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An alternative method for q-counting directed column-convex polyominoes

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Abstract

The area+perimeter generating function of directed column-convex polyominoes will be written as a quotient of two expressions, each of which involves powers of q of all kinds: positive, zero and negative. The method used in the proof applies to some other classes of column-convex polyominoes as well. At least occasionally, that method can do the case $q = 1$ too.

1. General discussion

The model of a self-avoiding polygon (SAP) on the step-set $\{\mathrm{N},\mathrm{S},\mathrm{E},\mathrm{W}\}$ has its origins in various physical and chemical contexts. The things that would be most interesting to know about SAPs are: what is the number of SAPs whose perimeter is p, or the number whose area is n, or the number whose perimeter is p and area is n. But these questions are all open, and there are little chances to answer any of them in the near future.

Hoping to get insight into the above-mentioned difficult problems, scientists started studying various simplified, but still nontrivial SAP models. Such models proved to be a rich vein of appealing exact results. Here we shall recall only the necessary minimum of those results. For comprehensive surveys of the subject, see e.g. [24,8].

Our approaching considerations will involve three models: the directed column-convex (dcc-) polyominoes, the parallelogram polyominoes, and the directed convex polyominoes. Let us see, for example, what is known about the dcc-polyominoes.

First, the number of dcc-polyominoes of area n is the Fibonacci number F_{2n-1} [20,10].

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Second, the number of dcc-polyominoes with c columns, $2v$ vertical edges and d L-shaped corners is

$$\frac{1}{c}\binom{c}{d}\binom{c+v-2}{v-d-1}\binom{c+v-d-1}{v}$$

[14,10]. The generating function (gf) for these numbers is algebraic of degree three.

Third, let $V(x,y,q)$ be the gf for dcc-polyominoes in which the variables x, y and q mark horizontal edges, vertical edges and area respectively. The function V was expressed by Bousquet–Mélou [7] as

$$V = y^2 \frac{\sum_{i=1}^{\infty} \frac{x^{2i}(y^2-1)^{i-1}q^{i(i+1)/2}}{(q)_{i-1}(y^2q)_{i-1}(y^2q)_i}}{1-\sum_{i=1}^{\infty} \frac{x^{2i}(y^2-1)^{i-1}q^{i(i+1)/2}}{(q)_i(y^2q)_{i-1}(y^2q)_i}}. \tag{1}$$

Formula (1) is the only expression for V that can be found in the literature.

Remark. The method which produced formula (1) is markedly versatile. Besides the dcc-polyominoes, that method can also handle e.g. stack, parallelogram, directed convex, convex, and column-convex polyominoes (see [7,6,15]).

The first thing we are going to do here is to forget all about formula (1) and derive the function V anew. By carrying out this oddly sounding plan we wish to give a transparent first illustration for an *alternative q-counting method.*

The basic idea of that new method is to combine Delest's [9] coding for column-convex polyominoes with a factorization of lattice paths used in Gessel [18]. It should be mentioned, however, that the formulas obtained in such a way are somewhat different from those derived in [7,6,15]. Namely, whereas the formulas of Bousquet-Mélou [7,6] and Feretić and Svrtan [15] involve only positive and zero powers of q, in our formulas negative powers of q are present too.

To appetize the reader for finding those 'negative' formulas, let us mention some historical facts: the first to discover a result of (nearly) that kind was Pólya [23] in 1938. Pólya's formula, and so too the akin results of Fédou and Rouillon [13], Flajolet [17], Gessel [18] and Goulden and Jackson [19] concern the parallelogram polyominoes. As far as the other polyomino models are concerned, by now almost no results of the 'negative' sort have been derived. (To my knowledge, the only exception is a result about directed diagonally convex polyominoes given in [16]).

To a certain extent, the present paper provides a remedy for that. Namely, our alternative method has fairly wide scope. To illustrate that feature, we shall, after the dcc-polyominoes (which are q-enumerated in Sections 3–5), q-enumerate also the parallelogram polyominoes (Section 6) and the directed convex polyominoes (Section 7). The latter species of polyominoes will be enumerated by perimeter as well (Section 8).

2. Definitions, conventions and notations

2.1. Directed column-convex polyominoes

Let $x=(1,0)$, $\bar{x}=(-1,0)$, $y=(0,1)$, and $\bar{y}=(0,-1)$. Suppose we have paths π_1 and π_2 such that

(i) π_1 lies in $\{x,y\}^*$, starts with an x-step, and ends with a y-step;

(ii) π_2 lies in $\{x,y,\bar{y}\}^*$, has no factors $y\bar{y}$ or $\bar{y}y$, starts with a y-step, and ends with an x-step;

(iii) π_1 and π_2 have the same origin and the same terminus, but are internally disjoint.

Let P be the plane figure bounded by π_1 and π_2. The figure P is called a *directed column-convex polyomino* (*dcc-polyomino*, Fig. 1, left). The paths π_1 and π_2 are the *lower border* and the *upper border* of P, respectively. The ith column of P is the part of P that lies between the vertical lines passing through the ends of the ith x-step of π_1. We denote the minimal and the maximal ordinate of the ith column of P by $y_i(P)$ and $Y_i(P)$, respectively. If no ambiguity need be feared, we suppress the 'P' and simply write y_i and Y_i.

A *parallelogram polyomino* is a dcc-polyomino whose upper border makes no $\bar{y}$-steps. A *directed convex polyomino* is a dcc-polyomino which has convex intersection with every horizontal straight line. Or equivalently, a directed convex polyomino is a dcc-polyomino whose upper border either makes no $\bar{y}$-steps, or can be written as a product $u\cdot\bar{y}\cdot v$, where $u\in\{x,y\}^*$ and $v\in\{x,\bar{y}\}^*$.

