$\times\Bigg(\prod_{j=l_2+1}^{n_{i_2}}(\rho^2_{i_2})(I_{\mu^2_{c(i_2,j)}})\rho^2_{i_2}(T^{\Phi}_{i_2,j})\Bigg)\Bigg)\Big(\prod_{k>i_2}f_k\Big)I_{\beta^2}\Bigg)\Bigg\rangle,\quad(11)$$

(the base points in [7] are immaterial due to Lemma 3 (3))

$$=\sum_{\substack{\mu^1_1,\dots,\mu^1_{m_1}\\ \mu^2_1,\dots,\mu^2_{m_2}}}\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\sum_{l_1=1}^{n^1_{i_1}}\sum_{l_2=1}^{n^2_{i_2}}\sum_{\beta^1,\beta^2}\sum_{p\in S^1_{i_1,i_1}\cup S^2_{i_2,i_2}}\varepsilon(p)\Big(\prod_{k<i_1}f_k\Big)$$
$$\times tr_{V^1_{i_1}}\Bigg(\Bigg(\prod_{j=0}^{l_1-1}\rho^1_{i_1}(I_{\mu^1_{c(i_1,j)}})\rho^1_{i_1}(T^{\Phi}_{i_1,j})\Bigg)\rho^1_{i_1}(I_{\mu^1_{c(i_1,l_1)}}I_{\beta^1})$$
$$\times\rho^1_{i_1}(T^{\Phi}_{i_1,l_1})\Bigg(\prod_{j=l^1_{i_1}+1}^{n^1_{i_1}}\rho^1_{i_1}(I_{\mu^1_{c(i_1,j)}})\rho^1_{i_1}(T^{\Phi}_{i_1,j})\Bigg)\Bigg)\Big(\prod_{k>i_1}f_k\Big)\Big(\prod_{k<i_2}f_k\Big)$$
$$\times tr_{V^2_{i_2}}\Bigg(\Bigg(\prod_{j=0}^{l_2-1}\rho^2_{i_2}(I_{\mu^2_{c(i_2,j)}})\rho^2_{i_2}(T^{\Phi}_{i_2,j})\Bigg)\rho^2_{i_2}(I_{\mu^2_{c(i_2,i_2)}}I_{\beta^2})$$
$$\times\rho^2_{i_2}(T^{\Phi}_{i_2,l_2})\Bigg(\prod_{j=l_2+1}^{n^2_{i_2}}\rho^2_{i_2}(I_{\mu^2_{c(i_2,j)}})\rho^2_{i_2}(T^{\Phi}_{i_2,j})\Bigg)\Bigg)\Big(\prod_{k>i_2}f_k\Big)\langle I_{\beta^1},I_{\beta^2}\rangle.$$

Now we use the remark at the beginning of the proof to get

$$
\begin{aligned}
&\sum_{\mu^1_1,\dots,\mu^1_{m_1}} \sum_{\beta^1,\beta^2} \sum_{\mu^2_1,\dots,\mu^2_{m_2}} \delta_{\beta^1,\beta^2}\left(\prod_{k<i_1} f_k\right)\\
&\times tr_{V^1_{i_1}}\left(\left(\prod_{j=0}^{l_1-1}\rho^1_{i_1}(I_{\mu^1_{c(i_1,j)}})\rho^1_{i_1}(T^{\Phi}_{i_1,j})\right)\rho^1_{i_1}(I_{\mu^1_{c(i_1,l_1)}}I_{\beta^1})\rho^1_{i_1}(T^{\Phi}_{i_1,l_1})\right.\\
&\times\left.\left(\prod_{j=l_1+1}^{n^1_{i_1}}\rho^1_{i_1}(I_{\mu^1_{c(i_1,j)}})\rho^1_{i_1}(T^{\Phi}_{i_1,j})\right)\right)\left(\prod_{k>i_1}f_k\right)\left(\prod_{k<i_2}f_k\right)\\
&\times tr_{V^2_{i_2}}\left(\left(\prod_{j=0}^{l_2-1}\rho^2_{i_2}(I_{\mu^2_{c(i_2,j)}})\rho^2_{i_2}(T^{\Phi}_{i_2,j})\right)\rho^2_{i_2}(I_{\mu^2_{c(i_2,l_2)}}I_{\beta^2})\rho^2_{i_2}(T^{\Phi}_{i_2,l_2})\right.\\
&\times\left.\left(\prod_{j=l_2+1}^{n^2_{i_2}}\rho^2_{i_2}(I_{\mu^2_{c(i_2,j)}})\rho^2_{i_2}(T^{\Phi}_{i_2,j})\right)\right)\left(\prod_{k>i_2}f_k\right)\\
&=f_{\rho_1\cup\rho_2,D_1\cup_p D_2}([\Phi]),\ p\in S^1_{i_1,l_1}\cap S^2_{i_2,l_2}.
\end{aligned}
\tag{12}
$$

