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Five-coloring graphs on the Klein bottle

Nathan Chenette [1], Luke Postle [1,3], Noah Streib [1], Robin Thomas [2], Carl Yerger [3]

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA

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ABSTRACT

We exhibit an explicit list of nine graphs such that a graph drawn in the Klein bottle is 5-colorable if and only if it has no subgraph isomorphic to a member of the list.

1. Introduction

All graphs in this paper are finite, undirected and simple. By a *surface* we mean a compact, connected 2-dimensional manifold with empty boundary. The classification theorem of surfaces (see e.g. [16]) states that each surface is homeomorphic to either S_g, the surface obtained from the sphere by adding g handles, or N_k, the surface obtained from the sphere by adding k cross-caps. Thus $S_0 = N_0$ is the sphere, S_1 is the torus, N_1 is the projective plane and N_2 is the Klein bottle.

In this paper we study a specific instance of the following more general question: Given a surface Σ and an integer $t \geqslant 0$, which graphs drawn in Σ are t-colorable?

Heawood [11] proved that if Σ is not the sphere, then every graph in Σ is t-colorable as long as $t \geqslant H(\Sigma) := \lfloor (7 + \sqrt{24\gamma + 1})/2 \rfloor$, where γ is the *Euler genus of* Σ, defined as $\gamma = 2g$ when $\Sigma = S_g$ and $\gamma = k$ when $\Sigma = N_k$. Incidentally, the assertion holds for the sphere as well, by the Four-Color Theorem [2,4,3,21]. Ringel and Youngs (see [20]) proved that the bound is best possible for all surfaces except the Klein bottle. Dirac [5] and Albertson and Hutchinson [1] improved Heawood's result by

E-mail address: thomas@math.gatech.edu (R. Thomas).

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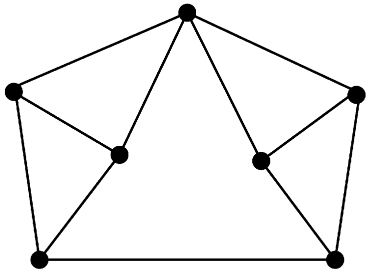

Fig. 1. The graph H_7.

showing that every graph in Σ is actually $(H(\Sigma)-1)$-colorable, unless it has a subgraph isomorphic to the complete graph on $H(\Sigma)$ vertices.

We say that a graph is $(t+1)$-*critical* if it is not t-colorable, but every proper subgraph is. Dirac [6] also proved that for every $t \geqslant 8$ and every surface Σ there are only finitely many t-critical graphs on Σ. Using a result of Gallai [9] it is easy to extend this to $t=7$. In fact, the result extends to $t=6$ by the following deep theorem of Thomassen [26].

Theorem 1.1. *For every surface Σ there are only finitely many* 6-*critical graphs in Σ.*

Thus for every $t \geqslant 5$ and every surface Σ there exists a polynomial-time algorithm to test whether a graph in Σ is t-colorable. What about $t=3$ and $t=4$? For $t=3$ the t-coloring decision problem is NP-hard even when Σ is the sphere [10], and therefore we do not expect to be able to say much. By the Four-Color Theorem the 4-coloring decision problem is trivial when Σ is the sphere, but it is open for all other surfaces. A result of Fisk [8] can be used to construct infinitely many 5-critical graphs on any surface other than the sphere, but the structure of 5-critical graphs on surfaces appears complicated [19, Section 8.4].

Thus the most interesting value of t for the t-colorability problem on a fixed surface seems to be $t=5$. By the Four-Color Theorem every graph in the sphere is 4-colorable, but on every other surface there are graphs that cannot be 5-colored. Albertson and Hutchinson [1] proved that a graph in the projective plane is 5-colorable if and only if it has no subgraph isomorphic to K_6, the complete graph on six vertices. Thomassen [24] proved the analogous (and much harder) result for the torus, as follows. If K, L are graphs, then by $K+L$ we denote the graph obtained from the union of a copy of K with a disjoint copy of L by adding all edges between K and L. The graph H_7 is depicted in Fig. 1 and the graph T_{11} is obtained from a cycle of length 11 by adding edges joining all pairs of vertices at distance two or three.

Theorem 1.2. *A graph in the torus is* 5-*colorable if and only if it has no subgraph isomorphic to K_6, C_3+C_5, K_2+H_7, or T_{11}.*

Our objective is to prove the analogous result for the Klein bottle, stated in the following theorem. The graphs $L_1, L_2, \dots, L_6$ are defined in Fig. 2. Lemma 4.2 explains how most of these graphs arise in the proof.

Theorem 1.3. *A graph in the Klein bottle is* 5-*colorable if and only if it has no subgraph isomorphic to K_6, C_3+C_5, K_2+H_7, or any of the graphs $L_1, L_2, \dots, L_6$.*

Theorem 1.3 settles a problem of Thomassen [26, Problem 3]. It also implies that in order to test 5-colorability of a graph G drawn in the Klein bottle it suffices to test subgraph isomorphism to one of the graphs listed in Theorem 1.3. Using the algorithms of [7] and [17] we obtain the following corollary.

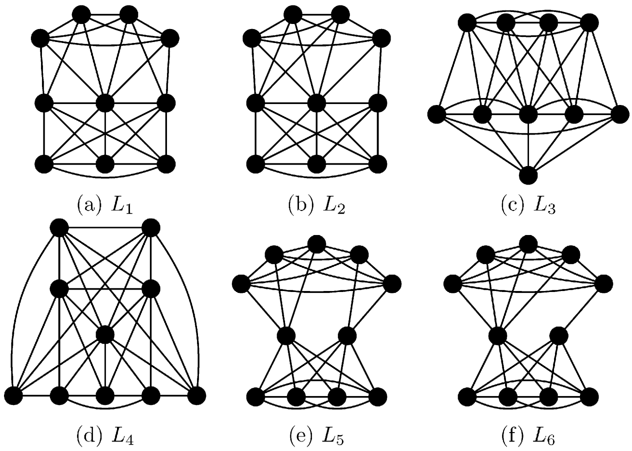

Fig. 2. The graphs $L_1, L_2, \ldots, L_6$.

Corollary 1.4. *There exists an explicit linear-time algorithm to decide whether an input graph embeddable in the Klein bottle is* 5*-colorable.*

It is not hard to see that with the sole exception of K_6, none of the graphs listed in Theorem 1.3 can be a subgraph of an Eulerian triangulation of the Klein bottle. Thus we deduce the following theorem of Král', Mohar, Nakamoto, Pangrác and Suzuki [14].

Corollary 1.5. *An Eulerian triangulation of the Klein bottle is* 5*-colorable if and only if it has no subgraph isomorphic to* K_6.

It follows by inspection that each of the graphs from Theorem 1.3 has a subgraph isomorphic to a subdivision of K_6. Thus we deduce the following corollary.

Corollary 1.6. *If a graph in the Klein bottle is not* 5*-colorable, then it has a subgraph isomorphic to a subdivision of* K_6.

This is related to Hajós' conjecture, which states that for every integer $k \geqslant 1$, if a graph G is not k-colorable, then it has a subgraph isomorphic to a subdivision K_{k+1}. Hajós' conjecture is known to be true for $k = 1, 2, 3$ and false for all $k \geqslant 6$. The cases $k = 4$ and $k = 5$ remain open. In [27, Conjecture 6.3] Thomassen conjectured that Hajós' conjecture holds for every graph in the projective plane or the torus. His results [24] imply that it suffices to prove this conjecture for $k = 4$, but that is still open. Likewise, one might be tempted to extend Thomassen's conjecture to graphs in the Klein bottle; Corollary 1.6 then implies that it would suffice to prove this extended conjecture for $k = 4$.

Thomassen proposed yet another related conjecture [27, Conjecture 6.2] stating that every graph which triangulates some surface satisfies Hajós' conjecture. He also pointed out that this holds for $k \leqslant 4$ for every surface by a deep theorem of Mader [15], and that it holds for the projective plane and the torus by [24]. Thus Corollary 1.6 implies that Thomassen's second conjecture holds for graphs in the Klein bottle. For general surfaces the conjecture was disproved by Mohar [18]. Qualitatively stronger counterexamples were found by Rödl and Zich [22].

Our proof of Theorem 1.3 follows closely the argument of [24], and therefore we assume familiarity with that paper. We proceed as follows. The result of Sasanuma [23] that every 6-regular graph in the Klein bottle is 5-colorable (which follows from the description of all 6-regular graphs on the Klein bottle) allows us to select a minimal counterexample G_0 and a suitable vertex $v_0 \in V(G_0)$ of degree five. If every two neighbors of v_0 are adjacent, then G_0 has a K_6 subgraph and the result holds. We may therefore select two non-adjacent neighbors x and y of v_0. Let G_{xy} be the graph obtained from G_0 by deleting v_0, identifying x and y and deleting all resulting parallel edges. If G_{xy} is 5-colorable, then so is G_0, as is easily seen. Thus we may assume that G_{xy} has a subgraph isomorphic to one of the nine graphs on our list, and it remains to show that either G_0 can be 5-colored, or it has a subgraph isomorphic to one of the nine graphs on the list. That occupies most of the paper.

We would like to acknowledge that Theorem 1.3 was independently obtained by Kawarabayashi, Král', Kynčl, and Lidický [12]. Their method relies on a computer search. The result of this paper forms part of the doctoral dissertation [29] of the last author.

2. Lemmas

Our first lemma is an adaptation of [24, Theorem 6.1, Claim (8)].

Lemma 2.1. *Let G be a graph in the Klein bottle that is not 5-colorable and has no subgraph isomorphic to K_6, $C_3 + C_5$, or $K_2 + H_7$. Then G has at least 10 vertices, and if it has exactly 10, then it has a vertex of degree nine.*

Proof. We follow the argument of [24, Theorem 6.1, Claim (8)]. Let G be as stated, and let it have at most ten vertices. We may assume, by replacing G by a suitable subgraph, that G is 6-critical. By a result of Gallai [9] it follows that G is of the form $H_1 + H_2$, where H_i is k_i-critical, $k_1 \leqslant k_2$, and $k_1 + k_2 = 6$. If $k_1 = k_2 = 3$, then we obtain that G is isomorphic to either K_6 or $C_3 + C_5$, a contradiction. So $k_1 \leqslant 2$ and therefore G has a vertex adjacent to all other vertices. Now, suppose for purposes of contradiction that $|V(G)| \leqslant 9$. If $k_1 = 1$, then $|V(H_2)| \leqslant 8$ and so H_2 is of the form $H_2' + H_2''$, where $H_2' = K_2$ or K_1. Thus we may assume that $k_1 = 2$ and that H_2 is 4-critical. By the results of [9] and [28], the only 4-critical graphs with at most seven vertices are K_4, $K_1 + C_5$, H_7 and M_7, where M_7 is obtained from a 6-cycle, $x_1x_2 \dots x_6x_1$ by adding an additional vertex v and edges $x_1x_3, x_3x_5, x_5x_1, vx_2, vx_4, vx_6$. However, G has no subgraph isomorphic to $K_2 + K_4 = K_6$, $K_2 + (K_1 + C_5) = C_3 + C_5$, or $K_2 + H_7$. This implies that G is isomorphic to $K_2 + M_7$. The latter graph has nine vertices and 27 edges, and so triangulates the Klein bottle. However, $K_2 + M_7$ has a vertex whose neighborhood is not Hamiltonian, a contradiction. □

Our next lemma is an extension of [24, Lemma 4.1], which proves the same result for cycles of length at most six. If C is a subgraph of a graph G and c is a coloring of C, then we say that a vertex $v \in V(G) - V(C)$ *sees a color α on C* if v has a neighbor $u \in V(C)$ such that $c(u) = \alpha$.

