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Hedlund metrics and the stable norm

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ABSTRACT

The real homology of a compact Riemannian manifold M is naturally endowed with the stable norm. The stable norm on $H_1(M,\mathbb{R})$ arises from the Riemannian length functional by homogenization. It is difficult and interesting to decide which norms on the finite-dimensional vector space $H_1(M,\mathbb{R})$ are stable norms of a Riemannian metric on M. If the dimension of M is at least three, I. Babenko and F. Balacheff proved in [I. Babenko, F. Balacheff, Sur la forme de la boule unité de la norme stable unidimensionnelle, Manuscripta Math. 119 (3) (2006) 347–358] that every polyhedral norm ball in $H_1(M,\mathbb{R})$, whose vertices are rational with respect to the lattice of integer classes in $H_1(M,\mathbb{R})$, is the stable norm ball of a Riemannian metric on M. This metric can even be chosen to be conformally equivalent to any given metric. In [I. Babenko, F. Balacheff, Sur la forme de la boule unité de la norme stable unidimensionnelle, Manuscripta Math. 119 (3) (2006) 347–358], the stable norm induced by the constructed metric is computed by comparing the metric with a polyhedral one. Here we present an alternative construction for the metric, which remains in the geometric framework of smooth Riemannian metrics.

1. Introduction

On every compact Riemannian manifold M the real homology vector spaces $H_m(M;\mathbb{R})$ are endowed with a natural norm $\|\cdot\|_s$, called *stable norm*. This concept appeared for the first time in Federer [4] and was named *stable norm* in Gromov [5]. The stable norm on $H_1(M;\mathbb{R})$ arises directly from the Riemannian metric on the manifold M. The following equality for an integral class $v\in H_1(M;\mathbb{R})$ (see [5])

$$\|v\|_s := \inf\{n^{-1}L(\gamma) \mid \gamma \text{ is a closed curve representing } nv,\ n\in\mathbb{N}\}$$

allows a description of this object that is geometrically very intuitive: the stable norm describes the geometry of the Abelian covering $\bar{M}$ of M from a point of view from which fundamental domains look arbitrarily small. Knowing the unit ball of this norm, one can decide on existence and properties of some of the minimal geodesics relative to the Riemannian Abelian covering of the manifold; these are curves in M whose lifts to the Riemannian Abelian covering minimize arc length between each two of their points. Bangert has presented in [3] a Riemannian metric on the 3-torus $\mathbb{T}^3$, such that the unit ball of the induced stable norm on $H_1(\mathbb{T}^3;\mathbb{R})\simeq\mathbb{R}^3$ is a symmetric octahedron. Furthermore, Babenko and Balacheff have shown in [1] that, given a compact Riemannian manifold (M,ρ) of dimension greater than 2, for every centrally symmetric and convex polytope in $H_1(M;\mathbb{R})$ with nonempty interior, such that the directions of its vertices are rational, there is a Riemannian metric on M that is conformal to ρ and induces the given polytope as unit ball of the stable norm. Here we propose an alternative Riemannian metric, satisfying the same conditions. Our construction is a generalization of the Hedlund metric in [3]. The idea, that can be already found in the original paper of Hedlund [6] and is also used

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in [1], is to construct a metric that is "small" in tubular neighborhoods of disjoint closed curves representing the vertices of the polytope, and much "bigger" everywhere else. The convexity properties of the polytope play a decisive role in our computation of the stable norm induced by the Hedlund metric.

Bangert and Hedlund use such metrics in order to illustrate their results on minimal geodesics. Here we focus only on the proof of the theorem of Babenko and Balacheff [1]. In fact, if we wanted to show results on minimal geodesics, we would need to specify the definition of the Hedlund metric we give here. A discussion of the minimal geodesics for such metrics (with additional assumptions) was made in Jotz [7].

Outline of the paper. The construction of tubular neighborhoods of curves will be recalled in the next section. A lemma on the existence of representatives for cohomology classes with "good" properties on the tubular neighborhood will be stated. The construction of the Riemannian metric will be given in the following section, and the formula for the corresponding stable norm will be computed.

Notations. In the following M will denote a compact smooth manifold with $\dim M \geqslant 3$, and ρ a Riemannian metric on M. Let $\bar{M}$ denote the Abelian covering of M. More precisely $\bar{M}$ is the subcovering of the universal covering whose group of deck transformations is the set $H_1(M;\mathbb{Z})_{\mathbb{R}}$ of integer classes in $H_1(M;\mathbb{R})$. We denote by $p:\bar{M}\to M$ the covering map and by $\bar{\rho}:=p^*\rho$ the pull-back metric. If $h:\pi_1(M)\to H_1(M;\mathbb{Z})$ denotes the Hurewicz homomorphism (see [9]) and T the torsion subgroup of $H_1(M;\mathbb{Z})$, then the Abelian covering can be described as the quotient manifold by the action of the normal subgroup $h^{-1}(T)\subseteq \pi_1(M)$ of the fundamental group on the universal cover $\tilde{M}$ of M. Hence the operation

$$\Phi\colon H_1(M;\mathbb{Z})_{\mathbb{R}}\times \bar{M}\to \bar{M}$$
$$(v,m)\mapsto \Phi(v,m)=: m+v$$

of $H_1(M;\mathbb{Z})_{\mathbb{R}}$ on $\bar{M}$ is Abelian and torsionfree (that is why we choose to use this $+$-notation).