Finally we deduce

$$
\begin{aligned}
&\{f_{\rho_1,D_1},f_{\rho_2,D_2}\}\\
&=\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\sum_{l_1=1}^{n^1_{i_1}}\sum_{l_2=1}^{n^2_{i_2}}\sum_{p\in S^1_{i_1,i_1}\cap S^2_{i_2,i_2}}\varepsilon(p)f_{\rho_1\cup\rho_2,D_1\cup_p D_2}\\
&=\sum_{p\in D_1\cap D_2}\varepsilon(p)f_{\rho_1\cup\rho_2,D_1\cup_p D_2}\\
&=f_{\rho_1\cup\rho_2,\{D_1,D_2\}}.
\end{aligned}
\tag{13}
$$

The proof is complete. □

We will see in the next section (Proposition 7) that for the classical groups $G=GL(N,\mathbb{C}),O(N,\mathbb{C})$, and $Sp(2N,\mathbb{C})$ the functions $f_{\rho,D}$ where all the colourings are by the standard representation, linearly generate the space of all algebraic functions on moduli space. Hence it is interesting to note the following: by colouring each component with the standard representation we get a map $D\mapsto(\rho_{\mathrm{std}},D)$ from $ch(\Sigma)$ to $ch(\Sigma)^G$. Consider the map $f:ch(\Sigma)\to C(\mathcal{M}^G)$ which is given by $D\mapsto f_{\rho_{\mathrm{std}},D}$.

Proposition 5. *Let $\pi_i:ch(\Sigma)\to l(\Sigma)_i$, $i=1,2$, denote the maps defined at the end of the previous section. Then the following diagrams are commutative:*

$$
\begin{array}{ccc}
ch(\Sigma) & \xrightarrow{\pi_1} & l(\Sigma)_1\\
\downarrow f & & \downarrow f\\
C(\mathcal{M}^{GL(N,\mathbb{C})}) & \xrightarrow{id} & C(\mathcal{M}^{GL(N,\mathbb{C})})
\end{array}
\qquad
\begin{array}{ccc}
ch(\Sigma) & \xrightarrow{\pi_2} & l(\Sigma)_2\\
\downarrow f & & \downarrow f\\
\mathscr{C}(\mathcal{M}^{G}) & \xrightarrow{id} & C(\mathcal{M}^{G})
\end{array}
$$

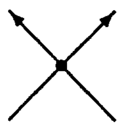
where all the colourings are by standard representations of the corresponding groups and in the second diagram $G=O(N,\mathbb{C})$ or $G=Sp(2N,\mathbb{C})$.

Compare this with [12], Theorems 3.13 and 3.14.

Proof. 1. For $SL(N,\mathbb{C})$ in the standard representation,

$$\sum I_\mu \otimes I_\mu = \sum E_{ij} \otimes E_{ji} - \frac{1}{N+1} Id \otimes Id$$

hence for (a suitably normalized inner product on) $GL(N,\mathbb{C})$ we have $\sum I_\mu \otimes I_\mu = \sum E_{ij} \otimes E_{ji} = P_{V \otimes V}$, so that



$$\mapsto P \circ P = Id.$$

2. Recall $\pi_2 = \Phi_2 \circ \Phi_1$. Since changing orientation of a component has the same effect on $f_{D,\rho}$ as dualizing the corresponding representation and $V \cong V^*$ for $G = O(N,\mathbb{C})$ and $G = Sp(2N,\mathbb{C})$, $f_{D_\rho^\varepsilon}$ is independent of the orientation (ε) of the components of D. In addition we have

$$\sum I_\mu \otimes I_\mu = \sum E_{ij} \otimes E_{ji} - \sum E_{ij} \otimes E_{ij} = P - \Theta$$

where Θ is the contraction: $\Theta : x \otimes y \mapsto \langle x, y \rangle Id$, the inner product and Id are given via the bilinear form preserved by G. Hence a crossing with a chord maps to $P \circ (P - \Theta) = Id - P \circ \Theta$ which is exactly the effect of the map Φ_1. □