Lemma 2.2. *Let G be a plane graph with an outer cycle C of length $k \leqslant 7$, and let c be a 5-coloring of $G[V(C)]$. Then c cannot be extended to a 5-coloring of G if and only if $k \geqslant 5$ and the vertices of C can be numbered $x_1, x_2, \dots, x_k$ in order such that one of the following conditions hold:*

- (i) *some vertex of $G - V(C)$ sees five distinct colors on C,*
- (ii) *$G - V(C)$ has two adjacent vertices that both see the same four colors on C,*
- (iii) *$G - V(C)$ has three pairwise adjacent vertices that each see the same three colors on C,*
- (iv) *G has a subgraph isomorphic to the first graph shown in Fig. 3, and the only pairs of vertices of C colored the same are either $\{x_5, x_2\}$ or $\{x_5, x_3\}$, and either $\{x_4, x_6\}$ or $\{x_4, x_7\}$,*
- (v) *G has a subgraph isomorphic to the second graph shown in Fig. 3, and the only pairs of vertices of C colored the same are exactly $\{x_2, x_6\}$ and $\{x_3, x_7\}$,*
- (vi) *G has a subgraph isomorphic to the third graph shown in Fig. 3, and the only pairs of vertices of C colored the same are exactly $\{x_2, x_6\}$ and $\{x_3, x_7\}$.*

Proof. Clearly, if one of (i)–(vi) holds, then c cannot be extended to a 5-coloring of G. To prove the converse we will show, by induction on $|V(G)|$, that if none of (i)–(vi) holds, then c can be extended to a 5-coloring of G. Since c extends if $|V(G)| \leqslant 4$, we assume that $|V(G)| \geqslant 5$, and that the lemma holds for all graphs on fewer vertices. We may also assume that $V(G) \neq V(C)$, and that every vertex

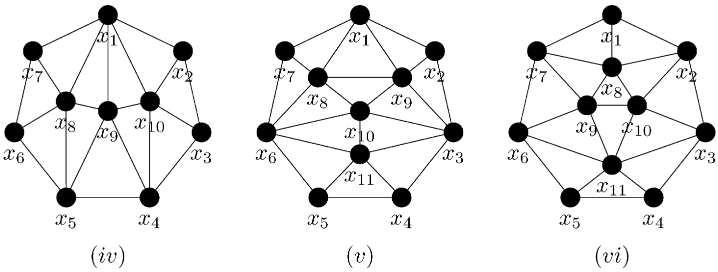

Fig. 3. Graphs that have non-extendable colorings.

of $G - V(C)$ has degree at least five, for we can delete a vertex of $G - V(C)$ of degree at most four and proceed by induction. Likewise, we may assume that

($*$) *the graph G has no cycle of length at most four whose removal disconnects G.*

This is because if a cycle C' of length at most four separates G, then we first delete all vertices and edges drawn in the open disk bounded by C' and extend c to that graph by induction. Then, by another application of the induction hypothesis we extend the resulting coloring of C' to a coloring of the entire graph G. Thus we may assume ($*$).

Let v be a vertex of $G - V(C)$ joined to m vertices of C, where m is as large as possible. Then we may assume that $m \geqslant 3$, for otherwise c extends to a 5-coloring of G by the theorem of [25].

Since (i) does not hold, the coloring c extends to a 5-coloring c' of the graph $G' := G[V(C) \cup \{v\}]$. Let D be a facial cycle of G' other than C, and let H be the subgraph of G consisting of D and all vertices and edges drawn in the disk bounded by D. If c' extends to H for every choice of D, then c extends to G, and the lemma holds. We may therefore assume that D was chosen so that c' does not extend to H. By the induction hypothesis H and D satisfy one of (i)–(vi).

If H and D satisfy (i), then there is a vertex $w \in V(H) - V(D)$ that sees five distinct colors on D. Thus w has at least four neighbors on C, and hence $m \geqslant 4$. It follows that every bounded face of the graph $G[V(C) \cup \{v, w\}]$ has size at most four, and hence $V(G) = V(C) \cup \{v, w\}$ by ($*$). Since (i) and (ii) do not hold for G, we deduce that c can be extended to a 5-coloring of G, as desired.

If H and D satisfy (ii), then there are adjacent vertices $v_1, v_2 \in V(H) - V(D)$ that see the same four colors on D. It follows that $m \geqslant 3$, and similarly as in the previous paragraph we deduce that $V(G) = V(C) \cup \{v, v_1, v_2\}$. It follows that c can be extended to a 5-coloring of G: if both v_1 and v_2 are adjacent to v we use that G does not satisfy (i), (ii), or (iii); otherwise we use that G does not satisfy (i), (ii), or (iv).

If H and D satisfy (iii), then there are three pairwise adjacent vertices of $v_1, v_2, v_3 \in V(H) - V(D)$ that see the same three colors on D. It follows in the same way as above that $V(G) = V(C) \cup \{v, v_1, v_2, v_3\}$. If v sees at most three colors on C, then c extends to a 5-coloring of G, because there are at least two choices for $c'(v)$. Thus we may assume that v sees at least four colors. It follows that $m = 4$, because $k \leqslant 7$. Since G does not satisfy (v) or (vi) we deduce that c extends to a 5-coloring of G.

If H and D satisfy (iv), then there are three vertices of $V(H) - V(D)$ forming the first subgraph in Fig. 3. But at least one of these vertices has four neighbors on C, and hence $m \geqslant 4$, contrary to $k \leqslant 7$.

Finally, if H and D satisfy (v) or (vi), then H has a subgraph isomorphic to the second or third graph depicted in Fig. 3, and the restriction of c' to D is uniquely determined (up to a permutation of colors). Since D has length seven, it follows that $m \leqslant 3$, and hence $c'(v)$ can be changed to a different value, contrary to the fact that the restriction of c' to D is uniquely determined. □

The following lemma is shown in [23].

Lemma 2.3. *All 6-regular graphs embeddable on the Klein bottle are 5-colorable.*

The next lemma is an adaptation of [24, Lemma 5.2] for the Klein bottle.

Lemma 2.4. *Let G be isomorphic to $C_3 + C_5$, let S be a cycle in G of length three with vertex-set $\{z_0, z_1, z_2\}$, and let u_1 be a vertex in $G \backslash V(S)$ adjacent to z_0. Let G' be obtained from G by splitting z_0 into two non-adjacent vertices x and y such that u_1 and at most one more vertex u_0 in G' is adjacent to both x and y and such that yz_1z_2x is a path in G'. Let G'' be obtained from G' by adding a vertex v_0 and joining v_0 to x, y, u_1, z_1, z_2. If G'' is not 5-colorable and can be drawn in the Klein bottle, then either $G' \backslash x$ or $G' \backslash y$ has a subgraph isomorphic to $C_3 + C_5$ or G'' is isomorphic to L_4.*

Proof. We follow the argument of [24, Lemma 5.2]. If one of x, y has the same neighbors in G' as z_0 does in G, say x, then $G' \backslash y$ has a subgraph isomorphic to $C_3 + C_5$, as desired. Thus we can assume that z_0 has two neighbors in G such that one is a neighbor in G' of x but not y and the other is a neighbor in G' of y but not x.

The vertices x, y have degree at least five in G'', for if say y had degree at most four, then $G'' \backslash y \backslash v_0$ would not be 5-colorable (because G'' is not), and yet it is a proper subgraph of $C_3 + C_5$, a contradiction. It follows that z_0 has degree at least six in G. Let G consist of a 5-cycle $p_1p_2p_3p_4p_5p_1$ and a 3-cycle $q_1q_2q_3q_1$ and the 15 edges p_iq_j where $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 5$. Since the degree of z_0 in G is at least 6, we have $z_0 \in \{q_1, q_2, q_3\}$. The remainder of the proof is an analysis based on which vertices are z_0, z_1, z_2.

First suppose that z_0, z_1, z_2 are q_3, q_1, q_2, respectively. If both u_0 and u_1 are in $\{p_1, p_2, p_3, p_4, p_5\}$, then we can color y, z_1, z_2, x with $2, 1, 2, 1$, respectively. We can color the remaining vertices with colors $3, 4, 5$ as the remaining vertices are v_0 and a 5-cycle, and in this case v_0 is only adjacent to one of the vertices of the 5-cycle. If $u_1 = p_1$ and $u_0 = z_1$, then we color y, z_1, z_2, x, u_1 by $2, 1, 2, 3, 4$, respectively. Since some neighbor of z_0 in G is not a neighbor of y in G'', some vertex in $\{p_2, p_3, p_4, p_5\}$ can obtain color 3 and the remaining vertices may be colored with colors 4 and 5.

Now consider the case where z_0, z_1, z_2 are q_1, p_1, p_2, respectively and u_0 is not in $\{z_1, z_2\}$. Color y, z_1, z_2, x, u_0, u_1 by $2, 1, 2, 1, 3, 4$, respectively. We can extend this to a 5-coloring of G'', coloring v_0 last, except (up to symmetry) in the following three cases. If $u_0 = q_2$ and $u_1 = p_4$, color q_3 by the same color as x or y and recolor either z_1 or z_2 by 4 and color the remaining vertices color 5. If $u_0 = p_3$ and $u_1 = p_4$, then color q_3 by color 1 or 2 and recolor z_1 or z_2 color 4. Then we can color p_5, q_2 with colors 3 and 5 respectively. If $u_1 = p_3$ and $u_0 = p_5$, color q_3 by 1 or 2 and recolor one of z_1, z_2 by 3 and recolor p_3, p_4, p_5, q_2 by $4, 3, 4, 5$, respectively.

Now suppose that z_0, z_1, z_2 are q_1, p_1, p_2, respectively and u_0 is in $\{p_1, p_2\}$. Without loss of generality let $u_0 = p_2$. Suppose that $u_1 \in \{p_3, p_4, p_5\}$. Then we can color y, z_1, z_2, x by $2, 4, 3, 1$ and can color u_1 by 3, except when $u_1 = p_3$, in which case we color u_1 by 4. Next, color one of q_2, q_3 color 1 or 2. If both q_2, q_3 can be colored 1, 2 then the rest of the coloring follows. So we can assume that q_2, q_3 are colored by 2, 5, respectively and both q_2, q_3 are adjacent to x. (The argument is analogous if q_2, q_3 are both adjacent to y.) Since y has degree at least four in G', at least one vertex in $\{p_3, p_4, p_5\} \backslash \{u_1\}$ is joined to y and is colored 1. With possibly a swapping of the colors of z_1 and z_2, we can now complete the 5-coloring.