The de Rham cohomology vector space $H^1_{\mathbf{dR}}(M)$ is isomorphic to the dual of $H_1(M;\mathbb{R})$ [8, de Rham theorem]. In the following, we will use this isomorphism without mentioning it.

Given a Riemannian metric g on M, we will write g^* for its dual metric. The space of 1-forms on M (respectively on $\bar{M}$) will be denoted by $\Omega^1(M)$ (respectively $\Omega^1(\bar{M})$). We will denote by $\|\cdot\|_x$ (or also simply $\|\cdot\|$) the norm on T_xM induced by the considered metric on M (we will also use this notation for the norm on $T_{\bar{x}}\bar{M}$, $\bar{x}\in\bar{M}$, induced from the corresponding metric on $\bar{M}$). For a curve $\gamma: I\to M$, $L(\gamma)$ will be the length induced from the given metric on M and for a curve $\bar{\gamma}: I\to\bar{M}$, $\bar{L}(\bar{\gamma})$ the length induced from the corresponding periodic metric on $\bar{M}$.

Given a polytope P, we will call the set $\{\sum_{i=1}^k \alpha_i v_i \mid \alpha_i\geqslant 0\}$ the *cone over the face* S of the polytope if $v_1,\dots,v_k$ are the vertices of P lying in this face (i.e. $S=\{\sum_{i=1}^k\alpha_i v_i\mid \alpha_i\geqslant 0$ and $\sum_{i=1}^k\alpha_i=1\}$).

An integer class v in $H_1(M;\mathbb{Z})_{\mathbb{R}}$ will be called *indivisible* if the equation $v=n\cdot v'$, $n\in\mathbb{Z}$ and $v'\in H_1(M;\mathbb{Z})_{\mathbb{R}}$ yields $n=\pm 1$.

2. Tubular neighborhoods of curves, adapted one-forms

Tubular neighborhoods and semi-geodesic coordinates. Let $\gamma:[0,1]\to M$ be a regular, simple closed curve. In the following, such a curve will be called *admissible*. We can write $\gamma:\mathbb{S}^1\to M$ and assume the curve γ is parametrized proportionally to arc length.

For $\varrho>0$, let $V_\varrho(\Gamma)$ denote the bundle of balls of radius ϱ in the normal bundle $\pi: N\Gamma\to\Gamma$ of the embedded submanifold $\Gamma:=\gamma(\mathbb{S}^1)$ in M. Analogously, if $I\subseteq\mathbb{S}^1$ is an interval, then $V_\varrho(\gamma(I))=V_\varrho(\Gamma)\cap\pi^{-1}(\gamma(I))$. We choose $\varrho>0$ small enough such that the normal exponential map E restricted to $V_\varrho(\Gamma)$ is a diffeomorphism onto an open neighborhood $U_\varrho(\Gamma)\subseteq M$ of Γ (and similarly $U_\varrho(\gamma(I))=E(V_\varrho(\gamma(I)))$). Such an open set $U_\varrho(\Gamma)$ is called the *tubular neighborhood* (of radius ϱ) of Γ.

Choose an orthogonal frame $(E_1,\dots,E_m)$ on $U\subseteq M$ open, such that for all $x=\gamma(t)$ in $\Gamma\cap U$,

$$E_1|_x=\dot{\gamma}(t)$$

and, consequently, $(E_2|_x,\dots,E_m|_x)$ forms a basis for $N_x\Gamma$. Assume that the open set U is such that $U_\varrho(\Gamma)\cap U=U_\varrho(\gamma(I))$ for an open interval $I\subseteq\mathbb{S}^1$. The diffeomorphism

$$\varphi\colon U_\varrho\big(\gamma(I)\big)\to I\times B^{m-1}_\varrho\subseteq\mathbb{R}^m$$
$$x\mapsto\big(s(x),\varphi_2(x),\dots,\varphi_m(x)\big),$$

where $\varphi_j(x)$ and $s(x)$ are such that

$$E^{-1}(x)=\sum_{j=2}^m\varphi_j(x)\cdot E_j|_{\gamma(s(x))}\in\mathcal{V}_\varrho,$$

will be called a *semi-geodesic chart for* $U_\varrho(\Gamma)$. A particularity of this chart is that $\partial_1^\varphi|_x=\dot{\gamma}(t)$ and, for $j=2,\dots,m$, $\partial_j^\varphi|_x=E_j|_x$ holds for all $x=\gamma(t)\in\Gamma\cap U$ (note that $\Gamma\cap U=\gamma(I)$).