5. SURJECTIVITY OF THE POISSON HOMOMORPHISM

Definition 6. A *string chord diagram* with n strings is a graph consisting of disjoint oriented segments $I_i \cong [0,1]$, $i \in \{1, \dots, n\}$, and disjoint chords C_j, $j \in \{1, \dots, m\}$, such that
1. the endpoints of the chords are distinct,
2. $\bigcup_j \partial C_j = (\bigcup_i S_i) \cap (\bigcup_j C_j)$.

The segments I_i are called *strings.*

A string chord diagrams with n strings and m chords will be denoted by $S_{n,m}$.

Let $\mathfrak{g}$ be a Lie algebra with a symmetric bilinear $\mathfrak{g}$-invariant form on $\mathfrak{g}^*$. Denote by t the corresponding element in $\mathfrak{g} \otimes \mathfrak{g}$. We will always assume that $\mathfrak{g} \subsetneq U(\mathfrak{g})$ is imbedded into its universal enveloping algebra and that $t \in \mathfrak{g}^{\otimes 2} \subsetneq U(\mathfrak{g})^{\otimes 2}$. If $\mathfrak{g}$ is a semisimple Lie algebra we can choose t to be the element dual to the Killing form.

Define the map

$$\phi : \{\text{string chord diagrams with } n \text{ strings}\} \to U(\mathfrak{g})^{\otimes n}$$

$$S \mapsto \phi(S)$$

as follows:
1. Denote by$(v_1^{(i)}, \dots, v_{k_i}^{(i)})$ the endpoints of the chords which lie on the ith string ordered with respect to the orientation of I_i (hence $\sum_{i=1}^n k_i = 2m$).
2. Choose a representation for t in the form $t = \sum_{i \in J} t_i \otimes t_i$.

3. Consider the set of multiindices $(i(v_1^{(1)}), \ldots, i(v_{k_1}^{(1)}), \ldots, i(v_{k_n}^{(n)}))$ where $i(v_b^{(a)}) = i(v_d^{(c)})$ if $v_b^{(a)}, v_d^{(c)}$ are endpoints of the same chord and $i(v) \in J$. Call these multiindices admissible. An admissible multiindex is the same as a labelling of the chords in the string chord diagram.

4. Define

$$\phi(S) := \sum_{\text{admissible multindices}} t_{i(v_1^{(1)})} \otimes \cdots \otimes t_{i(v_{k_1}^{(1)})} \otimes \cdots \otimes t_{i(v_{k_n}^{(n)})}.$$

Since this sum can be written in tensorial form this map is clearly independent of the choice of representation of t in step 2.

Proposition 6. *The elements $\phi(S)$ are invariant with respect to the diagonal action of $\mathfrak{g}$ in $U(\mathfrak{g})^{\otimes n}$ for all diagrams S with n strings.*

Proof. This proposition follows from the invariance of t. □

Proposition 7. *The elements $\phi(S)$ for diagrams S with n strings generate the space of all elements in $End(V^{\otimes n})$ which are invariant with respect to the diagonal action of G if G is one of the groups $GL(N,\mathbb{C}), O(N,\mathbb{C}), Sp(2N,\mathbb{C})$ and V its standard representation.*

Proof. This proposition follows from classical invariant theory, see e.g. [10] Appendix F or [20], which tells us that the invariants are compositions of permutations of factors in the tensor products and contractions and the fact that we can generate all these by the action of $\sum I_\mu \otimes I_\mu$ (see the proof of Proposition 5). □

Corollary 8. *The conclusion of Proposition 7 holds for the groups $SL(N,\mathbb{C})$ and $SO(2N+1,\mathbb{C})$.*