Suppose that z_0, z_1, z_2 are q_1, p_1, p_2, respectively and $u_0 = p_2$ and $u_1 = q_2$. Color y, z_1, z_2, x, q_2 colors $2, 1, 3, 1, 4$, respectively. If q_3 can be colored 2, then color p_3, p_4, p_5, v_0 colors $5, 3, 5, 5$, respectively. So we may assume that q_3 is adjacent to y. Then color q_3 by 5. If we can color $\{p_3, p_4, p_5\}$ by colors $\{1, 2, 3\}$, then color v_0 with 5. If not, then p_3, p_4 are adjacent to the same vertex in $\{x, y\}$. Since x has degree at least four in G' and only u_0, u_1 are adjacent to both x, y, that vertex must be x. We may assume that p_5 is adjacent to y since otherwise we color p_3, p_4, p_5 by $2, 3, 2$, respectively. It follows that G'' is isomorphic to L_4 by an isomorphism that maps the vertices $z_1, y, q_3, z_2, q_2, p_5, p_4, p_3, x, v_0$ to the vertices of L_4 in order, where the vertices of L_4 are numbered by reference to Fig. 2, starting at top left and moving horizontally to the right one row at a time.

Now, consider the case when z_0, z_1, z_2 are q_1, q_2, p_1, respectively. If $u_0 \notin \{z_1, z_2\}$, then color $y, z_1, z_2, x, p_2, p_3, p_4, p_5, q_3$ by $2, 1, 2, 1, 3, 4, 3, 4, 5$, respectively. If $u_0 = p_1$, color y, z_1, z_2, x by $2, 1, 3, 1$, respectively. If q_3 is not adjacent to y, then color q_3 by 2 and the vertices p_2, p_3, p_4, p_5 colors 4 and 5. If q_3 is adjacent to y, color q_3 by 5. Since x has degree at least four in G', some vertex in $\{p_2, \ldots, p_5\}$ can be colored 2. The other vertices in this set could then be colored with colors 3 and 4. Thus assume that $u_0 = q_2 = z_1$. Color y, z_1, z_2, x, u_1 by $2, 3, 2, 1, 4$ and we now will try and extend this coloring. If q_3 can be colored 1, then color p_2, p_3, p_4, p_5 by colors 4 and 5. So we assume that q_3 is adjacent to x. If $u_1 = p_3$, then recolor z_2 by color 4 and color q_3 by 2. Since y also has degree at least four in G', it must be adjacent to at least one of p_4, p_5, which we color 1. The remaining vertices of $\{p_1, \ldots, p_5\}$ are colored 5. If $u_1 = q_3$, then we color one of p_2 or p_5 color 1 if possible and complete the coloring by using 5 for two vertices in $\{p_2, p_3, p_4, p_5\}$. Now assume that both p_2 and p_5 are joined to x. Since y has degree at least four in G' it follows that y is adjacent to p_3 and p_4. We now claim that G'' is not embeddable on the Klein bottle. Notice that if an embedding of this graph exists, it must be that it is a triangulation as it has 10 vertices and 30 edges. Consider the induced embeddings of $G''\backslash p_2$, $G''\backslash p_5$ and $G''\backslash v_0$, respectively. The face of $G''\backslash p_2$ containing p_2 is bounded by a Hamiltonian cycle of $N_{G''}(p_2)$. There exist similarly constructed Hamiltonian cycles in $N_{G''}(p_5)$ and $N_{G''}(v_0)$. However, each of these cycles contains the edge xp_1. This would mean that xp_1 is part of three facial triangles, a contradiction.

Finally, consider the subcase where z_0, z_1, z_2, u_0, u_1 are q_1, q_2, p_1, q_2, p_2, respectively. Color y, z_1, z_2, x, u_1, q_3 by $2, 3, 2, 1, 4, 5$, respectively. We may assume that q_3 is adjacent to x else we can recolor q_3 by 1 and complete the coloring. Also, we can assume that p_5 is adjacent to x else we color p_5, p_4, by $1, 4$ and complete the coloring. Color p_5 by 4. The coloring can be completed unless p_3 and p_4 are both adjacent to the same vertex in $\{x, y\}$. Since y must have degree at least four in G', it follows that p_3 and p_4 are adjacent to y. It follows that G'' has a subgraph isomorphic to L_4 by an isomorphism that maps x, $z_2 = p_1$, q_3, $p_2 = u_1$, $q_2 = z_1 = u_0$, p_5, p_4, p_3, y, v_0 to the vertices of L_4 in order, using the same numbering of the vertices of L_4 as above. Thus G'' is isomorphic to L_4. $\square$

We also need a minor variation of the previous lemma, a case not treated in [24].

Lemma 2.5. *Let G be isomorphic to $C_3 + C_5$, let S be a cycle in G of length three with vertex-set $\{z_0, z_1, z_2\}$, and let u_1 be a vertex in $G\backslash V(S)$ adjacent to z_0. Let G' be obtained from G by adding an edge between two non-adjacent vertices neither of which is z_0, and then splitting z_0 into two non-adjacent vertices x and y such that u_1 is the only vertex in G' that is adjacent to both x and y and such that yz_1z_2x is a path in G'. Let G'' be obtained from G' by adding a vertex v_0 and joining v_0 to x, y, u_1, z_1, z_2. If G'' is not 5-colorable and can be drawn in the Klein bottle, then either $G'\backslash x$ or $G'\backslash y$ has a subgraph isomorphic to either $C_3 + C_5$ or K_6.*

Proof. If one of x, y has the same neighbors in G' as z_0 does in G, say x, then $G'\backslash y$ has a subgraph isomorphic to $C_3 + C_5$, as desired. Thus we can assume that z_0 has two neighbors in G such that one is a neighbor in G' of x but not y and the other is a neighbor in G' of y but not x. We may assume that the vertices x, y have degree at least five in G'', for if say y had degree at most four, then $G''\backslash y\backslash v_0 = G'\backslash y$ would not be 5-colorable (because G'' is not), yet this is a proper subgraph of $C_3 + C_5$ plus an additional edge, and hence by Lemma 2.1 must contain either $C_3 + C_5$ or K_6 as a subgraph, as desired. Moreover, the sum of the degrees of x and y in G'' is at most 10 since z_0 has degree at most seven in G. Thus, z_0 must have degree seven in G while x and y must have degree five in G''.

Let G consist of a 5-cycle $p_1p_2p_3p_4p_5p_1$ and a 3-cycle $q_1q_2q_3q_1$ and the 15 edges p_iq_j where $1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 5$. Since the degree of z_0 in G is seven, we have $z_0 \in \{q_1, q_2, q_3\}$. Without loss of generality, let $z_0 = q_1$. Moreover, in G', there is an edge between two of the p's that are not adjacent in G. Without loss of generality, suppose that this edge is p_1p_3.

As u_1 is the only vertex in G' adjacent to both x and y, we have that x and z_1 are not adjacent. Consider the graph G_{xz_1} obtained from G'' by deleting v_0, identifying x and z_1 into a new vertex w, and deleting parallel edges. Now G_{xz_1} must not be 5-colorable, as otherwise we could color G''. Now G_{xz_1} must contain a 6-critical subgraph H. As y has degree at most four in G_{zx_1}, y is not in H. Thus $|V(H)| \leqslant 7$. By Lemma 2.1 we find that H is isomorphic to K_6. The vertex w must be in H as otherwise $G\backslash x$ would contain K_6 as a proper subgraph, a contradiction. The remaining five vertices of H induce a K_5. So these vertices must be q_2, q_3, p_1, p_2, p_3. Hence z_1 must be one of p_4 or p_5.

A similar argument shows that y and z_2 are not adjacent and that the analogously defined graph G_{yz_2} must contain a subgraph H' isomorphic to K_6 with vertices q_2, q_3, p_1, p_2, p_3 and the new vertex of G_{yz_2}. Hence z_2 must be one of p_4 or p_5. Without loss of generality, suppose that $z_1 = p_4$ and $z_2 = p_5$. As there are edges between w and p_1, p_2, the edges xp_1 and xp_2 must be present in G'. Similarly, the edges yp_3 and yp_2 must be in G'. Hence $u_1 = p_2$. Finally, as x and y have degree four in G' and exactly one of $p_4 = z_1$ and $p_5 = z_2$ is adjacent to x and exactly one is adjacent to y, we may assume without loss of generality that x is adjacent to q_2 and y is adjacent to q_3.

It is straightforward to color G''. Color q_2 and y with color 5; color q_3 and x with color 4. Color p_2 and p_4 with color 1. Color p_3 and p_5 with color 3. Color p_1 and v_0 with color 2. This 5-coloring of G'' contradicts the hypothesis of the lemma. □

We also need an adaptation of [24, Lemma 5.3] for the Klein bottle. We leave the similar proof to the reader.

Lemma 2.6. *Let G be isomorphic to $K_2 + H_7$, let S be a cycle in G of length three with vertex-set $\{z_0, z_1, z_2\}$, and let u_1 be a vertex in $G\backslash V(S)$ adjacent to z_0. Let G' be obtained from G by splitting z_0 into two non-adjacent vertices x and y such that u_1 and at most one more vertex u_0 in G' is joined to both x and y and such that yz_1z_2x is a path in G'. Let G'' be obtained from G' by adding a vertex v_0 and joining v_0 to x, y, u_1, z_1, z_2. If G'' is not 5-colorable and can be drawn in the Klein bottle, then $G'\backslash x$ or $G'\backslash y$ has a subgraph isomorphic to $K_2 + H_7$.*

We also need a similar variation of the previous lemma to handle a case not treated in [24].

Lemma 2.7. *Let G be isomorphic to $K_2 + H_7$, let S be a cycle in G of length three with vertex-set $\{z_0, z_1, z_2\}$, and let u_1 be a vertex in $G\backslash V(S)$ adjacent to z_0. Let G' be obtained from G by adding an edge between two non-adjacent vertices neither of which is z_0, and then splitting z_0 into two non-adjacent vertices x and y such that u_1 is the only vertex in G' that is adjacent to both x and y and such that yz_1z_2x is a path in G'. Let G'' be obtained from G' by adding a vertex v_0 and joining v_0 to x, y, u_1, z_1, z_2. If G'' is not 5-colorable and can be drawn in the Klein bottle, then either $G'\backslash x$ or $G'\backslash y$ has a subgraph isomorphic to either $K_2 + H_7$ or K_6.*

Proof. If one of x, y has the same neighbors in G' as z_0 does in G, say x, then $G'\backslash y$ has a subgraph isomorphic to $K_2 + H_7$, as desired. Thus we can assume that z_0 has two neighbors in G such that one is a neighbor in G' of x but not y and the other is a neighbor in G' of y but not x. The vertices x, y have degree at least five in G'', for if say y had degree at most four, then $G''\backslash y\backslash v_0 = G'\backslash y$ would not be 5-colorable (because G'' is not), and yet this is a proper subgraph of $K_2 + H_7$ plus an additional edge and by Lemma 2.1 must contain $K_2 + H_7$, $C_3 + C_5$ or K_6 as a subgraph, a contradiction.

Hence x and y have degree at least four in G' and so z_0 has degree at least seven in G. Moreover, the sum of the degrees of x and y is at most 9 since z_0 has degree at most eight in G. Thus, z_0 must have degree eight in G. Without loss of generality we may assume that x has degree five and y has degree four in G'.