The map s is defined globally on $U_\varrho(\Gamma)$ and we have the identity

$$ds|_{\gamma(t)}\big(\dot\gamma(t)\big) = \frac{d}{dt} s\circ\gamma(t) = \frac{d}{dt}t = 1 \tag{1}$$

for all t in $\mathbb{S}^1$.

Let $\gamma_1,\dots,\gamma_N$ be disjoint admissible loops and choose $\varrho>0$ so that the construction above is possible for all the curves $\gamma_1,\dots,\gamma_N$ simultaneously. Choose furthermore ε with $\varrho>\varepsilon>0$ such that the tubular neighborhoods with radius ε of the curves are disjoint. Set $\Gamma_j=\gamma_j(\mathbb{S}^1)$, $\Gamma=\bigcup_{j=1}^N\Gamma_j$, and $U_\varepsilon(\Gamma):=\bigcup_{j=1}^N U_\varepsilon(\Gamma_j)$. Then there exists a bump-function ζ on M for the tubular neighborhoods, i.e., ζ is a smooth function such that the following holds:

$$\zeta(y)=\begin{cases}1, & y\in U_\varepsilon(\Gamma)\\ 0, & y\in M\setminus U_\varrho(\Gamma).\end{cases} \tag{2}$$

"Good" one-forms. Choose a connected fundamental domain F_0 for the action of $H_1(M;\mathbb{Z})_\mathbb{R}$ on $\bar M$. Denote by $\bar\gamma_j$ the lift of γ_j to $\bar M$ such that $\bar\gamma_j(0)\in F_0$ (note that γ_j is here considered as a smooth 1-periodic curve $\gamma_j:\mathbb{R}\to M$). Write $\bar\Gamma_i=\bar\gamma_i(\mathbb{R})$ and $U_\varrho(\bar\Gamma_i)$ the corresponding lift to $\bar M$ of $U_\varrho(\Gamma_i)$. Hence $U_\varrho(\bar\Gamma_i)$ is the tubular neighborhood of radius ρ of $\bar\Gamma_i$. The notion of a semi-geodesic chart for $U_\varrho(\bar\Gamma_i)$ makes also sense here, and $\bar s_i:U_\varrho(\bar\Gamma_i)\to\mathbb{R}$ exists with $\bar s_i(\bar\gamma_i(t))=t$ for all $t\in\mathbb{R}$. Since the covering map $p:\bar M\to M$ is a local isometry,

$$\bar x\in \exp_{\bar M}(N_{\bar\gamma_i(t)}\bar\Gamma_i) \Leftrightarrow p(\bar x)\in\exp_M(N_{p\circ\bar\gamma_i(t)}\Gamma_i)$$

holds for all $\bar x\in U_\varrho(\bar\Gamma_i)$ and

$$(p^*ds_i)|_{U_\varrho(\bar\Gamma_i)} = d\bar s_i. \tag{3}$$

Define $L_i=\bar\Gamma_i+H_1(M;\mathbb{Z})_\mathbb{R}$ and $U_\varrho(L_i)=U_\varrho(\bar\Gamma_i)+H_1(M;\mathbb{Z})_\mathbb{R}$, as well as $L=\bigcup_{j=1}^N L_j$ and $U_\varrho(L)=\bigcup_{j=1}^N U_\varrho(L_j)$. Choose ε with $0<\varepsilon<\varrho$ and define $U_\varepsilon(\bar\Gamma_i)$, $U_\varepsilon(L_i)$ and $U_\varepsilon(L)$ as above. The connected components of L will be called *lines* in the following.

Proposition 2.1. *Let $v_1,\dots,v_N$ be indivisible integer classes in $H_1(M;\mathbb{Z})_\mathbb{R}$, that span $H_1(M;\mathbb{R})$ as a real vector space. Let $\gamma_1,\dots,\gamma_N$ be disjoint admissible representatives of those classes, and $U_\varepsilon(\Gamma_1),\dots,U_\varepsilon(\Gamma_N)$ disjoint tubular neighborhoods of these curves. Furthermore let $\lambda\in H^1_{dR}(M)$ be an arbitrary cohomology class. Then there exists a one-form ω representing λ such that*:

$$\omega|_x=\lambda(v_i)\,ds_i|_x \quad \text{for } x\in U_\varepsilon(\Gamma_i),\ i=1,\dots,N.$$

Proof. The function $\bar s_j$ is defined on $U_\varrho(\bar\Gamma_j)$ for $j=1,\dots,N$. Set $\bar s_j=0$ on $U_\varrho(\bar\Gamma_i)$ for $i\neq j$ and define:

$$s_\lambda: U_\varrho(L)\to\mathbb{R}$$
$$x=x_0+v_0\mapsto\sum_{i=1}^N\lambda(v_i)\bar s_i(x_0)+\lambda(v_0).$$