Let P be a dcc-polyomino. If the boundary of P consists of j horizontal steps and k vertical steps, we say that the *horizontal and vertical perimeters* of P are j and k, respectively, and we write $h(P)=j$, $v(P)=k$. If the area of P is n, we write $a(P)=n$.

For Ω a family of dcc-polyominoes, we define the *generating function* (gf) of Ω to be the formal sum

$$\mathrm{gf}(\Omega)=\sum_{P\in\Omega}x^{h(P)}y^{v(P)}q^{a(P)}.$$

The case $q=1$ of $\mathrm{gf}(\Omega)$ is the *perimeter gf* of Ω.

We denote the set of all dcc-polyominoes by $\mathscr{V}$, and we put $V=\mathrm{gf}(\mathscr{V})$.

2.2. Lattice paths

Let $\mathscr{W}$ be the set of the paths on the step-set $\{x,y,\bar{y}\}$ which begin on the x-axis and have no factors $y\bar{y}$ or $\bar{y}y$.

Let $w\in\mathscr{W}$. If $|w|_x=n$, then w has a unique factorization

$$w=u_1\cdot x\cdot u_2\cdot x\cdots u_n\cdot x\cdot u_{n+1},$$

where $u_i\in\{y\}^*\cup\{\bar{y}\}^*$, for every i. We call the paths u_i *nests of* w. Clearly enough, by the *odd nests of* w we mean the nests $u_1,u_3,u_5,\ldots$, while by the *even nests of* w we mean the nests $u_2,u_4,u_6,\ldots$. An x-step of w is *odd* (resp. *even*) when it comes

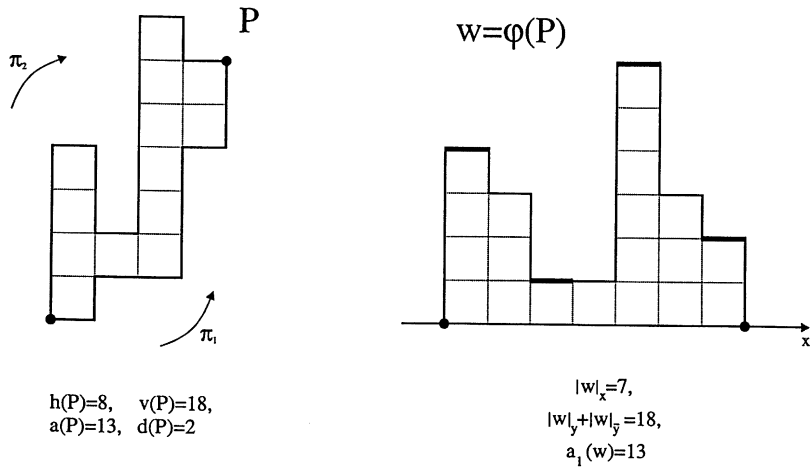

Fig. 1. A dcc-polyomino P and its code.

after an odd (resp. even) nest of w. The *rank* of w (denoted $r(w)$) is defined to be the ordinate of the terminus of w. We write $a_1(w)$ (resp. $a_2(w)$) for the sum of the ordinates of the odd (resp. even) x-steps of w. Finally, with $\mathscr{S} \subseteq \mathscr{W}$ we associate two generating functions, $\mathrm{gf}_1(\mathscr{S})$ and $\mathrm{gf}_2(\mathscr{S})$, defined by

$$\langle x^i y^j q^n t^z \rangle \mathrm{gf}_1(\mathscr{S}) = |\{w \in \mathscr{S}\colon |w|_x = i,\ |w|_y + |w|_{\bar y} = j,\ a_1(w) = n,\ r(w) = z\}|,$$

$$\langle x^i y^j q^n t^z \rangle \mathrm{gf}_2(\mathscr{S}) = |\{w \in \mathscr{S}\colon |w|_x = i,\ |w|_y + |w|_{\bar y} = j,\ a_2(w) = n,\ r(w) = z\}|.$$

(The symbol $\langle u^k \rangle f(u)$ means the coefficient of u^k in $f(u)$.)

2.3. Notations for products

Assuming from now on any empty product to be one, we write

$$(a)_n = \prod_{i=0}^{n-1} (1 - q^i a) \quad (n \in \mathbb{N}_0) \quad \text{and} \quad \begin{bmatrix} n \\ j \end{bmatrix} = \frac{(q^{n-j+1})_j}{(q)_j} \quad (n, j \in \mathbb{N}_0).$$

The second one of the above two items is a *q-binomial coefficient*, also called a *Gaussian polynomial.*

Gaussian polynomials have a long history and numerous applications. In this paper, however, we shall only need a few basic results (in fact, identities) about them. Any such identity will be quoted before use, and will be numbered (A.3.3.7) or similarly; this (A.3.3.7) means that the result in question appears in Andrews' book [1], and is numbered (3.3.7) therein.

3. A coding for dcc-polyominoes

The first step of our method is to encode the dcc-polyominoes.

Let $\mathscr{V}_{cvn}$ be the set of dcc-polyominoes which have c columns, $2v$ vertical edges and area n. With $P \in \mathscr{V}_{cvn}$ we associate a path $\varphi(P) \in \mathscr{W}$ which

(i) starts and ends on the x-axis, and

(ii) has $2c-1$ x-steps, whose ordinates are, from left to right:

$$Y_1 - y_1, \quad Y_1 - y_2, \quad Y_2 - y_2, \quad Y_2 - y_3, \quad \ldots, \quad Y_c - y_c. \tag{2}$$

(See Fig. 1 for an example.)