We label $K_2 + H_7$ as follows. The two degree eight vertices are q_1, q_2. The degree six vertex is p_1. The degree fives are $p_2, p_3, p_4, p_5, p_6, p_7$, where p_2, p_3, p_4 and p_5, p_6, p_7 are triangles, p_4p_5 is an edge, and p_2, p_3, p_6, p_7 are adjacent to p_1. Since the degree of z_0 in G is eight, we have $z_0 \in \{q_1, q_2\}$. Without loss of generality, let $z_0 = q_1$. Moreover, in G', there is an edge between two of the p's that are not adjacent in G.

As u_1 is the only vertex in G' adjacent to both x and y, we have that x and z_1 are not adjacent. Consider the graph G_{xz_1} obtained from G'' by deleting v_0, identifying x and z_1 into a new vertex w, and deleting parallel edges. Now G_{xz_1} must not be 5-colorable, as otherwise we could 5-color G''. Thus G_{xz_1} must contain a 6-critical subgraph H. As y has degree at most four in G_{zx_1}, y is not in H. Thus $|V(H)| \leqslant 8$. By Lemma 2.1 we find that H is isomorphic to K_6 or $C_3 + C_5$. The vertex w must be in H as otherwise G'' would contain a proper subgraph that is not 5-colorable, a contradiction.

Let $J = G'\backslash\{x, y, z_1\}$. If H is isomorphic to $C_3 + C_5$, then J must contain a subgraph isomorphic to $K_2 + C_5$, because q_2 and p_1 are the only vertices of $G\backslash z_0 = G\backslash q_1$ that could have degree at least six. Thus there must be two degree six vertices in J. These must be q_2 and p_1. The other five vertices

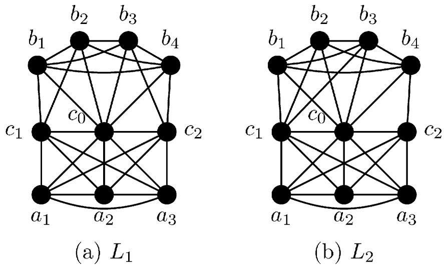

Fig. 4. The graphs L_1 and L_2 with their vertices labeled.

must be neighbors of p_1 and yet must form a C_5. This is impossible. So H must be isomorphic to K_6. Now J must contain K_5 as a subgraph. This can only happen if one of the edges p_1p_4 or p_1p_5 is present in G'. Without loss of generality suppose that p_1p_4 is present in G'. Then H must consist of the vertices $w, q_2, p_1, p_2, p_3, p_4$. So z_1 must be one of p_5, p_6, p_7. It follows that x is adjacent to p_2 and p_3.

Similarly, as u_1 is the only vertex in G' adjacent to both x and y, we have that y and z_2 are not adjacent. Consider the graph G_{yz_2} obtained from G'' by deleting v_0, identifying y and z_2 into a new vertex w', and deleting parallel edges. Now G_{yz_2} must not be 5-colorable, as otherwise we could 5-color G''. Now G_{yz_2} must contain a 6-critical subgraph H'. Thus $|V(H')| \leqslant 9$. By Lemma 2.1 we find that H' is isomorphic to K_6, $C_3 + C_5$, or $K_2 + H_7$. The vertex w' must be in H as otherwise G'' would contain a proper subgraph that is not 5-colorable, a contradiction.

Suppose that x is not H'. The previous argument for H shows that H' is isomorphic to K_6, that H' consists of $w', q_2, p_1, p_2, p_3, p_4$ and that y is adjacent to p_2 and p_3. But then there are two vertices, p_2 and p_3, adjacent to both x and y, a contradiction.

So x is in H'. Now the neighbors of x must be in H'. Specifically, p_2 and p_3 are in H'. Note that p_2 and p_3 are not equal to z_2 as they are not adjacent to p_5, p_6 or p_7. Meanwhile, at least one of p_2, p_3 is not adjacent to y. Without loss of generality, suppose that p_2 is not adjacent to y. Now p_2 has degree five in G' and hence degree at most five in G_{yz_2}. Thus the neighbors in G' of p_2 and all edges incident in G' with p_2 must be in H'.

If H' is isomorphic to K_6, then it follows that x must be adjacent to all of $H \backslash w'$ as well as z_2. That is, x must be adjacent to all the neighbors of p_2, namely q_2, p_1, p_3, p_4. Now G' contains K_6 as a subgraph, a contradiction. If H' is isomorphic to $C_3 + C_5$, then xp_2p_3 is a triangle in H'. Thus one of these vertices must have degree seven in H'. However, x and p_2 have degree five in G' while p_3 has degree at most six, a contradiction.

Thus H' is isomorphic to $K_2 + H_7$. As H' has nine vertices, z_1 must be in H' and have degree five. As z_1 is adjacent to y but not adjacent to x, z_1 has degree five in G'. However, z_1 is adjacent to z_2. So z_1 has degree four in G_{yz_2} and so has degree at most four in H', a contradiction. □

Lemma 2.8. *Let G be a graph drawn in the Klein bottle, and let $c, d \in V(G)$ be such that $G \backslash c$ does not embed in the projective plane, and G does not embed in the torus. Then every closed curve in the Klein bottle intersecting G in a subset of $\{c, d\}$ separates the Klein bottle.*

Proof. Let ϕ be a closed curve in the Klein bottle intersecting G in a subset of $\{c, d\}$, and suppose for a contradiction that it does not separate the Klein bottle. Then ϕ is either one-sided or two-sided. If ϕ is one-sided, then it intersects $G \backslash c$ in at most one vertex, and hence the Klein bottle drawing of $G \backslash c$ can be converted into a drawing of $G \backslash c$ in the projective plane, a contradiction. Thus ϕ is two-sided, but then the drawing of G can be converted into a drawing of G in the torus, again a contradiction. □

Lemma 2.9. *Let G be L_1 or L_2 with its vertices numbered as in Fig. 4, and let it be drawn in the Klein bottle. Then*

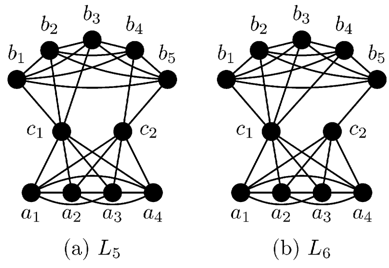

Fig. 5. The graphs L_5 and L_6 with their vertices labeled.

(i) *every face is bounded by a triangle, except for exactly one, which is bounded by a cycle of length five with vertices c_1, a_i, c_2, b_j, b_k in order for some indices i, j, k, and*
(ii) *for $i = 0, 1, 2$ the vertices a_1, a_2, a_3 appear consecutively in the cyclic order around c_i (but not necessarily in the order listed), and so do the neighbors of c_i that belong to $\{b_1, b_2, b_3, b_4\}$.*

Proof. Let $i \in \{1, 2\}$. There are indices j, k such that a_j and b_k are both adjacent to c_i and are next to each other in the cyclic order around c_i. Let f_i be the face incident with both the edges $c_i a_j$ and $c_i b_k$. We claim that the walk bounding f_i includes at most one occurrence of c_i and no occurrence of c_0. Indeed, otherwise we can construct a simple closed curve either passing through f_i and intersecting G in c_i only (if c_i occurs at least twice in the boundary walk of f_i), or passing through f_i and a neighborhood of the edge $c_i c_0$ and intersecting G in c_i and c_0 (if c_0 occurs in the boundary walk of f_i). By Lemma 2.8 this simple closed curve separates the Klein bottle. It follows from the construction that it also separates G, contrary to the fact that $G \backslash \{c_i, c_0\}$ is connected. This proves our claim that the walk bounding f_i includes at most one occurrence of c_i and no occurrence of c_0.

Since the boundary of f_i includes a subwalk from a_j to b_k that does not use c_i, we deduce that c_{3-i} belongs to the facial walk bounding f_i. But the neighbors of c_1 and c_2 in $\{b_1, b_2, b_3, b_4\}$ are disjoint, and hence f_i has length at least five. By Euler's formula $f_1 = f_2$, this face has length exactly five, and every other face is bounded by a triangle. This proves (i). Statement (ii) also follows, for otherwise there would be another face with the same properties as $f_1 = f_2$, and yet we have already shown that this face is unique. $\square$

Lemma 2.10. *Let G be L_5 or L_6 with its vertices numbered as in Fig. 5, and let it be drawn in the Klein bottle. Then*

(i) *every face is bounded by a triangle, except for exactly two, which are bounded by cycles C_1, C_2 of length five, each with vertices c_1, a_i, c_2, b_j, b_k in order for some indices i, j, k,*
(ii) *if $G = L_5$, then $C_1 \cap C_2$ consists of the vertices c_1, c_2, and if $G = L_6$, then $C_1 \cap C_2$ consists of the vertices c_1, c_2, b_5 and the edge $c_2 b_5$, and*
(iii) *for $i = 1, 2$ the vertices a_1, a_2, a_3, a_4 appear consecutively in the cyclic order around c_i (but not necessarily in the order listed), and so do the neighbors of c_i that belong to $\{b_1, b_2, b_3, b_4, b_5\}$.*

Proof. The proof is similar to the proof of Lemma 2.9. There are distinct pairs (j_1, k_1) and (j_2, k_2) of indices such that a_{j_i} and b_{k_i} are both adjacent to c_1 and are next to each other in the cyclic order around c_1. Let f_i be the face incident with both $c_1 a_{j_i}$ and $c_1 b_{k_i}$. We claim that the walk bounding f_i includes at most one occurrence of c_1. For if not, then there is a simple closed curve ϕ that passes through f_i and intersects G in c_1 only. But since L_5 and L_6 are not embeddable in the torus and $L_5 \backslash c_1$ and $L_6 \backslash c_1$ are not embeddable in the projective plane, it follows from Lemma 2.8 that ϕ separates the Klein bottle. By construction, ϕ also separates G, a contradiction, as $G \backslash c_1$ is connected. This proves our claim that the walk bounding f_i includes at most one occurrence of c_1. Thus the walk bounding f_i includes c_2, and it follows similarly that c_2 occurs in that walk at most once. We deduce that f_1 and f_2 are distinct and have length at least five. Euler's formula implies that f_1, f_2 have length exactly five, and that every other face is bounded by a triangle. It follows that conditions (i), (ii) and (iii) hold. □

3. Reducing to K_6

If v is a vertex of a graph G, then we denote by $N_G(v)$, or simply $N(v)$ if the graph can be understood from the context, the open neighborhood of the vertex v; that is, the subgraph of G induced by the neighbors of v. Sometimes we will use $N(v)$ to mean the vertex-set of this subgraph. We say that a vertex v in a graph G embedded in a surface has a *wheel neighborhood* if the neighbors of v form a cycle C in the order determined by the embedding, and the cycle C is null-homotopic. (The cycle C need not be induced.)