Doing so, each element $U_\varrho(L_j)$ is written $x=x_0+v_0$ with $x_0\in U_\varrho(\bar\Gamma_j)\cap F_0$ and $v_0\in H_1(M;\mathbb{Z})_\mathbb{R}$. For $x\in U_\varrho(\bar\Gamma_j)\cap F_0$ holds: $s_\lambda(x)=\lambda(v_j)\bar s_j(x)$. Thus, with the definition of s_λ, for $v=z\cdot v_j$ with $z\in\mathbb{Z}$:

$$s_\lambda(x+v)=\lambda(v_j)\bar s_j(x)+\lambda(v)=\lambda(v_j)\cdot\big(\bar s_j(x)+z\big)\overset{(3)}{=}\lambda(v_j)\cdot\bar s_j(x+v).$$

This leads to $s_\lambda|_{U_\varrho(\bar\Gamma_j)}=\lambda(v_j)\bar s_j$, and analogously: $s_\lambda|_{U_\varrho(\bar\Gamma_j)+v}=\lambda(v_j)\bar s_j\circ\Phi(-v,\cdot)+\lambda(v)$. Thus, s_λ is a smooth function.

Choose an arbitrary representative ω' for λ. Since ω' is closed, the 1-form $\tilde p^*\omega'\in\Omega^1(\tilde M)$ is also closed, where $\tilde p:\tilde M\to M$ is the universal covering of M. Since each closed 1-form on $\tilde M$ is exact, there exists $\tilde f\in C^\infty(\tilde M)$ such that $\tilde p^*\omega'=d\tilde f$. One can show easily that $\tilde f$ is invariant under the action of $h^{-1}(T)$ on $\tilde M$ and descends to $\bar f\in C^\infty(\bar M)$, i.e., $\tilde f=\bar f\circ q$ where $q:\tilde M\to\tilde M/h^{-1}(T)=\bar M$ is the projection. We have $p\circ q=\tilde p$ and $q^*d\bar f=d\tilde f=\tilde p^*\omega'=q^*(p^*\omega')$, which leads to $d\bar f=p^*\omega'$. Define $\bar g:=s_\lambda-\bar f|_{U_\varrho(L)}:U_\varrho(L)\to\mathbb{R}$. A computation shows that for all $x\in U_\varrho(L)$ and $v\in H_1(M;\mathbb{Z})_\mathbb{R}$, we have $\bar g(x+v)=\bar g(x)$ and the existence of $g:U_\varrho(\Gamma)\to\mathbb{R}$ with $\bar g=g\circ p$ follows.

The map g is smooth and we have on $U_\varrho(L)$:

$$p^*dg=d\bar g=\sum_{i=1}^N\lambda(v_i)\,d\bar s_i-d\bar f=p^*\left(\sum_{i=1}^N\lambda(v_i)\,ds_i-\omega'\right).$$

Since p is a surjective local diffeomorphism, the equality $dg=\sum_{i=1}^N\lambda(v_i)\,ds_i-\omega'$ follows.

Define now the smooth 1-form

$$\omega:=g\,d\zeta+(1-\zeta)\omega'+\zeta\sum_{i=1}^N\lambda(v_i)\,ds_i$$

with ζ as in (2). Using the fact that ω' is closed on $U_\varrho(\Gamma)$ and the properties of ζ, one can easily verify that ω is smooth and closed. Furthermore, for $x \in U_\varepsilon(\Gamma_j)$:

$$\omega|_x = g(x)\,d\zeta|_x + \bigl(1-\zeta(x)\bigr)\omega'|_x + \zeta(x)\cdot\sum_{i=1}^N \lambda(v_i)\,ds_i|_x = \lambda(v_j)\,ds_j|_x,$$

as claimed. We get

$$[\omega](v_j) = \int_{\gamma_j}\omega = \lambda(v_j)\int_0^1 ds_j|_{\gamma_j(t)}\bigl(\dot\gamma_j(t)\bigr)\,dt \overset{(1)}{=} \lambda(v_j)$$

for $j = 1,\dots,N$. With $\operatorname{span}\{v_1,\dots,v_N\} = H_1(M;\mathbb{R})$, this yields that ω is a representative for λ. □

In the following, such a representative ω will be called a *good representative* of λ with respect to the family $\{v_1,\dots,v_N\}$.

3. Hedlund metrics

Let P be a centrally symmetric and convex polytope in $H_1(M;\mathbb{R})$ with nonempty interior, such that the directions of its vertices are rational. Such a polytope will be called *admissible*. We call $\tilde V_P = \{\tilde v_1,\dots,\tilde v_N,-\tilde v_1,\dots,-\tilde v_N\}$ the set of vertices of P.

Let $v_1,\dots,v_N$ be indivisible integer classes such that $v_i = \varepsilon_i\tilde v_i$ with $\varepsilon_i > 0$, $i = 1,\dots,N$. Define $V_P := \{v_1,\dots,v_N,-v_1,\dots,-v_N\}$ and let J_i be the subset of V_P consisting of the indivisible integer classes corresponding to the vertices belonging to the ith face S_i of P. In order to simplify the notation, we assume without loss of generality that $J_1 = \{v_1,\dots,v_k\}$ for an integer $k \leqslant N$. The norm $|\cdot|$ on $H_1(M;\mathbb{R})$, whose unit ball is P, is given as follows (for vectors lying in the cone over the face S_1):

$$v = \sum_{j=1}^k \alpha_j\tilde v_j \quad\text{with}\quad \sum_{j=1}^k \alpha_j = 1 \quad\text{and all}\quad \alpha_j \geqslant 0 \Rightarrow |v| = 1 \tag{4}$$

or generally

$$v = \sum_{j=1}^k \alpha_j\tilde v_j \quad\text{with all}\quad \alpha_j \geqslant 0 \Rightarrow |v| = \sum_{j=1}^k \alpha_j$$

and likewise for every other face of P.