On account of the geometry of dcc-polyominoes, the numbers displayed in (2) are all positive, which means that the internal vertices of $\varphi(P)$ all lie in the half-plane $y>0$.

The first differences of sequence (2) are

$$y_1 - y_2, \quad Y_2 - Y_1, \quad y_2 - y_3, \quad \ldots, \quad Y_c - Y_{c-1}. \tag{3}$$

Since $y_1 \leqslant y_2 \leqslant y_3 \leqslant \cdots$, we see that each even x-step of $\varphi(P)$ stands on the same or lower level than the last x-step before it. Hence the even nests of $\varphi(P)$ all lie in $\{\bar{y}\}^*$. Further, the absolute value of any of the numbers in (3) is the length of an internal nest of $\varphi(P)$. So

$$\begin{aligned} |\varphi(P)|_y + |\varphi(P)|_{\bar{y}} = {} & (Y_1 - y_1) + |y_1 - y_2| + |Y_2 - Y_1| \\ & + |y_2 - y_3| + \cdots + |Y_c - Y_{c-1}| + (Y_c - y_c). \end{aligned} \tag{4}$$

In (4), we readily recognize the side after the equals sign: it is nothing else than the vertical perimeter of P. Thus $|\varphi(P)|_y + |\varphi(P)|_{\bar{y}} = 2v$.

Further, it is obvious that the ordinates of the odd x-steps of $\varphi(P)$ sum up to the area of P, i.e. to n.

Let $\mathscr{C}_{cvn}$ be the set of those $w \in \mathscr{W}$ which meet the following five conditions:

(i) the origin and terminus of w are on the x-axis, and all the internal vertices of w lie in the half-plane $y>0$,

(ii) all even nests of w lie in $\{\bar{y}\}^*$,

(iii) $|w|_x = 2c-1$,

(iv) $|w|_y + |w|_{\bar{y}} = 2v$,

(v) $a_1(w) = n$.

We have shown that φ maps the set $\mathscr{V}_{cvn}$ into $\mathscr{C}_{cvn}$. What is more, this mapping is readily seen to be a bijection.

Let

$$\mathscr{C}_0 = \bigcup_{c,v,n \geqslant 1} \mathscr{C}_{cvn}.$$

(The family $\mathscr{C}_0$ consists of those $w \in \mathscr{W}$ which possess the properties (i) and (ii) and have an odd number of x-steps.)

For all $c, v, n \in \mathbb{N}$ we have

$$\langle x^{2c} y^{2v} q^n \rangle V = \langle x^{2c-1} y^{2v} q^n \rangle \mathrm{gf}_1(\mathscr{C}_0)$$

which means that

$$V = x \cdot \mathrm{gf}_1(\mathscr{C}_0). \tag{5}$$

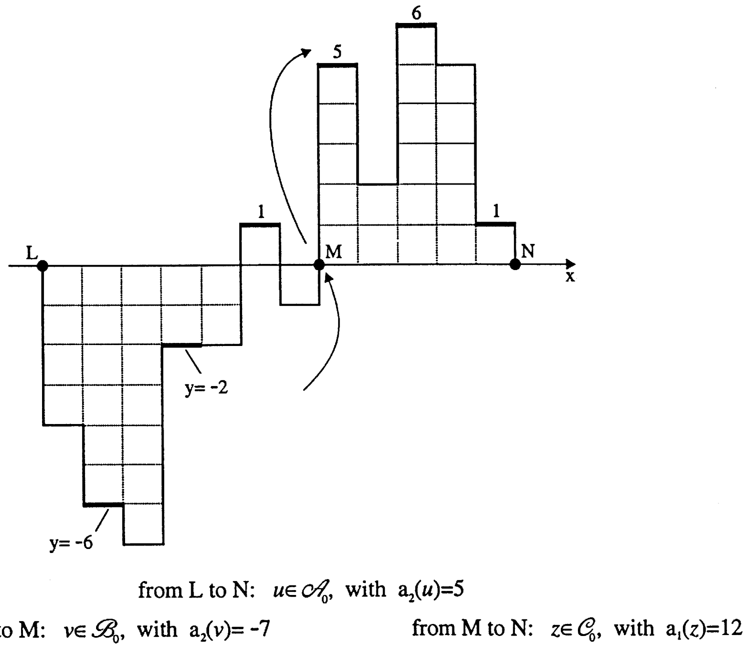

Fig. 2. A path $u \in \mathscr{A}_0$ has a unique factorization $u = vz$, where $v \in \mathscr{B}_0$ and $z \in \mathscr{C}_0$.

4. A factorization of lattice paths

It turns out advantageous to regard the paths of $\mathscr{C}_0$ as right factors of certain other lattice paths. The relevant definitions follow.

Let $\mathscr{A}$ be the set of those $u \in \mathscr{W}$ which possess the properties:

(i) $|u|_x$ is a nonzero even number,

(ii) the odd nests of u lie in $\{\bar{y}\}^*$,

(iii) the last (odd) nest of u is nonempty.

Let $\mathscr{A}_0 = \{u \in \mathscr{A}\colon r(u) = 0\}$.

Further, let $\mathscr{B}$ be the set of those $v \in \mathscr{W}$ which possess the properties:

(i) $|v|_x$ is an odd number,

(ii) the odd nests of v lie in $\{\bar{y}\}^*$,

(iii) the last (even) nest of v lies in $\{y\}^*$.

Let $\mathscr{B}_0 = \{v \in \mathscr{B}\colon r(v) = 0\}$.