Let G_0 be a graph drawn in the Klein bottle such that G_0 is not 5-colorable and has no subgraph isomorphic to any of the graphs listed in Theorem 1.3. Let a vertex $v_0 \in V(G_0)$ of degree exactly five be chosen so that each of the following conditions hold subject to all previous conditions:

(i) $|V(G_0)|$ is minimum,
(ii) the clique number of $N(v_0)$, the neighborhood of v_0, is maximum,
(iii) the number of largest complete subgraphs in $N(v_0)$ is maximum,
(iv) the number of edges in $N(v_0)$ is maximum,
(v) $|E(G_0)|$ is minimum,
(vi) the number of homotopically-trivial triangles containing v_0 is maximum.

In those circumstances we say that the pair (G_0, v_0) is an *optimal pair*. Given an optimal pair (G_0, v_0) we say that a pair of vertices v_1, v_2 is an *identifiable pair* if v_1 and v_2 are non-adjacent neighbors of v_0. If v_1, v_2 is an identifiable pair, then we define $G_{v_1v_2}$ to be the graph obtained from G_0 by deleting all edges incident with v_0 except v_0v_1 and v_0v_2, contracting the edges v_0v_1 and v_0v_2 into a new vertex z_0, and deleting all resulting parallel edges. This also defines a drawing of $G_{v_1v_2}$ in the Klein bottle.

We now introduce notation that will be used throughout the rest of the paper. Let G_0' be obtained from G_0 by deleting all those edges that got deleted during the construction of $G_{v_1v_2}$. That means all edges incident with v_0 except v_0v_1 and v_0v_2 and all those edges of G_0 that got deleted because they became parallel to another edge. Thus if a vertex v of G_0 is adjacent to both v_1 and v_2, then G_0' will include exactly one of the edges vv_1, vv_2. Thus the edges of $G_0' \backslash v_0$ may be identified with the edges of $G_{v_1v_2}$, and in what follows we will make use of this identification. Now if J is a subgraph of $G_{v_1v_2}$ with $z_0 \in V(J)$, then let $\hat{J}$ be the corresponding subgraph of G_0'; that is, $\hat{J}$ has vertex-set $\{v_0, v_1, v_2\} \cup V(J) - \{z_0\}$ and edge-set $\{v_0v_1, v_0v_2\} \cup E(J)$. Let $\hat{R}_1$ and $\hat{R}_2$ be the two faces of $\hat{J}$ incident with v_0, and let R_1, R_2 be the corresponding two faces of J. We call R_1, R_2 the *hinges* of J. Finally, let $\hat{R}$ be the face of $\hat{J} \backslash v_0$ containing v_0.

Lemma 3.1. *Let (G_0, v_0) be an optimal pair, and let v_1, v_2 be an identifiable pair. Then $G_{v_1v_2}$ has no subgraph isomorphic to $C_3 + C_5$ or $K_2 + H_7$.*

Proof. Suppose for a contradiction that there exists a subgraph J of $G_{v_1v_2}$ such that $J = C_3 + C_5$ or $J = K_2 + H_7$. Let us recall that z_0 is the vertex of G_0 that arises from the identification of v_1 and v_2. Since J is not 5-colorable the choice of G_0 implies that $z_0 \in V(J)$. Thus we apply the notation introduced prior to this lemma. Let R_1, R_2 be the hinges of J, let $\hat{R}_1$ be bounded by the walk $v_1u_1u_2 \ldots u_kv_2v_0$, and let $\hat{R}_2$ be bounded by the walk $v_2z_1z_2 \ldots z_mv_1v_0$. Then $k, m \geqslant 2$. We may assume that $k \leqslant m$ and that G_0 is drawn on the Klein bottle such that $k + m$ is minimized. Since $|E(J)| = 3|V(J)| - 1$ it follows that J has exactly one face bounded by a 4-cycle and all other faces are bounded by 3-cycles. So $k = 2$ and $m \leqslant 3$. Furthermore, if $m = 3$, then all faces of $\hat{J}$ other than $\hat{R}_1$ and $\hat{R}_2$ are triangles; otherwise at most one face other than $\hat{R}_1$ and $\hat{R}_2$ is bounded by a cycle of length four. It follows that $z_1 \neq u_2$, for otherwise the cycle $z_1z_2 \ldots z_mv_1u_1$ of G_0 has length at most five and bounds a disk containing v_0 and v_2, contrary to Lemma 2.2. Similarly, $u_1 \neq z_m$. Since J has no parallel edges we deduce that $z_1 \neq u_1$ and $u_2 \neq z_m$. It follows that the vertices $v_1, v_2, u_1, u_2, z_1, z_m$ are pairwise distinct. However, if $m = 3$, then possibly $z_2 \in \{u_1, u_2\}$. Finally, all vertices of G_0 are either in $\hat{J}$ or inside one of the faces $\hat{R}_1, \hat{R}_2$ of J by Lemma 2.2.

Next we claim that z_0 has degree at least six. Indeed, otherwise z_0 is contained in the open disk bounded by a walk w of J of length at most six (because J has at most one face that is not a triangle). But W is also a walk in G_0, and the disk it bounds includes v_0, v_1, v_2. But v_1 is not adjacent to v_2, contrary to Lemma 2.2. This proves our claim that z_0 has degree at least six.

We now make a couple of remarks about vertices of degree five in J. If $J = C_3 + C_5$, then J has five vertices of degree five, and the neighborhood of each is isomorphic to K_5^-. If $J = K_2 + H_7$, then J has six vertices of degree five; four of them have neighborhoods isomorphic to K_5^- and the remaining two have neighborhoods isomorphic to $K_5 - E(P_3)$.

Let us say a vertex v of degree five in J is *good* if its neighborhood in J has the property that there are at least two triangles disjoint from any given vertex. Thus J either has five good vertices, or it has exactly four, and they induce a matching of size two. It follows from the definition of optimal pair that if $N(v_0)$ has at most one triangle, then the degree of each good vertex of J must be at least six in G_0.

Note that if z_0 is a vertex of degree six in $K_2 + H_7$, then all the vertices of degree five have a K_4 in their neighborhood disjoint from z_0. Hence if $N(v_0)$ does not contain a K_4, each vertex of degree five in J must have degree at least six in G_0.

We now condition on the cases of Lemma 2.2 for $\hat{R}$. Suppose that case (i) holds. First consider the case that $m = 3$. Let us say that two vertices of G_0 are *adjacent through a face* f of $\hat{J}$ if the edge joining them lies in f. We condition on the number of edges incident with v_0 through $\hat{R}_1$. Suppose there are two such edges. Hence v_0 is adjacent to u_1 and u_2 through $\hat{R}_1$. Further suppose that v_0 is adjacent to z_2 through $\hat{R}_2$. If z_2 is adjacent to z_0 in J, then without loss of generality suppose that v_1 is adjacent to z_2 but not through $\hat{R}_2$. Redrawing the edge through $\hat{R}_2$ contradicts condition (vi) of an optimal pair.

So we may assume that z_2 is not adjacent to z_0. It follows that $J = K_2 + H_7$ and that z_0 must be the vertex of degree six in $K_2 + H_7$, because that is the only vertex of degree at least six in $C_3 + C_5$ or $K_2 + H_7$ that has a non-neighbor. However, v_1 is not adjacent to u_2 and v_2 is not adjacent to u_1. So $N(v_0)$ does not contain a K_4. But then the vertices of degree five in J must be a subset of $\{u_1, u_2, z_2\}$, a contradiction.

So we may assume without loss of generality that v_0 is adjacent to z_1 through $\hat{R}_2$. Now we may apply Lemmas 2.4, 2.5, 2.6 and 2.7 to $G_0 + v_1 z_1$, where $G_0 + v_1 z_1$ denotes the graph obtained from G_0 by adding the edge $v_1 z_1$ if v_1 is not adjacent to z_1 in G_0 and $G_0 + v_1 z_1 = G_0$ otherwise. We find that either $G_0 \backslash v_1 \backslash v_0$ or $G_0 + v_1 z_1 \backslash v_2 \backslash v_0$ contains a subgraph H isomorphic to $K_6, C_3 + C_5$ or $K_2 + H_7$, or $G_0 + v_1 z_1$ is isomorphic to L_4. In the latter case, G_0 is 5-colorable or isomorphic to L_4, a contradiction. In the former case, note that $G_0 \backslash v_0$ has a proper 5-coloring that does not extend to a 5-coloring of G_0 and hence in this coloring all of the neighbors of v_0 must receive different colors. This yields a 5-coloring of H, a contradiction.

Suppose v_0 is incident with exactly one edge through $\hat{R}_1$. Without loss of generality we may assume that v_0 is adjacent to u_2 through $\hat{R}_1$. Suppose that v_0 is adjacent to z_1 and z_2. If z_2 is adjacent to z_0 in J, we may apply Lemmas 2.4, 2.5, 2.6 and 2.7 to $G_0 + v_1 u_2$. We find that either $G_0 \backslash v_1 \backslash v_0$ or $G_0 + v_1 u_2 \backslash v_2 \backslash v_0$ contains a subgraph H isomorphic to $K_6, C_3 + C_5$ or $K_2 + H_7$, or $G_0 + v_1 u_2$ is isomorphic to L_4. In the latter case, G_0 is 5-colorable, a contradiction. In the former case, note that $G_0 \backslash v_0$ has a proper 5-coloring that does not extend to a 5-coloring of G_0 and hence in this coloring all of the neighbors of v_0 must receive different colors. This yields a 5-coloring of H, a contradiction. So we may assume that z_2 is not adjacent to z_0 in J. As v_1 is not adjacent to z_1 and v_2 is not adjacent to z_2, $N(v_0)$ does not contain a K_4. But then the vertices of degree five in J would have to be a subset of $\{z_1, z_2, u_2\}$, a contradiction.

If v_0 is adjacent to z_2 and z_3, then a similar but easier argument applies as above. Let us assume next that v_0 is adjacent to z_1 and z_3. Note that z_0 must have degree at least seven in J for v_1 and v_2 to have degree at least five in G_0 in this case. As z_0 has degree at most eight in J, at least one of v_1 or v_2 has degree five in G_0. If v_1 has degree five, consider $G_{v_2 z_3}$, defined as before. This graph contains a subgraph H isomorphic to a graph listed in Theorem 1.3. Since $v_0 \notin V(H)$, the vertex v_1 has degree four in $G_{v_2 z_3}$ and hence $v_1 \notin V(H)$. It follows that the graph obtained from H by deleting the new vertex of $G_{v_2z_3}$ is a proper subgraph of $J\backslash z_0$. Consequently, H is isomorphic to a proper subgraph of J, a contradiction. If v_2 has degree five in G_0, we consider $G_{v_1z_1}$ similarly to obtain a contradiction.

Finally suppose v_0 is not incident with any edge through $\hat{R}_1$. Hence v_0 is adjacent to z_1 z_2, and z_3 through $\hat{R}_2$. Then same argument as in the preceding paragraph applies.

We may assume that $m = 2$. We may assume without loss of generality that v_0 is adjacent to u_1 and u_2 through $\hat{R}_1$ and to z_1 through $\hat{R}_2$. Now we may apply Lemmas 2.4, 2.5, 2.6 and 2.7 to $G_0 + v_1z_1$. We find that either $G_0\backslash v_1\backslash v_0$ or $G_0 + v_1z_1\backslash v_2\backslash v_0$ contains a subgraph H isomorphic to $K_6, C_3 + C_5$ or $K_2 + H_7$, or $G_0 + v_1z_1$ is isomorphic to L_4. In the latter case, G_0 is 5-colorable, a contradiction. In the former case, note that $G_0\backslash v_0$ has a proper 5-coloring that does not extend to a 5-coloring of G_0 and hence in this coloring all of the neighbors of v_0 must receive different colors. This yields a 5-coloring of H, a contradiction. This concludes the case when (i) of Theorem 2.2 holds.