Since P is convex, there exists for each face S_i of P an element λ_i of $H^1_{dR}(M) \simeq H^1(M,\mathbb{R})$ such that

$$\lambda_i(\tilde v_j)\begin{cases} = 1, & v_j = \varepsilon_j\tilde v_j \in J_i \\ < 1, & v_j = \varepsilon_j\tilde v_j \notin J_i \end{cases}$$

(i.e. $\lambda_i \equiv 1$ on the plane defined by the face S_i and λ_i is smaller on the rest of the polytope). Now, since P is symmetric, $-\lambda_i$ is the 1-form corresponding to $-S_i$ and we get in fact:

$$-1 < \lambda_i(\tilde v_j) < 1 \quad\text{for } \pm v_j \notin J_i. \tag{5}$$

We get an alternative definition for the norm:

$$v \in \bigoplus_{j=1}^k \mathbb{R}_{\geqslant 0}\cdot v_j \Rightarrow |v| = \lambda_1(v), \tag{6}$$

and likewise for every other face of P.

The metrics defined below will be called *Hedlund metrics* because such a metric first appears in Hedlund's paper [6] in the case $M = \mathbb{T}^3$:

Definition 3.1. Let P be an admissible polytope with vertices $\{\tilde v_1,\dots,\tilde v_N,-\tilde v_1,\dots,-\tilde v_N\}$. Let $v_1,\dots,v_N \in H_1(M,\mathbb{Z})_{\mathbb{R}}$ be the indivisible integer classes such that $\varepsilon_i\tilde v_i = v_i$ for some $\varepsilon_i > 0$, $i = 1,\dots,N$. Choose disjoint admissible curves $\gamma_1,\dots,\gamma_N$ representing the classes $v_1,\dots,v_N$. For each face S_i of P, let η_i be a good representative of λ_i with respect to the family $\{v_1,\dots,v_N\}$. A *Hedlund metric* associated to P on (M,ρ) is a Riemannian metric g that is conformal to ρ and such that its dual metric g^* satisfies:

(H1) $g^*_{\gamma_i(t)}(ds_i|_{\gamma_i(t)}, ds_i|_{\gamma_i(t)}) = \max_{x\in U_\varepsilon(\Gamma_i)} g^*_x(ds_i|_x, ds_i|_x) = \frac{1}{\varepsilon_i^2}$ for all $t \in [0,1]$ and $g^*_x(ds_i|_x, ds_i|_x) < \frac{1}{\varepsilon_i^2}$ for $x \in U_\varepsilon(\Gamma_i)\setminus\Gamma_i$ and all $i \in \{1,\dots,N\}$.

(H2) $g_x^*(\eta_i|_x, \eta_i|_x) \leqslant 1$ for all $i = 1, \ldots, N$ and $x \notin U_\varepsilon(\Gamma)$.

Remark that for orientable compact surfaces of positive genus, it is not possible to choose disjoint loops representing the vertices of the polytope. In fact, it is shown in Bangert [3] that in the case of the 2-torus, the stable norm induced by a Riemannian metric on $\mathbb{T}^2$ has always a strictly convex unit ball. Yet, Massart shows in [10] that this is not true in general: the stable norm induced by a smooth Finsler metric on a closed, orientable surface has neither to be strictly convex, nor smooth. For a *non-orientable* surface, the analogon to Theorem 3.5 can be found in Balacheff and Massart [2]: they show that if M is a closed non-orientable surface equipped with a Riemannian metric, then there exists in every conformal class a metric on M whose stable norm has a polyhedron as its unit ball.

Existence and properties of such a metric.

Proposition 3.2. *On every compact Riemannian manifold (M, ρ) with $\dim M \geqslant 3$ and for every admissible polytope P in $H_1(M, \mathbb{R})$ there exists a Hedlund metric associated to P on (M, ρ).*