Now, let $u \in \mathscr{A}_0$. Consider the factorization $u = vz$, where v is the longest among such left factors of u which are different from u and have rank zero. A little thought shows that here we have $v \in \mathscr{B}_0$ and $z \in \mathscr{C}_0$. (See Fig. 2.) Evidently,

$$|u|_x = |v|_x + |z|_x \quad \text{and} \quad |u|_y + |u|_{\bar{y}} = (|v|_y + |v|_{\bar{y}}) + (|z|_y + |z|_{\bar{y}}).$$

Since $|v|_x$ is an odd number, the odd x-steps of z are even x-steps of u, and consequently $a_2(u) = a_2(v) + a_1(z)$. Furthermore, the factorization just described is actually a bijection between $\mathscr{A}_0$ and the cartesian product $\mathscr{B}_0 \times \mathscr{C}_0$.

Putting these remarks together, we find

$$\mathrm{gf}_2(\mathscr{A}_0) = \mathrm{gf}_2(\mathscr{B}_0) \cdot \mathrm{gf}_1(\mathscr{C}_0). \tag{6}$$

From (5) and (6) it follows that

$$V = x \cdot \frac{\mathrm{gf}_2(\mathscr{A}_0)}{\mathrm{gf}_2(\mathscr{B}_0)}. \tag{7}$$

5. Computations

As we see, now we need to compute the functions $\mathrm{gf}_2(\mathscr{A}_0)$ and $\mathrm{gf}_2(\mathscr{B}_0)$. A good way to do that is to compute $\mathrm{gf}_2(\mathscr{A})$ and $\mathrm{gf}_2(\mathscr{B})$ first, and then read off the coefficients of t^0.

In what follows, for k a negative integer, we write y^k to mean $\bar{y}^{(-k)}$.

Now, the family $\mathscr{A}$ consists of all paths of the form

$$u = y^{n_1}x \cdot y^{n_2}x \cdot y^{n_3}x \cdot y^{n_4}x \cdots y^{n_{2i-1}}x \cdot y^{n_{2i}}x \cdot y^{n_{2i+1}},$$

with $i \in \mathbb{N}$, the odd-indexed n's up through n_{2i-1} nonpositive, n_{2i+1} strictly negative, and the even-indexed n's arbitrary integers. It is easy to see that for such a u we have

$$a_2(u) = i \cdot n_1 + i \cdot n_2 + (i-1) \cdot n_3 + (i-1) \cdot n_4 + \cdots + n_{2i-1} + n_{2i}$$

and

$$\mathrm{gf}_2(\{u\}) = x^{2i} \cdot (y^{|n_1|} q^{in_1} t^{n_1}) \cdot (y^{|n_2|} q^{in_2} t^{n_2}) \cdot (y^{|n_3|} q^{(i-1)n_3} t^{n_3}) \cdot (y^{|n_4|} q^{(i-1)n_4} t^{n_4})$$
$$\cdots (y^{|n_{2i-1}|} q^{n_{2i-1}} t^{n_{2i-1}}) \cdot (y^{|n_{2i}|} q^{n_{2i}} t^{n_{2i}}) \cdot (y^{|n_{2i+1}|} t^{n_{2i+1}}).$$

Now we sum this latter equation over $i \geqslant 1$ and over all legal values of $n_1, \ldots, n_{2i+1}$. Using the evaluations

$$\sum_{n=-\infty}^{0} y^{|n|} q^{kn} t^n = \frac{1}{1 - yq^{-k}t^{-1}}$$

and

$$\sum_{n \in \mathbb{Z}} y^{|n|} q^{kn} t^n = \frac{1 - y^2}{(1 - yq^{-k}t^{-1}) \cdot (1 - yq^k t)},$$

which are valid for every $k \in \mathbb{Z}$, we find that

$$\mathrm{gf}_2(\mathscr{A}) = \sum_{i=1}^{\infty} \frac{x^{2i}(1-y^2)^i y t^{-1}}{(a)_i (b)_i (b)_{i+1}}, \tag{8}$$

where $a = yqt$ and $b = yq^{-i}t^{-1}$.

Remark. Handling formal power series with positive and negative exponents can sometimes be tricky (and lead to errors). But here we are pretty safe, in that we are dealing with formal power series in x and y, the coefficients of which are Laurent polynomials in q and t.

Our next move is expanding $\mathrm{gf}_2(\mathscr{A})$ in a series in powers of t. What enables us to do that is the well-known result

$$\frac{1}{(c)_n} = \sum_{r=0}^{\infty} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} \cdot c^r \quad (n \in \mathbb{N}). \tag{A.3.3.7}$$

Applying (A.3.3.7) turns formula (8) into

$$\mathrm{gf}_2(\mathscr{A}) = \sum_{i=1}^{\infty} \sum_{j,k,l=0}^{\infty} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+l \\ i \end{bmatrix} \cdot$$
$$\times x^{2i}(1-y^2)^i y^{j+k+l+1} q^{-i(k+l)+j} t^{j-k-l-1}.$$

To complete the derivation of $\mathrm{gf}_2(\mathscr{A}_0)$, it is now enough to take the coefficient of t^0. That gives the formula

$$\mathrm{gf}_2(\mathscr{A}_0) = \sum_{i,j=1}^{\infty} x^{2i}(1-y^2)^i y^{2j} q^{i+j-ij} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}$$
$$\times \sum_{k=0}^{j-1} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+j-k-1 \\ i \end{bmatrix}. \tag{9}$$

The function $\mathrm{gf}_2(\mathscr{B}_0)$ is found in much the same way as $\mathrm{gf}_2(\mathscr{A}_0)$. So we omit the derivation and merely state that

$$\mathrm{gf}_2(\mathscr{B}_0) = x \cdot \left\{ \frac{1}{1-y^2} + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} x^{2i}(1-y^2)^i y^{2j} q^{-ij} \begin{bmatrix} i+j \\ j \end{bmatrix} \right.$$
$$\left. \times \sum_{k=0}^{j} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+j-k \\ i \end{bmatrix} \right\}. \tag{10}$$