For cases (iv)–(vi), we have that $m = 3$. Note that case (vi) cannot happen as v_0 must be adjacent to v_1 and v_2, which are distance three on the boundary of $\hat{R}$. For cases (iv) and (v), there are in each case two possibilities, up to symmetry, as to which internal vertex is v_0. In all cases, it is easy to check that $N(v_0)$ is triangle-free. All vertices of degree five in J must then have degree six in G_0, since their neighborhood in G_0 has a triangle and would thus contradict that (G_0, v_0) is an optimal pair. To have higher degree in G_0, these vertices must be a subset of $\{u_1, u_2, z_1, z_2, z_3\}$. As $K_2 + H_7$ has six vertices of degree five, $J = C_3 + C_5$. Furthermore, u_1, u_2, z_1, z_2, z_3 are all distinct and induce C_5. We now color G_0 as follows. We may assume without loss of generality that z_2 is adjacent to v_2 but not to v_1. Color z_1 and z_3 by 1. Color z_2, v_1, and u_2 by 2. Color u_1 and v_2 by 3. Finally color the other two vertices of C_3 using colors 4 and 5. As only three colors appear on the boundary of $\hat{R}$, this coloring extends to G_0 by Lemma 2.2, a contradiction.

Suppose that case (iii) happens. Suppose that $m = 3$. We may assume without loss of generality that v_0 is adjacent to u_1. If $u_1 \neq z_2$, then $N(v_0)$ is triangle-free. Hence the good vertices of J must be a subset of $\{u_1, z_1, z_2, z_3\}$, which do not induce a matching, a contradiction. Thus $u_1 = z_2$ and we color G_0 by [24, Lemma 5.1(a)]. For $m = 2$, case (iii) cannot happen as v_0 must be adjacent to v_1 and v_2, which are distance three on the boundary of $\hat{R}$.

So finally we may assume case (ii). Let v_0' be the other vertex in the interior of $\hat{R}$. Suppose that $m = 3$. Further suppose that $z_2 \in \{u_1, u_2\}$. Note that if $N(v_0)$ has at most one triangle, then all good vertices of J must have degree six in G_0. However, they must be a subset of $\{u_1, u_2, z_1, z_3\}$. Hence there are at most four good vertices in J and they do not induce a perfect matching, a contradiction. So $N(v_0)$ has at least two triangles.

Suppose that one of v_0 or v_0' is adjacent to both z_1 and z_3. Now $N(v_0)$ has at most one triangle unless that vertex is v_0' which is also adjacent to z_2, and v_0 is adjacent to one of z_1 or z_3 through $\hat{R}_2$ as well as z_2 through $\hat{R}_1$. In that case, the hypotheses of [24, Lemma 5.1(c)] are satisfied and we can extend that coloring to a coloring of G_0 by Lemma 2.2, a contradiction.

So we may suppose that neither v_0 or v_0' is adjacent to both z_1 and z_3. Thus v_0' must be adjacent to both v_1 and v_2. Without loss of generality, we may assume that one of v_0 or v_0' is adjacent to both z_1 and z_2 through $\hat{R}_2$. Now $N(v_0)$ will have at most one triangle unless $z_2 = u_2$. Let G be the graph obtained from $G_0\backslash\{v_0, v_0'\}$ by adding the edge u_1z_1. It follows that G is not 5-colorable, because every 5-coloring of G can be extended to a 5-coloring of G_0 by Lemma 2.2. Since G has fewer vertices than G_0, it follows that G has a subgraph G' isomorphic to one of the graphs listed in Theorem 1.3. But the edge u_1z_1 belongs to G', because $G\backslash u_1z_1$ is 5-colorable.

On the other hand, we claim that the edge u_1z_1 belongs to no facial triangle of G'. Indeed, if it did, say it belonged to a facial triangle u_1z_1q, then either $qz_1v_2u_2u_1q$ or $qz_1z_2v_1u_1q$ would be a contractible 5-cycle with more than one vertex in its interior, contradicting Lemma 2.2. Thus u_1z_1 belongs to no facial triangle of G'. But there are only two graphs among those listed in Theorem 1.3 that have an embedding with an edge that does not belong to a facial triangle, namely, K_6 and L_6. But G' has at most 10 vertices, because it is obtained from J by splitting one vertex, and hence G' is isomorphic to K_6. We have $u_1, z_1 \in V(G')$, but u_1 is not adjacent to v_2 (in G_0 and hence in G') and z_1 is not adjacent to v_1, because there are no exceptional vertices and no parallel edges. Thus

$v_1, v_2 \notin V(G')$. It follows that G' can be obtained from J by first deleting a vertex of degree at least six (and some other vertices) and then adding an edge. This is impossible because $J = C_3 + C_5$ or $J = K_2 + H_7$.

Thus $z_2 \notin \{u_1, u_2\}$. Suppose that one of v_0 or v'_0 is adjacent to both z_1 and z_3. Now $N(v_0)$ is triangle-free. Thus all the vertices of degree five in J must have degree six in G_0. As these are subset of $\{u_1, u_2, z_1, z_2, z_3\}$, we find that $J = C_3 + C_5$. Moreover u_1, u_2, z_1, z_2, z_3 are distinct and induce C_5. As in cases (iv) and (v), we may color so that the boundary of $\hat{R}$ only uses colors 1, 2, and 3 and then color v_0 and v'_0 with colors 4 and 5, a contradiction.

So we may suppose that neither v_0 or v'_0 is adjacent to both z_1 and z_3. Thus v'_0 must be adjacent to both v_1 and v_2. Without loss of generality, we may assume that one of v_0 or v'_0 is adjacent to both z_1 and z_2 through $\hat{R}_2$. Now $N(v_0)$ has at most one triangle and hence the good vertices of J must have degree six in G_0. However, z_3 is not adjacent to either v_0 or v'_0. Thus the good vertices must be a subset of $\{u_1, u_2, z_1, z_2\}$. Hence $J = K_2 + H_7$. Color G_0 as follows. Consider a 5-coloring of $G_0 \backslash \{v_0, v'_0\}$. If $\{u_1, u_2\}$ and $\{z_1, z_2\}$ do not receive the same pair of colors, then we may extend this coloring to G_0 by Lemma 2.2. So we may assume they are colored with colors 1 and 2. But then no other vertex of $J - z_0$ must receive colors 1 or 2. By swapping the colors of u_1 and u_2 if necessary, we may assume that u_2 and z_1 have the same color. We may now recolor v_1 with this color and extend the coloring to G_0 by Lemma 2.2.

We may now assume that $m = 2$. It follows that v_0 is adjacent to u_1 and u_2 and that v'_0 is adjacent to z_1, z_2, v_1, v_2 and v_0. Now $N(v_0)$ has at most one triangle. Moreover $N(v_0)$ is triangle-free unless the edge v_1u_2 or the edge v_2u_1 is in the interior of the unique facial 4-cycle in J. So let us suppose that $N(v_0)$ has a triangle. Without loss of generality suppose the edge v_1u_2 is present. All of the good vertices of J must be degree six in G_0. These must be a subset of $\{u_1, u_2, z_1, z_2\}$. Hence, $J = K_2 + H_7$ and these vertices induce a perfect matching. Repeating the argument from the above paragraph, color G_0 as follows. Consider a 5-coloring of $G_0 \backslash \{v_0, v'_0\}$. If $\{u_1, u_2\}$ and $\{z_1, z_2\}$ do not receive the same pair of colors, then we may extend this coloring to G_0 by Lemma 2.2. So we may assume they are colored with colors 1 and 2. But then no other vertex of $J - z_0$ must receive colors 1 or 2. By swapping the colors of u_1, u_2 if necessary, we may assume that u_2 and z_1 have the same color. We may now recolor v_1 with this color and extend the coloring to G_0 by Lemma 2.2.

So $N(v_0)$ is triangle-free. All vertices of degree five in J must be degree six in G_0. Such vertices are a subset of $\{u_1, u_2, z_1, z_2, x_1, x_2\}$ where x_1, x_2 are the ends of an edge in the interior of the facial 4-cycle of J. Let us assume that $J = K_2 + H_7$. There are six vertices of degree five; hence, all of these vertices are distinct. As x_1 is not adjacent to x_2 in J, we may assume without loss of generality that x_1 is a good vertex, while x_2 may be good or not. Color G_0 as follows. Consider a 5-coloring of $G_0 \backslash \{v_0, v'_0\}$. If $\{u_1, u_2\}$ and $\{z_1, z_2\}$ do not receive the same pair of colors, then we may extend this coloring to G_0 by Lemma 2.2. So we may assume they are colored with colors 1 and 2.

We claim that one of the pairs $\{u_1, u_2\}$ and $\{z_1, z_2\}$ only sees two other colors in $J \backslash z_0$. Suppose not. If x_2 is good, then it follows that the other two vertices of degree five receive the same color but they are adjacent, a contradiction. If x_2 is not good, then as the pair which contains two good vertices sees all the colors 3, 4, and 5 then x_1 will also see 3, 4, and 5, as well as 1, 2 from the pair which contains one good vertex and one not good vertex. Hence x_1 cannot receive a color, a contradiction. Now consider the pair, say $\{u_1, u_2\}$ that only sees 2 other colors, say colors 3 and 4. As v_1 and v_2 are not colored the same, one of v_1 or v_2 must not be colored 5. Without loss of generality suppose v_1 is not colored 5. Then recolor u_1 with color 5 and extend this coloring to G_0 by Lemma 2.2.

So we may assume that $J = C_3 + C_5$. Suppose that at least one of $\{u_1, u_2, z_1, z_2\}$ is not a vertex of degree five in J. Then it must be exactly one, say u_1. Consider a 5-coloring of $G_0 \backslash \{v_0, v'_0\}$. If u_1 is not colored the same as one of $\{z_1, z_2\}$, then this coloring extends to G_0 by Lemma 2.2. However, as u_1 is not a vertex of degree five, it is adjacent to all of $J - z_0$ and hence to z_1 and z_2, it cannot be colored the same as z_1 or z_2. Thus we may assume that all of $\{u_1, u_2, z_1, z_2\}$ are vertices of degree five in J.