Proof. Given the admissible polytope P, choose disjoint admissible curves $\gamma_1, \ldots, \gamma_N$ representing the indivisible integer classes $v_1, \ldots, v_N$ corresponding to its vertices $\tilde{v}_1, \ldots, \tilde{v}_N$. Let $\varepsilon_1, \ldots, \varepsilon_N$ be the coefficients as in Definition 3.1. For each face S_i of P, $i = 1, \ldots, l$, let η_i be a good representative for λ_i. Set

$$\Omega := \max_{\substack{j=1,\ldots,l \\ x \in M \setminus U_\varepsilon(\Gamma)}} \rho_x^*(\eta_j|_x, \eta_j|_x)$$

and

$$\Omega_i := \max\left\{ \max_{\substack{j=1,\ldots,l \\ x \in U_\varrho(\Gamma_i)}} \frac{\rho_x^*(\eta_j|_x, \eta_j|_x)}{\rho_x^*(ds_i|_x, ds_i|_x)}, \varepsilon_i^2 \right\}$$

for $i = 1, \ldots, N$. Define:

$$\begin{aligned} h_i \colon U_\varrho(\Gamma_i) &\to (0, \infty) \\ x &\mapsto \frac{1}{\varepsilon_i^2 \rho_x^*(ds_i|_x, ds_i|_x)} \cdot \exp\bigl(-C_i \cdot \ell(x)^2\bigr) \end{aligned}$$

where $C_i := \ln(\Omega_i / \varepsilon_i^2) \cdot \frac{1}{\varepsilon^2} > 0$ and $\ell(x)$ is the distance from to x to its "projection" $\gamma_i(s_i(x)) \in \Gamma_i$. Define the smooth function $F : M \to (0, \infty)$ by

$$F(x) = \zeta(x) \cdot \sum_{i=1}^{N} h_i(x) + \bigl(1 - \zeta(x)\bigr) \cdot \frac{1}{\Omega},$$

where ζ is a smooth bump function as in (2). It is then easy to verify that the metric g defined by

$$g_x^* = F(x) \rho_x^* \quad \text{for all } x \in M$$

is a Hedlund metric associated to P. □

Proposition 3.3. *It results immediately from Definition 3.1 and from the properties of an admissible polytope that*

$$\|\eta_i\|^* := \max_{x \in M} \|\eta_i|_x\|_x^* = 1 \tag{7}$$

for each face S_i of P.

Proof. Here again, we assume that $i = 1$. The arguments are the same for every other face of P. Outside of $U_\varepsilon(\Gamma)$, Definition 3.1 yields $\|\eta_1|_x\|_x^* \leqslant 1$. With

$$\|\eta_1|_x\|_x^* = \begin{cases} \varepsilon_j \|ds_j|_x\|_x^* = 1, & x \in \Gamma_j \text{ and } j = 1, \ldots, k \\ \varepsilon_j \|ds_j|_x\|_x^* < 1, & x \in U_\varepsilon(\Gamma_j) \setminus \Gamma_j \text{ and } j = 1, \ldots, k \\ |\lambda_1(v_j)| \cdot \|ds_j|_x\|_x^* = \varepsilon_j |\lambda_1(\tilde{v}_j)| \cdot \frac{1}{\varepsilon_j} \overset{(5)}{<} 1, & x \in U_\varepsilon(\Gamma_j) \text{ and } j > k, \end{cases}$$

this proves the statement. □

For the proof of the following lemma, we need to compute the lengths of the chosen admissible curve $\gamma_1, \ldots, \gamma_N$ relative to the new metric. Choose $x = \gamma_i(t) \in \Gamma_i$ and a semi-geodesic chart φ around x. Recall the construction of such a chart; the matrix representing ρ relative to the orthogonal basis $(\dot\gamma_i(t), \partial_2^\varphi|_x, \ldots, \partial_m^\varphi|_x)$ of T_xM is diagonal. Hence, because g is conformal to ρ, the matrix representing g relative to the same basis is diagonal, too. Since the covectors $(ds_i|_x, d\varphi_2|_x, \ldots, d\varphi_m|_x)$ form a dual basis of T_x^*M, we obtain

$$g_x\big(\dot\gamma_i(t), \dot\gamma_i(t)\big) = \frac{1}{g_x^*(ds_i|_x, ds_i|_x)},$$

using the fact that the matrix representing g_x in the basis $(\dot\gamma_i(t), \partial_2^\varphi|_x, \ldots, \partial_m^\varphi|_x)$ is inverse to the matrix representing g_x^* in the dual basis. But because of (H1) in Definition 3.1, we have $g_x^*(ds_i|_x, ds_i|_x) = \frac{1}{\varepsilon_i^2}$. Hence, this leads to:

$$L(\gamma_i) = \int_0^1 \varepsilon_i\, dt = \varepsilon_i. \tag{8}$$

It is possible to show that γ_i is even the shortest curve representing v_i: Assume, without loss of generality, that $v_i \in J_1$ and choose an arbitrary curve $c : [0,1] \to M$ representing v_i. We have $\lambda_1(v_i) = \varepsilon_i$ and hence

$$\varepsilon_i = \int_c \eta_1 = \int_0^1 \eta_1|_{c(t)}\big(\dot c(t)\big)\, dt \leqslant \int_0^1 \|\eta_1|_{c(t)}\|^* \big\|\dot c(t)\big\|\, dt \overset{(7)}{\leqslant} \int_0^1 1 \cdot \big\|\dot c(t)\big\|\, dt = L(c).$$

Lemma 3.4. *There is a constant $C = C(M, P)$ such that for each face S_i of P, every $w \in \bigoplus_{v \in J_i} \mathbb{N} \cdot v$ and every $x \in \bar M$, the distance from x to $x + w$ is bounded above by $\lambda_i(w) + C$.*