Combining (7), (9) and (10), we establish the following:

Theorem 1. *The* gf *for directed column-convex polyominoes is given by*

$$V = \frac{\sum_{i,j=1}^{\infty} x^{2i}(1-y^2)^i y^{2j} q^{i+j-ij} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} \cdot \sum_{k=0}^{j-1} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+j-k-1 \\ i \end{bmatrix}}{1/(1-y^2) + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} x^{2i}(1-y^2)^i y^{2j} q^{-ij} \begin{bmatrix} i+j \\ j \end{bmatrix} \cdot \sum_{k=0}^{j} \begin{bmatrix} i+k-1 \\ k \end{bmatrix} \begin{bmatrix} i+j-k \\ i \end{bmatrix}}. \tag{11}$$

Thus we have got the ‘negative’ counterpart of formula (1).

It is plausible that the equivalence of formulas (1) and (11) can also be proved in a direct (no-polyominoes) way. However, we have not (yet?) found such a direct way out.

6. q-enumeration of parallelogram polyominoes

To q-enumerate the dcc-polyominoes, we first encoded them, then found a factorization of lattice paths, and then completed the job by carrying out computations. In this section we shall see that the parallelogram (p-)polyominoes also lend themselves to that kind of treatment.

First, coding. Recall that in Section 3 we defined a mapping φ. Now let $\mathscr{F}_0$ be the set of the images of p-polyominoes under φ. The set $\mathscr{F}_0$ is easy to characterize. Indeed, it consists of just those paths $w \in \mathscr{W}$ which have the properties:

(i) the origin and terminus of w are on the x-axis, and all the internal vertices of w lie in the half-plane $y > 0$,

(ii) $|w|_x$ is an odd number,

(iii) all odd nests of w lie in $\{y\}^*$,

(iv) all even nests of w lie in $\{\bar{y}\}^*$.

Next we factorize lattice paths. Of course, the first thing to do is to tell exactly what path-families are we going to consider. So, the relevant definitions are these.

Let $\mathscr{D}$ be the set of those $u \in \mathscr{W}$ which possess the properties:

(i) $|u|_x$ is a nonzero even number,

(ii) the odd nests of u lie in $\{\bar{y}\}^*$,

(iii) the last (odd) nest of u is nonempty,

(iv) the even nests of u lie in $\{y\}^*$.

Let $\mathscr{D}_0 = \{u \in \mathscr{D}\colon r(u) = 0\}$.

Further, let $\mathscr{E}$ be the set of those $v \in \mathscr{W}$ which possess the properties:

(i) $|v|_x$ is an odd number,

(ii) the odd nests of v lie in $\{\bar{y}\}^*$,

(iii) the even nests of v lie in $\{y\}^*$.

Let $\mathscr{E}_0 = \{v \in \mathscr{E}\colon r(v) = 0\}$.

Now, for every $u \in \mathscr{D}_0$ there exists exactly one pair (v, z) such that $v \in \mathscr{E}_0$, $z \in \mathscr{F}_0$, and $u = vz$. (As in Section 4, the path v is the longest among such left factors of u which are different from u and have rank zero.) This fact immediately implies that $\mathrm{gf}_2(\mathscr{D}_0) = \mathrm{gf}_2(\mathscr{E}_0) \cdot \mathrm{gf}_1(\mathscr{F}_0)$.

Let P be the gf for the p-polyominoes. Since $P = x \cdot \mathrm{gf}_1(\mathscr{F}_0)$, we have

$$P = x \cdot \frac{\mathrm{gf}_2(\mathscr{D}_0)}{\mathrm{gf}_2(\mathscr{E}_0)}.$$

At this point, there is no other way but to dive into the calculations.

To derive $\mathrm{gf}_2(\mathscr{E}_0)$, we first note that the family $\mathscr{E}$ consists of all paths of the form

$$y^{n_1}x \cdot y^{n_2}x \cdot y^{n_3}x \cdot y^{n_4}x \cdots y^{n_{2i+1}}x \cdot y^{n_{2i+2}}$$

with $i \in \mathbb{N}_0$, the odd-indexed n's nonpositive, and the even-indexed n's nonnegative. Having determined what does a single path of the above form contribute to $\mathrm{gf}_2(\mathscr{E})$, we sum those contributions over all $i \in \mathbb{N}_0$ and over all legal values of $n_1, \ldots, n_{2i+2}$.

Thus we find that

$$\mathrm{gf}_2(\mathscr{E}) = x \cdot \sum_{i=0}^{\infty} \frac{x^{2i}}{(yt)_{i+1}(yq^{-i}t^{-1})_{i+1}}. \tag{12}$$

With the aid of (A.3.3.7), we manage to express (12) as

$$\mathrm{gf}_2(\mathscr{E}) = x \cdot \sum_{i,j,k=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+k \\ k \end{bmatrix} x^{2i} y^{j+k} q^{-ik} t^{j-k}$$

and then we are almost finished, since

$$\mathrm{gf}_2(\mathscr{E}_0) = \langle t^0 \rangle \quad \mathrm{gf}_2(\mathscr{E}) = x \cdot \sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij}.$$

The function $\mathrm{gf}_2(\mathscr{D}_0)$ is derived similarly, and the result is

$$\mathrm{gf}_2(\mathscr{D}_0) = \sum_{i,j=1}^{\infty} \begin{bmatrix} i+j-1 \\ i \end{bmatrix} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} x^{2i} y^{2j} q^{i+j-ij}.$$

Putting the pieces together, we arrive at the following conclusion:

Theorem 2. *The* gf *for parallelogram polyominoes is given by*

$$P = \frac{\sum_{i,j=1}^{\infty} \begin{bmatrix} i+j-1 \\ i \end{bmatrix} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} x^{2i} y^{2j} q^{i+j-ij}}{\sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij}}. \tag{13}$$

It would be exaggerated to say that our formula (13) is new, because it can be obtained from the result of Goulden and Jackson's [19, exercise 5.5.2.b] by the simple change of variables $z = y^2$, $q = q$, and $s = x^2 y^{-2}$.