Proof. This holds since $GL(N,\mathbb{C})$ and $O(2N+1,\mathbb{C})$ are central extensions of $SL(N,\mathbb{C})$ and $SO(2N+1,\mathbb{C})$ respectively. □

Remark 5. Proposition 7 does not hold for $SO(2N,\mathbb{C})$. Recall that $O(2N)$ is the extension of $SO(2N)$ by the diagram automorphism which can be taken to be

$$\begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix}$$

in the standard representation. Since the diagram automorphism of $\mathfrak{d}_N$ acts on V and commutes with the action of $\sum I_\mu \otimes I_\mu$, every intertwiner of $V^{\otimes 2}$ of the form $\Phi(S)$ will be $O(2N,\mathbb{C})$-invariant. But $V^{\wedge N}$ is irreducible as an $O(2N)$-module, hence dim $(\Phi^{V^{\wedge N}}(S)) = 1$ whereas it is reducible as an $SO(2N)$-module, so that $\dim(End^{SO(2N)}(V^{\wedge N})) > 1$. Hence there are more $SO(2N)$-intertwiners than $O(2N)$-intertwiners. In fact the missing intertwiners arise from the fact that

$$\begin{pmatrix} 0 & Id \\ Id & 0 \end{pmatrix}^{\otimes N}$$

commutes with the $SO(2N)$-action. This follows from [10] Appendix F since the above element corresponds to the determinant under the map $\Lambda^{2N} V \subsetneq V^{\otimes 2N} \cong End(V^{\otimes N})$. Note also that $O(2N)$ is the adjoint group for the Lie algebra $\mathfrak{d}_N$. The above suggests that for each simple Lie algebra element of the form $\sum I_\mu \otimes I_\mu$ generate all the intertwiners of $V^{\otimes n}$ for the adjoint group and all intertwiners should be given by the $\sum I_\mu \otimes I_\mu$ together with suitable diagram automorphisms. We thank Greg Kuperberg for this remark.

COROLLARY 9. *The conclusion of Proposition 7 holds for the compact real forms of the complex groups for which it holds.*

Proof. For complex representations of classical real Lie algebras, an orthonormal basis is also an orthonormal basis for its complexification and an invariant endomorphism is invariant for the complexification. The result follows since an irreducible representation of $\mathfrak{g}$ is an irreducible representation of $\mathfrak{g} \otimes \mathbb{C}$. □

Now we are able to show for certain groups that every algebraic function $g \in C(\mathcal{M}^G)$ is in the linear span of the functions constructed above from chord diagrams.

THEOREM 10. *The map $f: ch(\Sigma) \to C(\mathcal{M}^G)$ is surjective for the algebraic groups for which the conclusion of the above proposition holds.*

Proof. A basis in the space of all algebraic functions on $\mathcal{M}^G$ is given in [9] for each given triangulation of the surface Σ; we will show that the functions $f_{\rho,D}$ generate such a basis $\{\psi^G_{\alpha,\Gamma}\}$ for any triangulation. First, let us recall the construction of this basis in $C(\mathcal{M}^G)$ for [9]:

1. The 1-skeleton of the triangulation defines a graph $\Gamma \subseteq \Sigma$. The space $\mathcal{M}^G_\Gamma$ of graph connections Φ with trivial monodromy around the boundaries of the 2-cells of the triangualrtion is isomorphic to $\mathcal{M}^G$.
2. Choose a ciliation of Γ (a linear ordering of the edges at each vertex).
3. Associate an irreducible finite dimensional G-module to each edge $e \in E$ of Γ: $\pi_{V_e}: G \to End(V_e)$.
4. With each vertex v we associate a G-invariant map

$$\alpha_v: V(e_1^{\varepsilon_1}) \otimes \cdots \otimes V(e_n^{\varepsilon_n}) \to \mathbb{C}$$

where $\varepsilon_i = \pm$ depends on whether $\pm v \in \partial e_i$. The order of the tensor product is determined by the ciliation.