Proof. Recall the definitions of γ_i, Γ_i, $\bar\gamma_i$, $\bar\Gamma_i$, $i = 1, \ldots, N$, L and F_0. Let d be the distance induced on M by the Hedlund metric g and define

$$D := \max_{1 \leqslant i,j \leqslant N} \min_{\substack{x \in \Gamma_i \\ y \in \Gamma_j}} d(x, y),$$
$$\operatorname{diam}(M) := \max_{x,y \in M} d(x, y)$$

and choose a real positive number e such that $e > \max_{i=1,\ldots,N} \varepsilon_i$. Set

$$C := 2 \cdot \operatorname{diam}(M) + \kappa \cdot (D + e), \tag{9}$$

where $\kappa = \kappa(P)$ is the maximal number of vertices lying on a common face of P.

Without loss of generality, we assume that $w \in \bigoplus_{v \in J_1} \mathbb{N} \cdot v$, i.e., we can write $w = \sum_{i=1}^k n_i v_i$ with $n_1, \ldots, n_k \in \mathbb{N}$. We give a path from x to $x + w$ that has length bounded above by $\lambda_1(w) + C = \sum_{i=1}^k \varepsilon_i n_i + C$. Assume that $x \in F_0$ (otherwise, if $x \in F_0 + u$ with $u \in H_1(M; \mathbb{Z})_{\mathbb{R}}$, we can replace the path with startpoint $x - u$ as constructed below with its image under Φ_u). We join x with $x + w$ by a path that runs *as much as possible* in L with "changes of lines" that are *as short as possible*:

Choose $i_1 \in \{j \mid 1 \leqslant j \leqslant k,\ n_j \neq 0\}$ such that the point x_1 in $L \cap F_0$ with minimal distance from x lies in $\bar\Gamma_{i_1}$. Let τ_1 be the corresponding geodesic segment from x to x_1 with minimal length. This length $\bar L(\gamma_1)$ is smaller than $\operatorname{diam}(M)$. Let c_1 be the segment of $\bar\gamma_{i_1}$ connecting x_1 and $x_1 + n_{i_1} v_{i_1}$. This segment has length equal to

$$\bar L(c_1) = n_{i_1} \cdot L(\gamma_{i_1}) \overset{(8)}{=} n_{i_1} \cdot \varepsilon_{i_1}.$$

Now choose $i_2 \in \{j \mid 1 \leqslant j \leqslant k,\ n_j \neq 0\} \setminus i_1$ and $x_2 \in \bar\Gamma_{i_2} + n_{i_1} v_{i_1}$ such that x_2 is the point of $(L \setminus \bar\Gamma_{i_1}) \cap (F_0 + n_{i_1} v_{i_1})$ having minimal distance from $\bar\Gamma_{i_1} \cap (F_0 + n_{i_1} v_{i_1})$. Let x_1' be the point in $\bar\Gamma_{i_1} \cap (F_0 + n_{i_1} v_{i_1})$ at this minimal distance from x_2. Let c_1' be the section of $\bar\gamma_{i_1}$ connecting x_1 and x_1'; the length of c_1' lies in $[n_{i_1} \cdot \varepsilon_{i_1} - e, n_{i_1} \cdot \varepsilon_{i_1} + e]$. Let τ_2 be the minimal geodesic segment joining x_1' and x_2, it has length smaller than D. Now continue in this way; choose $i_3 \in \{j \mid 1 \leqslant j \leqslant k,\ n_j \neq 0\} \setminus \{i_1, i_2\}$ and $x_3 \in \bar\Gamma_{i_3} + n_{i_1} v_{i_1} + n_{i_2} v_{i_2}$ such that x_3 is the point of $(L \setminus (\bar\Gamma_{i_1} \cup \bar\Gamma_{i_2})) \cap (F_0 + n_{i_1} v_{i_1} + n_{i_2} v_{i_2})$ having minimal distance from $\bar\Gamma_{i_2} \cap (F_0 + n_{i_1} v_{i_1} + n_{i_2} v_{i_2})$. Let x_2' be the point in $\bar\Gamma_{i_2} \cap (F_0 + n_{i_1} v_{i_1} + n_{i_2} v_{i_2})$ at this minimal distance from x_3. The curve c_2' joining x_2 and x_2' on $\bar\Gamma_{i_2} + n_{i_1} v_{i_1}$ has length smaller than $n_{i_2} \cdot \varepsilon_{i_2} + e$.