Remark. In substance, there exist three 'positive' formulas for P. One of them is x–y asymmetric and involves onefold summations. Another formula is x–y symmetric, but involves twofold summations. Finally, there is also an x–y symmetric formula which, being stated in terms of continued fractions, looks quite different from the other two. The reader may refer to Bousquet-Mélou and Viennot [4] and Klarner and Rivest [21] for the first formula, to Bousquet-Mélou and Viennot [4] for the second, and to Bousquet-Mélou and Viennot [4] and Gessel [18] for the third.

Next of kin to (13) is the identity

$$x^2 + y^2 + P(x,y,q) + P(x,y,q^{-1}) = 1 - \frac{1}{\sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} \cdot x^{2i} y^{2j} q^{-ij}}. \tag{14}$$

The case $x = y$ of (14) was found by Pólya in 1938 [23]. The general case $x \neq y$ is more recent, and is due to Gessel [18].

The identity (14) can be derived from (13) without too great creativity. Indeed, by the definition of Gaussian polynomials, for $i, j \in \mathbb{N}_0$ it holds that

$$\begin{bmatrix} i+j \\ i \end{bmatrix}_{\text{with } q \text{ replaced by } 1/q} = \begin{bmatrix} i+j \\ i \end{bmatrix} \cdot q^{-ij}.$$

Using this fact, one readily finds from (13) that

$$P(x, y, q^{-1}) = \frac{\sum_{i,j=1}^{\infty} \begin{bmatrix} i+j-1 \\ i \end{bmatrix} \begin{bmatrix} i+j-1 \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij}}{\sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij}}. \tag{15}$$

(A good news: (13) and (15) have the same denominator.) Hence

$$x^2 + y^2 + P(x, y, q) + P(x, y, q^{-1}) = \frac{D_1 + D_2 + D_3 + D_4}{\sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} \cdot x^{2i} y^{2j} q^{-ij}}, \tag{16}$$

where

$D_1 = x^2 \cdot$ (the common denominator of (13) and (15)),

$D_2 = y^2 \cdot$ (that denominator),

$D_3 =$ the numerator of (13),

and

$D_4 =$ the numerator of (15).

Now there are two things to remember: first, that

$$\begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + q^j \cdot \begin{bmatrix} n-1 \\ j \end{bmatrix} = \begin{bmatrix} n \\ j \end{bmatrix} \quad (n, j \in \mathbb{N}) \tag{A.3.3.4}$$

and second, that the Gaussian polynomial n choose j equals the Gaussian polynomial n choose $n-j$ (A.3.3.2).

The proof then goes on as follows:

(1) Let S_1 be the part of D_1 in which j is zero, and let S_2 be the part of D_2 in which i is zero. Using (A.3.3.4) and sporadically also (A.3.3.2), we merge $D_1 - S_1$ and D_3 into one double sum (say D_5), and we merge $D_2 - S_2$ and D_4 into another one double sum (say D_6).

(2) We merge D_5 and D_6 into a certain double sum D_7.

(3) We incorporate S_1 and S_2 into D_7.

This three-step procedure outputs the formula

$$D_1 + D_2 + D_3 + D_4 = \left(\sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} x^{2i} y^{2j} q^{-ij} \right) - 1$$

which combines with (16) into the result we wanted to prove.

So, this is how identity (14) can be derived once (13) is known. However, it is fair to say that not all the roads to (14) go *via* (13). Indeed, Flajolet [17] gave an elegant derivation of (14) in which only the denominator of (13) is computed, and the job is then completed by taking a close look at some near relatives of the paths of $\mathscr{E}_0$.

7. q-enumeration of directed convex polyominoes

Our method is already well described, so this time we shall proceed *in medias res*.

Let $\mathscr{I}_0$ be the set of the images of directed convex (dc-) polyominoes under the coding φ (defined in Section 3). A path $w \in \mathscr{W}$ lies in $\mathscr{I}_0$ if and only if it holds that

(i) the origin and terminus of w are on the x-axis, and all the internal vertices of w lie in the half-plane $y > 0$,

(ii) $|w|_x$ is an odd number,

(iii) the odd nests of w behave unimodally. That is to say, up to the last of the odd nests which lie in $\{y\}^*\backslash\{\varepsilon\}$, no odd nest lies in $\{\bar{y}\}^*\backslash\{\varepsilon\}$,

(iv) the even nests of w all lie in $\{\bar{y}\}^*$.

To pave the rest of our way, now we define some new objects.

Let $\mathscr{G}$ be the set of those $u \in \mathscr{W}$ which possess the properties:

(i) $|u|_x$ is a nonzero even number,

(ii) the odd nests of u lie in $\{\bar{y}\}^*$,

(iii) the last (odd) nest of u is nonempty,

(iv) at least one of the even nests of w lies in $\{y\}^*\backslash\{\varepsilon\}$,

(v) the even nests of w obey the 'unimodality' rule: up to the last one which lies in $\{y\}^*\backslash\{\varepsilon\}$, none lies in $\{\bar{y}\}^*\backslash\{\varepsilon\}$.