5. For $[\Phi] \in \mathcal{M}^G$ choose a representative $\{\Phi(e)\} \in G^{\times E}$. We can regard every $\pi_{V_e}(\Phi(e))$ as an element in $End(V_e) = V_e \otimes V_e^*$ and (after reordering) $\prod_v \alpha_v$ is an element of $Hom(\bigotimes (V_e \otimes V_e^*), \mathbb{C})$, hence we can apply $\prod_v \alpha_v$ to $\prod_e \pi_{V_e}(\Phi(e))$.
6. Now we can make the following definition:

$$\psi^G_{\alpha,\Gamma}(\Phi) := \left(\prod_v \alpha_v\right)\left(\prod_e \pi_{V_e}(\Phi(e))\right)$$

or equivalently, if we regard $\prod_v \alpha_v$ as an element in $\bigotimes(V_e^* \otimes V_e)$ and hence $\prod_v \alpha_v^*$ as an element of $\bigotimes(V_e \otimes V_e^*) \cong End(\bigotimes_{e\in E} V_e)$, we can compose it with $\prod_e \pi_{V_e}(\Phi(e))$ to get

$$\psi^G_{\alpha,\Gamma}(\Phi) := tr_{(\otimes_{e\in E} V_e)}\left(\prod_v \alpha_v^*\right) \circ \left(\prod_e \pi_{V_e}(\Phi(e))\right).$$

7. By the G-invariance of α, $\psi^G_{\alpha,\Gamma}(\Phi) = \psi^G_{\alpha,\Gamma}([\Phi])$ is independent of the choice of Φ.

Now we proceed as follows: Let $F \subset E$ denote the set of edges such that the associated representation of G is non-trivial. For every non-trivial $V_k = V_{e_k}$ we choose an n_k such that V_k imbeds into $V^{\otimes n_k}$, where V again denotes the standard representation of the group G. By semisimplicity, $End(\bigotimes V_k) \subsetneq End(\bigotimes V^{\otimes n_k})$. This map, which we will denote by $\alpha \to \tilde{\alpha}$, preserves G-invariants.

Furthermore we can choose:

- for all k paths e_k^j, $j = 1, \dots, n_k$,
- an imbedding i of $[0, \# F] \times [0,1]$,
- paths $\gamma_{k,-}^j : \partial_+ e_k^j \to i(k + j/3n_k, 0)$, $\gamma_{k,+}^j : i(k + j/3n_k, 1) \to \partial_- e_k^j$

such that for $[\Phi]$ there is a representative Φ satisfying

$$T^\Phi(e_k^j) = T^\Phi(e_k)$$

and Φ is trivial on $i([0, \# F] \times [0,1])$ as well as on the $\gamma_{k,-}^j, \gamma_{k,+}^j$.

The conclusion of Proposition 7 says that we can choose a linear combination of chord diagrams with boundary $\{(k + j/3n_k, 0), (k + j/3n_k, 1)\}$ such that

$$\pi(\Phi(S)) = \widetilde{\prod_v \alpha_v^*}.$$

Using the e_k^j and the $\gamma_{k,-}^j, \gamma_{k,+}^j$ these can be closed up to a linear combination of chord diagrams $\sum l_p D_p$ which satisfies

$$\psi_{\alpha,\Gamma}^G([\Phi]) = \sum l_p f_{D_p}([\Phi]) = f_{\sum l_p D_p}(\Phi).$$

Hence the functions $f_{\rho, D}$ form a linear basis of $\mathscr{C}(\mathscr{M}^G)$. □

In the proof of Theorem 10 we only use the following facts:

1. There is a set of irreducible representations $\{V_i\}$ of G such that every irreducible representation of $G^{\times n}$ imbeds into some tensor product $V_{i_1} \otimes \cdots \otimes V_{i_k}$ $(k \geq n)$.
2. All G-endomorphisms of $V_{i_1} \otimes \cdots \otimes V_{i_k}$ are of the form $\Phi(S)$.