If $n_j \neq 0$ for $j = 1, \ldots, k$, our path will be the composition

$$\gamma := \tau_1 * c_1' * \tau_2 * c_2' * \cdots * c_k' * \tau_{i_{k+1}}$$

where τ_{k+1} is the path joining the last point in $L \cap (F_0 + \sum_{i=1}^k n_i v_i)$ with minimal distance from $x + w$ to $x + w$ and has length smaller than $\operatorname{diam}(M)$. Summing all the lengths of those segments we get

$$\begin{aligned}\bar L(\gamma) &\leqslant \operatorname{diam}(M) + n_{i_1} \cdot \varepsilon_{i_1} + e + D + n_{i_2} \cdot \varepsilon_{i_2} + e + D + \cdots + n_{i_k} \cdot \varepsilon_{i_k} + e + \operatorname{diam}(M) \\ &= \lambda_1(w) + k \cdot e + k \cdot D + 2 \cdot \operatorname{diam}(M) \leqslant \lambda_1(w) + C.\end{aligned}$$

Finally, if $n_j = 0$ for some $j \in \{1, \dots, k\}$, we need to make fewer changes of lines, and the inequality can be shown the same way. □

The stable norm and the main theorem. In the introduction of this paper, we gave the definition of the stable norm induced on $H_1(M;\mathbb{R})$ by a Riemannian metric g on M. We present here a way to compute the stable norm of a vector lying in $H_1(M;\mathbb{Z})_{\mathbb{R}}$: Define

$$f\colon H_1(M;\mathbb{Z})_{\mathbb{R}} \to \mathbb{R}_{\geqslant 0}$$
$$v \mapsto \inf\bigl\{L(\gamma) \mid \gamma \text{ closed curve representing } v\bigr\}$$

and $f_n : n^{-1}H_1(M;\mathbb{Z})_{\mathbb{R}} \to \mathbb{R}_{\geqslant 0}$, $f_n(v) = n^{-1}f(nv)$. In Bangert [3] it is shown that f_n converges uniformly on compact sets to the stable norm $\|\cdot\|_s$. Especially, we have: if $(v_n)_{n\in\mathbb{N}}$ is a sequence in $H_1(M;\mathbb{Z})_{\mathbb{R}}$ with $\lim_{n\to\infty}\frac{v_n}{n} = v \in H_1(M;\mathbb{R})$ (relative to the standard topology on the vector space $H_1(M;\mathbb{R}) \simeq \mathbb{R}^b$), then we have for the norm of v:

$$\|v\|_s = \lim_{n\to\infty}\frac{f(v_n)}{n}.$$

If $\bar{d}$ is the distance on $\bar{M}$ induced from p^*g, we have for $v \in H_1(M;\mathbb{Z})_{\mathbb{R}}$:

$$f(v) = \inf_{x\in\bar{M}} \bar{d}(x, x+v) = \min_{x\in F_0}\bar{d}(x, x+v)$$

because p^*g is a periodic metric and the closure of F_0 is a compact set. With $\lim_{n\to\infty}\frac{nv}{n} = v$, this yields:

$$\|v\|_s = \lim_{n\to\infty}\frac{f(nv)}{n}$$
$$= \lim_{n\to\infty}\frac{\min_{x\in F_0}\bar{d}(x, x+nv)}{n}.$$

Theorem 3.5. *The polytope P is the unit ball of the stable norm on $H_1(M;\mathbb{R})$ induced by an arbitrary Hedlund metric associated to P on M.*

Note that by Definition 3.1, the Hedlund metric is chosen in the conformal class of the given Riemannian metric ρ on M.

Proof. Let g be a Hedlund-metric associated to P. We show that for each $w \in \bigoplus_{j=1}^{k}\mathbb{N}\cdot v_j$, the stable norm of w is given by $\|w\|_s = \lambda_1(w)$. The proof of this works analogously for each other face of P. Consequently, this holds for all vectors in $H_1(M;\mathbb{R})$ that can be written as linear combinations of the vectors $v_1, \dots, v_N$ with rational coefficients, and then, by continuity, this holds for all vectors in $H_1(M;\mathbb{R})$. Let x be an arbitrary point in F_0 and let $n \in \mathbb{N}$. Let $\gamma : [0,1] \to \bar{M}$ be an arbitrary path from x to $x + nw$. We have

$$\lambda_1(nw) = \int_\gamma \eta_1 = \int_0^1 \eta_1|_{\gamma(t)}\bigl(\dot\gamma(t)\bigr)\,dt \leqslant \int_0^1 \|\eta_1|_{\gamma(t)}\|^*\bigl\|\dot\gamma(t)\bigr\|\,dt \overset{(7)}{\leqslant} \int_0^1 1\cdot\bigl\|\dot\gamma(t)\bigr\|\,dt = \bar{L}(\gamma).$$

With this and Lemma 3.4 we get

$$\lambda_1(n\cdot w) \leqslant \bar{d}(x, x+nw) \leqslant \lambda_1(n\cdot w) + C.$$

Thus

$$\lambda_1(n\cdot w) \leqslant \min_{x\in F_0}\bar{d}(x, x+nw) \leqslant \lambda_1(n\cdot w) + C,$$

and

$$\lambda_1(w) \leqslant \frac{\min_{x\in F_0}\bar{d}(x, x+nw)}{n} \leqslant \lambda_1(w) + \frac{C}{n}.$$

Letting n go to infinity, this yields $\|w\|_s = \lambda_1(w)$, as claimed. □

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