Let $\mathscr{G}_0 = \{u \in \mathscr{G}\colon r(u) = 0\}$.

Let K be the gf for the dc-polyominoes.

Now, every $u \in \mathscr{G}_0$ has a unique factorization $u = vz$ with $v \in \mathscr{E}_0$ and $z \in \mathscr{I}_0$. From this it follows easily that $\mathrm{gf}_2(\mathscr{G}_0) = \mathrm{gf}_2(\mathscr{E}_0) \cdot \mathrm{gf}_1(\mathscr{I}_0)$. Since $K = x \cdot \mathrm{gf}_1(\mathscr{I}_0)$, we have

$$K = x \cdot \frac{\mathrm{gf}_2(\mathscr{G}_0)}{\mathrm{gf}_2(\mathscr{E}_0)}.$$

As we know $\mathrm{gf}_2(\mathscr{E}_0)$ from Section 6, here we only need to find $\mathrm{gf}_2(\mathscr{G}_0)$. For that purpose, we first remark that the family $\mathscr{G}$ consists of all paths of the form

$$y^{n_1}x \cdot y^{n_2}x \cdots y^{n_{2k}}x \cdot y^{n_{2k+1}}x \cdot y^{n_{2k+2}}x \cdots y^{n_{2i}}x \cdot y^{n_{2i+1}}$$

with $i \in \mathbb{N}$, $k \in \{1, \ldots, i\}$, the odd-indexed n's through n_{2i-1} nonpositive, n_{2i+1} strictly negative, the even-indexed n's through n_{2k-2} nonnegative, n_{2k} strictly positive, and the even-indexed n's from n_{2k+2} on nonpositive. From this we find that

$$\mathrm{gf}_2(\mathscr{G}) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \frac{x^{2i} y^2 q^{i-k+1}}{(yq^{-i}t^{-1})_i (yq^{i-k+1}t)_k (yq^{-(i-k)}t^{-1})_{i-k+1}}. \tag{17}$$

The next two steps are expanding (17) by use of (A.3.3.7) and taking the coefficient of t^0. Thus we obtain

$$\mathrm{gf}_2(\mathscr{G}_0) = \sum_{i,j=1}^{\infty} \sum_{k=1}^{i} \sum_{l=1}^{j} \begin{bmatrix} i+l-2 \\ i-1 \end{bmatrix} \begin{bmatrix} j+k-2 \\ j-1 \end{bmatrix} \begin{bmatrix} i+j-k-l \\ i-k \end{bmatrix} \cdot x^{2i} y^{2j} q^{i+j-kl}.$$

Now we have all the ingredients for the following theorem.

Theorem 3. *The* gf *for directed convex polyominoes is given by*

$$K = \frac{\sum_{i,j=1}^{\infty} \sum_{k=1}^{i} \sum_{l=1}^{j} \begin{bmatrix} i+l-2 \\ i-1 \end{bmatrix} \begin{bmatrix} j+k-2 \\ j-1 \end{bmatrix} \begin{bmatrix} i+j-k-l \\ i-k \end{bmatrix} \cdot x^{2i} y^{2j} q^{i+j-kl}}{\sum_{i,j=0}^{\infty} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{bmatrix} i+j \\ j \end{bmatrix} \cdot x^{2i} y^{2j} q^{-ij}}. \tag{18}$$

Formula (18) firmly testifies that K is an x–y symmetric function.

Remark. At the current state of affairs, there exists, in substance, only one 'positive' formula for K. That formula was found by Bousquet-Mélou and Viennot [4], and was subsequently touched up by Bousquet-Mélou [6]. The touched-up version reads

$$K = y^2 \cdot \frac{\sum_{i=1}^{\infty} x^{2i} q^i / (y^2 q)_i \sum_{k=0}^{i-1} (-1)^k q^{(k-1)k/2} / (q)_k (y^2 q^{k+1})_{i-k-1}}{\sum_{i=0}^{\infty} (-1)^i x^{2i} q^{i(i+1)/2} / (q)_i (y^2 q)_i}.$$

8. Perimeter enumeration of directed convex polyominoes

Lin and Chang [22] have found that the dc-polyominoes have an algebraic perimeter gf, viz.

$$K(x, y, 1) = \frac{x^2 y^2}{\sqrt{1 - 2x^2 - 2y^2 + (x^2 - y^2)^2}}. \tag{19}$$

Is our method able to reproduce their result? [1]

Let us see... We have $K = x \cdot \mathrm{gf}_2(\mathscr{G}_0)/\mathrm{gf}_2(\mathscr{E}_0)$, and $\mathrm{gf}_2(\mathscr{E}_0)$ is no doubt a simpler function than $\mathrm{gf}_2(\mathscr{G}_0)$. So, to begin with, let us try to find the case $q = 1$ of $\mathrm{gf}_2(\mathscr{E}_0)$. [2]

From (12) we have

$$\mathrm{gf}_2(\mathscr{E}) = x \cdot \sum_{i=0}^{\infty} \frac{x^{2i}}{(1 - yt)^{i+1} (1 - yt^{-1})^{i+1}},$$

which the geometric series evaluation readily puts into the form

$$\mathrm{gf}_2(\mathscr{E}) = \frac{x}{(1 - yt)(1 - yt^{-1}) - x^2}.$$

For convenience, in $\mathrm{gf}_2(\mathscr{E})$ we now substitute $y^{-1}t$ for t. Thus we obtain a new function, viz.

$$E = \frac{x}{(1 - t)(1 - y^2 t^{-1}) - x^2}, \tag{20}$$

in which the coefficient of t^0 is the same as in $\mathrm{gf}_2(\mathscr{E})$. That is, $\langle t^0 \rangle E = \langle t^0 \rangle \mathrm{gf}_2(\mathscr{E}) = \mathrm{gf}_2(\mathscr{E}_0)$.