Remark 5 shows that we cannot always simply use tensor products of standard representations but we conjecture that there are always such representations:

CONJECTURE 11. *For every simply connected complex Lie group there is a set of irreducible representations $\{V_i\}$ of G such that every irreducible representation of $G^{\times n}$ imbeds into some tensor produce $V_{i_1} \otimes \cdots \otimes V_{i_k}$ and all G-endomorphisms of $V_{i_1} \otimes \cdots \otimes V_{i_k}$ are of the form $\Phi(S)$.*

More precisely we expect that the following sets of representations can be taken as the sets $\{V_i\}$:

- $SO(2N)$: $V, V_{2\omega_{n-1}}$
- $Spin(2N)$: $V_{\omega_{n-1}}, V_{\omega_n}$
- $Spin(2+1)$: V_{ω_n}
- $G2$: V_{ω_1}
- $F4$: V_{ω_4}
- $E6$: V_{ω_1}
- $E7$: V_{ω_7}
- $E8$: V_{ω_8}.

In [15] evidence is given that this holds for $G = G_2$.

In particular the above conjecture implies

CONJECTURE 12. *The map $f: ch(\Sigma)^G \to \mathscr{C}(\mathscr{M}^G)$ is surjective for all simple algebraic groups.*

Acknowledgements—We are grateful to Micheal Polyak for interesting discussions and to Greg Kuperberg for valuable remarks and conjectures.

REFERENCES

[1] A. ALEKSEEV, H. GROSSE and V. SCHOMERUS: Combinatorial quantization of the Hamiltonian Chern–Simons theory, Preprints I: hep-th/94/03, II: hep-th 94/04 (1994).
[2] J. EANDERSEN, J. MATTES and N. RESHETIKHIN: Quantization of the algebra of chord diagrams, in preparation.
[3] M. ATIYAH: Topological quantum field theories, *Publ. Math. IHES* **68** (1989), 175–186.
[4] M. ATIYAH and R. BOTT: The Yang–Mills equations over Riemann surfaces, *Phil. Trans. R. Soc. London Ser. A* **308** (1982), 523–615.
[5] D. BAR-NATAN: On the Vassiliev knot invariants, *Topology* **34** (1995), 423–472.
[6] J. BIRMAN and X.-S. LIN: Knot polynomials and Vassiliev's invariants, *Invent. Math.* **111** (1993), 225–270.
[7] I. BISWAS and K. GURUPRASAD: Principal bundles on open surfaces and invariant functions on Lie groups, *Int. J. Math.* **4** (1993), 535–544.
[8] E. BUFFENOIR and P. ROCHE: Two-dimensional lattice gauge theory based on a quantum group, CPHT preprint A302-05/94 (1994).
[9] V. FOCK and A. ROSLY: Poisson structure on moduli of flat connections and γ matrix, Preprint ITEP 92-72 (1992).
[10] W. FULTON and J. HARRIS: *Representation theory*, GTM **129**,. Springer, Berlin (1991).
[11] W. GOLDMAN: The symplectic nature of the fundamental group of surfaces, *Adv. Math.* **54** (1984), 200–225.
[12] W. GOLDMAN: Invariant functions on Lie groups and Hamiltonian flows of surface group representations, *Invent. Math.* **85** (1986), 263–302.
[13] Y. KARSHON: An algebraic proof for the symplectic structure of moduli space, *Proc. AMS* **116** (1992) 591–605.
[14] M. KONTSEVICH: Vassiliev's knot invariants, in I. M. Gelfand seminar, S. Gelfand and S. Gindikin, Eds, AMS, Providence, RI (1993), pp. 137–150.
[15] G. KUPERBERG: Spiders for rank 2 Lie algebras, University of Chicago preprint (1993).
[16] M. NARASIMHAN and S. SESHADRI: Stable and unitary vector bundles on compact Riemann surfaces, *Ann. Math.* **82** (1965) 540-567.
[17] N. RESHETIKHIN and V. TURAEV: Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547–597.
[18] G. SEGAL: The definition of conformal field theory, unpublished.
[19] S. SESHADRI: Space of unitary vector bundles on a compact Riemann surface, *Ann. Math.* **85** (1967), 303–336.
[20] H. WEYL: *The classical groups*, Princeton University Press, Princeton, NJ (1946).
[21] E. WITTEN: Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* **121** (1989), 351–399.

Department of Mathematics,
University of Aarhus
DK-8000 Aarhus C
Denmark

Department of Mathematics
UC Berkeley
CA 94720
U.S.A.

Department of Mathematics
UC Berkeley
CA 94720
U.S.A.