Consider a 5-coloring of $G_0 \backslash \{v_0, v'_0\}$. Now $\{u_1, u_2\}$ must receive the same colors as $\{z_1, z_2\}$, as otherwise this coloring extends to G_0 by Lemma 2.2. Now the other vertex of degree five in J, call this x_1 must receive a third color, say color 3. Meanwhile, the other two vertices of $J - z_0$ must receive new colors, namely, colors 4 and 5. Now if v_1 is not adjacent to any vertex of color 1, we may recolor v_1 by 1 and extend the coloring to G_0. Similarly with color 2 and the same applies for v_2 with colors 1 and 2. So we may assume that u_1 and z_1 are colored 1 and u_2 and z_2 are colored 2. Further, we may assume that u_1 and z_2 are adjacent to x_1. As x_1 must have degree six in G_0 there exists an edge x_1x_2 through the 4-cycle in J. As it is not a parallel edge, $x_2 \in \{v_1, v_2, u_2, z_1\}$. Thus at least one of u_2, z_1 is not adjacent to x_1. We may assume without loss of generality that u_2 is not adjacent to x_1. Now recolor u_2 by 3. If the resulting coloring of $G_0 \backslash v_0, v_0'$ is proper, then we may extend it to G_0 by Lemma 2.2, a contradiction. Thus u_2 must be adjacent to a vertex colored 3. As u_2 is not adjacent to x_1 nor to v_1, that vertex must be v_2. So recolor v_2 by color 2. The resulting coloring is proper as v_2 is not adjacent to z_2. This coloring extends to a coloring of G_0 by Lemma 2.2, a contradiction. □

Lemma 3.2. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, let J be a subgraph of $G_{v_1v_2}$ isomorphic to L_1, L_2, L_5 or L_6, and let R_1, R_2 be the hinges of J. If R_1 and R_2 share a vertex $u \neq z_0$ and at least one of them has length three, then the other one has length five and there exists an index $i \in \{1, 2\}$ such that $\hat{R}_1 \cup \hat{R}_2 \backslash \{v_0, v_i\}$ is a cycle in G_0 that bounds an open disk containing v_0 and v_i.*

Proof. By the symmetry we may assume that R_2 has length three. Thus u is adjacent to z_0 in J. Since R_1 is an induced cycle, the cycles R_1, R_2 share the edge z_0u. Thus $\hat{R}_1$, $\hat{R}_2$ share the edge v_iu for some $i \in \{1, 2\}$, and the second conclusion follows. By Lemma 2.2 the cycle $\hat{R}_1 \cup \hat{R}_2 \backslash \{v_0, v_i\}$ has length at least six, and hence R_1 has length five, as desired. □

We denote by K_5^- the graph obtained from K_5 by deleting an edge, and by $K_5 - P_3$ the graph obtained from K_5 by deleting two adjacent edges.

Lemma 3.3. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1v_2}$ isomorphic to L_1, L_2, L_5 or L_6. Then there exists a vertex $s \in V(G_0) - \{v_0\}$ of degree five such that*

(i) *$N_{G_0}(s)$ has a subgraph isomorphic to $K_5 - P_3$, and*
(ii) *if both hinges of J have length five, then $N_{G_0}(s)$ has a subgraph isomorphic to K_5^-.*

Proof. We only prove the first assertion, leaving the second one to the reader. A proof of the second assertion may be found in [29]. Assume that the notation is as in the paragraph prior to Lemma 3.1, and suppose first that $J = L_5$. Let the vertices of J be numbered as in Fig. 5. It follows from Lemma 2.10 that the indices of a_i and b_j can be renumbered so that the faces of J around c_1 are $a_1c_1a_2, a_2c_1a_3, a_3c_1a_4, a_4c_1b_3b_5c_2, b_3c_1b_2, b_2c_1b_1, b_1c_1a_1c_2b_4$, in order. Recall that z_0 is the vertex of J that results from the identification of v_1 and v_2. If $z_0 \neq c_1$, then one of the vertices a_2, a_3, b_2 is not incident with $\hat{R}_1$ or $\hat{R}_2$, and hence has the same neighbors in J and in G_0. It follows that such a vertex satisfies the conclusion of the lemma, as desired. We will use the same argument again later, whereby we will simply say that a certain vertex satisfies the conclusion of the lemma.

Thus we may assume that $z_0 = c_1$, and since we may assume that no vertex satisfies the conclusion of the lemma, we deduce that one of R_1 and R_2 is the face $a_2c_1a_3$ and the other is $b_1c_1b_2$ or $b_2c_1b_3$. Thus we may assume that R_1 is $a_2c_1a_3$ and R_2 is $b_1c_1b_2$. We may assume, by swapping v_1 and v_2, that the neighbors of v_1 in $\hat{J}$ are a_1, a_2, v_0, b_1 and that the neighbors of v_2 are a_3, a_4, b_3, b_2, v_0. Hence the face $\hat{R}$ is $v_1a_2a_3v_2b_2b_1$. Now v_1 is not adjacent to a_3 in G_0, for otherwise a_2 satisfies the conclusion of the lemma. We shall abbreviate this argument by $a_2 \Rightarrow v_1 \nsim a_3$. Similarly, we have $b_5 \Rightarrow b_3 \nsim c_2$ and $b_3 \Rightarrow v_2 \nsim b_5$. We shall define a 5-coloring c of $\hat{J} \backslash v_0$. Let $c(a_1) = c(v_2) = c(b_5) = 1$, $c(a_2) = c(b_1) = 2$, $c(a_3) = c(v_1) = 3$, $c(a_4) = 4$, and $c(c_2) = c(b_3) = 5$. Assume first that b_4 is adjacent to a_1. Then b_2 is not adjacent to v_1, for otherwise b_1 satisfies the conclusion of the lemma. Furthermore, there is no vertex of G in the face of $\hat{J}$ bounded by $b_1v_1a_1c_2b_4$. In that case we let $c(b_4) = 4$ and $c(b_2) = 3$. If b_4 is not adjacent to a_1, then we let $c(b_4) = 3$ and $c(b_2) = 4$. In either case it follows from Lemma 2.2 and the fact that v_0 is adjacent to v_1 and v_2 that c extends to a 5-coloring of G_0, a contradiction. This completes the case $J = L_5$.

If $J = L_6$ we proceed analogously. By Lemma 2.10 we may assume that the faces around c_1 are $a_1c_1a_2$, $a_2c_1a_3$, $a_3c_1a_4$, $a_4c_1b_4b_5c_2$, $b_4c_1b_3$ $b_3c_1b_2$, $b_2c_1b_1$ and $b_1c_1a_1c_2b_5$. If $z_0 \neq c_1$, or if one of R_1,

R_2 is not $a_1c_1a_2$ or $b_2c_1b_3$, then one of a_2, a_3, b_2, b_3 satisfies the conclusion of the lemma. Thus we may assume that R_1 is $a_2c_1a_3$ and R_2 is $b_2c_1b_3$. We may also assume, by swapping v_1 and v_2 that the neighbors of v_1 in $\hat{J}$ are a_1, a_2, v_0, b_1 and b_2 and the neighbors of v_2 in $\hat{J}$ are a_3, a_4, b_3, b_4, and v_0. Now $a_1 \Rightarrow v_1 \nsim y$, $b_4 \Rightarrow v_2 \nsim b_5$, $a_3 \Rightarrow a_2 \nsim v_2$, and $b_2 \Rightarrow b_3 \nsim v_1$. With these constraints in mind and recalling that v_0 is adjacent to v_1 and v_2, consider the following coloring: $c(a_4) = c(b_1) = 1$, $c(a_1) = c(b_2) = 2$, $c(b_3) = c(v_1) = c(c_2) = 3$, $c(a_3) = c(b_4) = 4$ and $c(b_5) = c(a_2) = c(v_2) = 5$. It follows from Lemma 2.2 that c extends to a 5-coloring of G_0, a contradiction. This completes the case $J = L_6$.

We now consider the case $J = L_1$. By Lemma 2.9 exactly one face of J, say F, is bounded by a cycle of length five, and the remaining faces are bounded by triangles. Furthermore, we may assume, by swapping b_1, b_2, and by permuting a_1, a_2, a_3 that the faces around c_1 in order are $F, b_2c_1b_1, b_1c_1c_0, c_0c_1a_1, a_3c_1a_1, a_2c_1a_3$. By swapping b_3, b_4 we may assume that the faces around c_2 are $F, b_3c_2b_4, b_4c_2c_0, c_0c_2a_\alpha, a_\beta c_2a_\alpha, a_\gamma c_2a_\beta$ for some distinct indices $\alpha, \beta, \gamma \in \{1, 2, 3\}$. Thus the face F is bounded by the cycle $c_1a_2c_2b_3b_2$, and hence $\gamma = 2$. Since $a_1c_0c_1, c_1c_0b_1, b_4c_0c_2$ and $c_2c_0a_\alpha$ are faces of J we deduce that the faces around c_0 in order are $a_1c_0c_1, c_1c_0b_1, b_1c_0b_i, b_ic_0b_j, b_jc_0b_4$, $b_4c_0c_2, c_2c_0a_\alpha, a_\alpha c_0a_\delta, a_\delta c_0a_1$ for some integers i, j, δ with $\{i, j\} = \{2, 3\}$ and $\delta \in \{2, 3\} - \{\alpha\}$. Since $\gamma = 2$ we have $\alpha \neq 2$, and hence $\alpha = 3$ and $\delta = 2$.

Now if $z_0 \neq c_0$, then one of the vertices $a_1, a_2, a_3, b_1, b_2, b_3, b_4$ satisfies the conclusion of the lemma, and hence we may assume that $z_0 = c_0$. Furthermore, it is not hard to see that one of the above vertices satisfies the conclusion of the lemma unless one of R_1, R_2 is $a_1c_0a_2$ or $a_2c_0a_3$ and the other is one of $b_1c_0b_i$, $b_ic_0b_j$, $b_jc_0b_4$. Thus by symmetry we may assume that R_1 is $a_1c_0a_2$ and that R_2 is one of $b_1c_0b_i$, $b_ic_0b_j$, $b_jc_0b_4$.

We may assume that in $\hat{J}$ the vertex v_1 is adjacent to c_1 and v_2 is adjacent to c_2. We see that $a_3 \Rightarrow c_1 \nsim c_2$ and $a_3 \Rightarrow a_1 \nsim v_2$. Furthermore, if R_2 is the face $b_1c_0b_i$, then $b_4 \Rightarrow b_1 \nsim v_2$, and if R_2 is the face $b_1c_0b_i$, then $b_1 \Rightarrow v_1 \nsim b_4$. Let c be the coloring of $\hat{J} \backslash v_0$ defined by $c(b_1) = c(v_2) = 1$, $c(b_i) = c(a_1) = 2$, $c(b_j) = c(v_1) = c(a_3) = 3$, $c(b_4) = c(a_2) = 4$, and $c(x) = c(y) = 5$, and let c' be obtained from c by changing the colors of the vertices v_1, v_2, a_2 to $4, 2, 1$, respectively. It follows from Lemma 2.2 by examining the three cases for R_2 separately that one of c, c' extends to a 5-coloring of G, a contradiction. This completes the case $G = L_1$.