[1] This interesting question was posed to us by one of the referees.

[2] The symbol which so far has been denoting a given gf_2, from now on will denote another thing, namely the case $q = 1$ of that gf_2.

The denominator of (20) is equal to $-t^{-1}[t^2-(1-x^2+y^2)t+y^2]$, and the roots of $t^2-(1-x^2+y^2)t+y^2=0$ are

$$r_1=\frac{1-x^2+y^2+\sqrt{\Delta}}{2} \quad \text{and} \quad r_2=\frac{1-x^2+y^2-\sqrt{\Delta}}{2},$$

with $\Delta=1-2x^2-2y^2+(x^2-y^2)^2$.

So we have $-t^{-1}[t^2-(1-x^2+y^2)t+y^2]=-t^{-1}\cdot(t-r_1)(t-r_2)$. It is now immediate that

$$E=\frac{x}{(r_1-t)(1-r_2t^{-1})}=x\cdot\left(\sum_{i=0}^{\infty}\frac{t^i}{r_1^{i+1}}\right)\cdot\left(\sum_{j=0}^{\infty}\frac{r_2^j}{t^j}\right),$$

and that

$$\mathrm{gf}_2(\mathscr{E}_0)=\langle t^0\rangle E=x\cdot\sum_{i=0}^{\infty}\frac{r_2^i}{r_1^{i+1}}=\frac{x}{r_1-r_2}=\frac{x}{\sqrt{1-2x^2-2y^2+(x^2-y^2)^2}}. \quad (21)$$

So, $\mathrm{gf}_2(\mathscr{E}_0)$ was pretty easy. Let us now try to work out the more complicated function $\mathrm{gf}_2(\mathscr{G}_0)$.

It follows from (17) that

$$\begin{aligned}\mathrm{gf}_2(\mathscr{G})&=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\frac{x^{2i}y^2}{(1-yt^{-1})^{2i-k+1}(1-yt)^k}\\&=\frac{y^2}{1-yt^{-1}}\sum_{k=1}^{\infty}\frac{1}{(1-yt)^k(1-yt^{-1})^{-k}}\sum_{i=k}^{\infty}\frac{x^{2i}}{(1-yt^{-1})^{2i}}\\&=\frac{x^2y^2(1-yt^{-1})}{(1-x-yt^{-1})(1+x-yt^{-1})[(1-yt)(1-yt^{-1})-x^2]}.\end{aligned}$$

Let G be the function which results from changing t to $y^{-1}t$ in $\mathrm{gf}_2(\mathscr{G})$. We have

$$\begin{aligned}G&=\frac{x^2y^2(1-y^2t^{-1})}{(1-x-y^2t^{-1})(1+x-y^2t^{-1})[(1-t)(1-y^2t^{-1})-x^2]}\\&=\frac{x^2y^2(1-y^2t^{-1})}{(1-x-y^2t^{-1})(1+x-y^2t^{-1})(r_1-t)(1-r_2t^{-1})}\\&=x^2y^2(1-y^2t^{-1})\cdot\sum_{i,j,k,l=0}^{\infty}\frac{r_2^j\cdot y^{2(k+l)}\cdot t^{i-j-k-l}}{r_1^{i+1}(1-x)^{k+1}(1+x)^{l+1}}.\end{aligned}$$

In order to get $\mathrm{gf}_2(\mathscr{G}_0)$, we now take the coefficient of t^0. This leaves us with two triple sums, which we then put together by changing i to $i+1$ in one of them. Next we replace $j+k$ by j, and we sum on i, j and k (in that order). Thus we obtain

$$\mathrm{gf}_2(\mathscr{G}_0)=\frac{x^2y^2r_1(r_1-y^2)}{[(r_1-y^2)^2-x^2r_1^2](r_1-r_2)}.$$

Using $r_1 + r_2 = 1 - x^2 + y^2$ and $r_1 r_2 = y^2$, we have that

$$\begin{aligned}(r_1 - y^2)^2 - x^2 r_1^2 &= (1 - x^2) r_1^2 - 2y^2 r_1 + y^4 \\ &= (r_1 + r_2 - y^2) r_1^2 - 2r_1^2 r_2 + y^2 r_1 r_2 \\ &= r_1 (r_1 - y^2)(r_1 - r_2).\end{aligned}$$

Therefore,

$$\mathrm{gf}_2(\mathscr{G}_0) = \frac{x^2 y^2 r_1 (r_1 - y^2)}{r_1 (r_1 - y^2)(r_1 - r_2)^2} = \frac{x^2 y^2}{(r_1 - r_2)^2} = \frac{x^2 y^2}{1 - 2x^2 - 2y^2 + (x^2 - y^2)^2}. \tag{22}$$

(This function is rational!)

Eventually, we plug (21) and (22) into the relation $K = x \cdot \mathrm{gf}_2(\mathscr{G}_0)/\mathrm{gf}_2(\mathscr{E}_0)$, and we quickly get the required result (19).

Incidentally, the number of dc-polyominoes with perimeter $2n + 4$ is $\binom{2n}{n}$ [5]. But clearly enough, $\binom{2n}{n}$ is also the number of those $2n$-step members of $\{y, \bar{y}\}^*$ which start and end on the x-axis. For (at least attempted) explanations of this curious coincidence, see [3,12].

9. For further reading

The following references are also of interest to the reader: [11] and [2].

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