Finally, let $J = L_2$. We proceed similarly as above, using Lemma 2.9. Let F be the unique face of J of size five. By renumbering a_1, a_2, a_3 and b_1, b_2, b_3 we may assume that the faces around c_1 are $F, b_3c_1b_2, b_2c_1b_1, b_1c_1c_0, c_0c_1a_1, a_1c_1a_3, a_3c_1a_2$. Then the faces around c_2 are $F, b_4c_2c_0$, $c_0c_2a_\alpha, a_\alpha c_2a_\beta, a_\beta c_2a_\gamma$ for some distinct integers $\alpha, \beta, \gamma \in \{1, 2, 3\}$. It follows that $\gamma = 2$ and that F is bounded by $c_1b_3b_4c_2a_2$. Since $b_1c_1c_0, c_0c_1a_1, b_4c_2c_0, c_0c_2a_\alpha$ are faces of J we deduce that $\alpha \neq 1$ (and hence $\alpha = 3$ and $\beta = 1$) and that the cyclic order of the neighbors of c_2 around c_2 is $c_1b_1b_ib_jb_4c_2a_3a_2a_1$ for some distinct integers $i, j \in \{2, 3\}$. (Recall that all faces incident with c_0 are triangles.) Since b_4 is adjacent to b_3 in the boundary of F we deduce that $i = 3$ and $j = 2$.

Similarly as above, it is easy to see that some a_i or b_j satisfies the conclusion of the lemma, unless $z_0 \in \{c_0, c_1\}$. Suppose first that $z_0 = c_1$. We may assume that R_1 is $b_1b_2c_1$ and R_2 is $a_1a_3c_1$, for otherwise some vertex satisfies the conclusion of the lemma. We may assume that v_1 is adjacent to a_2, a_3, b_2, b_3. We have $a_2 \Rightarrow v_1 \nsim c_2$, $a_1 \Rightarrow a_3 \nsim v_2$ and $b_2 \Rightarrow v_1 \nsim b_1$. Let $c(a_2) = c(b_2) = 1$, $c(a_3) = c(b_4) = c(v_2) = 2$, $c(a_1) = c(b_3) = 3$, $c(v_1) = c(b_1) = c(c_2) = 4$, and $c(c_0) = 5$. It follows from Lemma 2.2 that c extends to a 5-coloring of G_0, a contradiction. Thus we may assume that $z_0 = c_0$. Similarly as above we may assume that R_1 is $b_1b_3c_0$ or $b_3b_2c_0$ and that R_2 is $a_1a_2c_0$ or $a_2a_3c_0$. We may assume that v_1 is adjacent to a_1 and b_1. If R_2 is $a_1a_2c_0$, then we have $a_3 \Rightarrow c_1 \nsim c_2$ and $a_3 \Rightarrow a_1 \nsim v_2$. If R_2 is $a_2a_3c_0$, then $a_1 \Rightarrow c_1 \nsim c_2$ and $a_1 \Rightarrow a_3 \nsim v_2$. If R_1 is $b_1b_3c_0$, then $b_2 \Rightarrow b_1 \nsim v_2$. Let $c(a_1) = c(b_1) = c(v_2) = 1$, $c(b_3) = 2, c(a_2) = c(b_2) = 3$, $c(a_3) = c(b_4) = c(v_1) = 4$ and $c(c_1) = c(c_2) = 5$. It follows from Lemma 2.2 that c extends to a 5-coloring of G_0, a contradiction. $\square$

Lemma 3.4. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1v_2}$. Then J is not isomorphic to L_1, L_2, L_5 or L_6.*

Proof. Let G, v_0, v_1, v_2 and J be as stated, and suppose for a contradiction that J is isomorphic to L_1, L_2, L_5 or L_6. Let R_1, R_2 be the hinges of J, and let $\hat{J}$, $\hat{R}_1$ and $\hat{R}_1$ be as prior to Lemma 3.1. From Lemma 3.3 and conditions (ii)–(iv) in the definition of an optimal pair we deduce that

(1) *$N_{G_0}(v_0)$ has a subgraph isomorphic to $K_5 - P_3$,*

and

(2) *if both R_1 and R_2 have length five, then v_1, v_2 is the only non-adjacent pair of vertices in $N_{G_0}(v_0)$.*

Let v_3, v_4, v_5 be the remaining neighbors of v_0 in G_0. If at least two of them belong to the interior of $\hat{R}_1$ or $\hat{R}_2$, then they belong to the interior of the same face, say R_1, by (1). But then $\hat{R}_1$ is bounded by a cycle of length seven, and that again contradicts (1) by inspecting the outcomes of Lemma 2.2. Thus at most one of v_3, v_4, v_5 belongs to the interior of $\hat{R}_1$ or $\hat{R}_2$.

From the symmetry we may assume that the edges v_0v_4 and v_0v_5 belong to the face $\hat{R}_1$. We may also assume that v_5 belongs to the boundary of $\hat{R}_1$, and that if v_4 does not belong to the boundary of $\hat{R}_1$, then the edge v_0v_3 belongs to $\hat{R}_2$. We claim that v_4 belongs to the boundary of $\hat{R}_1$. To prove this suppose to the contradiction that v_4 belongs to the interior of $\hat{R}_1$. Then one of the edges v_1v_4, v_2v_4 does not belong to G_0, and so we may assume v_2v_4 does not. By (1) v_1, v_2 and v_2, v_4 are the only non-adjacent pairs of vertices in $N_{G_0}(v_0)$, and by (2) at least one of R_1, R_2 has length three. It follows that v_3 belongs to the boundary of $\hat{R}_1$, and the choice of v_4, v_5 implies that the edge v_0v_3 lies in the face $\hat{R}_2$. Thus v_3 belongs to the boundary of $\hat{R}_2$. By Lemma 3.2 there exists an index $i \in \{1, 2\}$ such that the cycle $R_1 \cup R_2 \backslash \{v_0, v_i\}$ bounds a disk containing v_0, v_i in its interior. By shortcutting this cycle through v_0 we obtain a cycle of G_0 of length at most four bounding a disk that contains the vertex v_i in its interior, contrary to Lemma 2.2. This proves our claim that v_4 belongs to the boundary of $\hat{R}_1$. We may assume that v_0, v_1, v_4, v_5, v_2 occur on the boundary of $\hat{R}_1$ in the order listed.

Let $e \in E(G_0)$ have ends either v_1, v_5, or v_2, v_4. Then $e \notin E(\hat{J})$, because the boundary of $\hat{R}_1$ is an induced cycle of $\hat{J}$. Moreover, e does not belong to the face $\hat{R}_1$, because the edges v_0v_4, v_0v_5 belong to that face. Thus e belongs to $\hat{R}_2$ or a face of $\hat{J}$ of length five. We claim that e does not belong to $\hat{R}_2$. To prove the claim suppose to the contrary that it does, and from the symmetry we may assume that $e = v_2v_4$. We now argue that not both R_1, R_2 are pentagons. Indeed, otherwise v_1 is adjacent to v_5 by (2), and the edge v_1v_5 belongs to $\hat{R}_2$, because there is no other face of length at least five to contain it. In particular, v_4, v_5 belong to the boundary of $\hat{R}_2$, and because the edges v_1v_5, v_2v_4 do not cross inside $\hat{R}_2$, the vertices v_1, v_0, v_2, v_4, v_5 occur on the boundary of $\hat{R}_2$ in the order listed. It now follows by inspecting the 5-cycles of L_5 and L_6 that this is impossible. Thus not both R_1, R_2 are pentagons. By Lemma 3.2 the cycle $\hat{R}_1 \cup \hat{R}_2 \backslash \{v_0, v_1\}$ bounds a disk with v_0, v_1 in its interior. By shortcutting this cycle using the chord v_2v_4 we obtain a cycle in G_0 of length at most five bounding a disk with at least two vertices in its interior, contrary to Lemma 2.2. This proves our claim that v_1v_5 and v_2v_4 do not lie in the face $\hat{R}_2$.

By (1) and the symmetry we may assume that $v_2v_4 \in E(G_0)$, and hence the edge v_2v_4 belongs to a face $\hat{F}$ of $\hat{J}$ such that $\hat{F} \neq \hat{R}_1, \hat{R}_2$. Let F be the corresponding face of J. Since F is bounded by an induced cycle, we deduce that v_4 is not adjacent to z_0 in J. Consequently, R_1 has length five. Thus R_1 and F have length five, and all other faces of J, including R_2, are triangles. In particular, $J = L_5$ or $J = L_6$, and v_1, v_5 are not adjacent in G_0 (because no face of $\hat{J}$ can contain the edge v_1v_5). By (1) v_1, v_2 and v_1, v_5 are the only non-adjacent pairs of vertices in $N_{G_0}(v_0)$. Condition (1) also implies that v_3 belongs to the boundary of $\hat{R}_2$. Using that and the fact that v_3 is adjacent to v_1 and v_2 in G_0, it now follows that there exists a vertex of $G_0 \backslash v_0$ whose neighborhood in G_0 has a subgraph isomorphic to $K_5 - P_3$. Finding such a vertex requires a case analysis reminiscent of but simpler than the proof of Lemma 3.3. We omit further details. The existence of such a vertex contradicts the fact that (G_0, v_0) is an optimal pair. □

Lemma 3.5. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1v_2}$. Then J is not isomorphic to L_3 or L_4.*

Proof. Let G_0, v_0, v_1, v_2 and J be as stated, and suppose for a contradiction that J is isomorphic to L_3 or L_4. Let R_1, R_2 be the hinges of J, and let $\hat{J}, \hat{R}_1, \hat{R}_2$ be as prior to Lemma 3.1. Since by Euler's formula J triangulates the Klein bottle, we deduce that the faces $\hat{R}_1, \hat{R}_2$ have size five, and every other face of $\hat{J}$ is a triangle. Let the boundaries of $\hat{R}_1$ and $\hat{R}_2$ be $v_1v_0v_2a_1b_1$ and $v_1v_0v_2cb_l$, respectively. Let the neighbors of v_1 in $\hat{J}$ in cyclic order be $v_0, b_1, b_2, \dots, b_l$, and let the neighbors of v_2 in $\hat{J}$ be $v_0, a_1, a_2, \dots, a_k, c$. Then $\deg_J(z_0) = k + l + 1$. Since J has no parallel edges the vertices $a_1, a_2, \dots, a_k, c, b_l, b_{l-1}, \dots, b_1$ are distinct, and since J is a triangulation they form a cycle, say C, in the order listed. Since v_1 is not adjacent to v_2 in G_0, Lemma 2.2 implies that $|V(C)| \geqslant 7$.

Let us assume that $|V(C)| = 7$. Then z_0 has degree seven, and hence $J = L_4$, because L_3 has no vertices of degree seven. Let X be the set of neighbors of z_0 in J. By inspecting the graph obtained from L_4 by deleting a vertex of degree seven, we find that for every $x \in X$, there exists a 5-coloring of $J \backslash z_0$ such that no vertex of $X - \{x\}$ has the same color as x. But this contradicts Lemma 2.2 applied to the subgraph of G_0 consisting of all vertices and edges drawn in the closed disk bounded by C, because $X = V(C)$. This completes the case when $|V(C)| = 7$.

Since L_3 and L_4 have no vertices of degree eight, it follows that $|V(C)| = 9$, and hence z_0 is the unique vertex of J of degree nine. From the symmetry between v_1 and v_2, we may assume that $\deg_{\hat{J}}(v_1) \leqslant 5$; in other words $l \leqslant 4$. The graph J is 6-critical. Since z_0 is adjacent to every other vertex of J, we deduce that $J \backslash z_0 \backslash x$ is 4-colorable for every vertex $x \in V(J) - \{z_0\}$, and hence

(1) *for every vertex $x \in V(J) - \{z_0\}$, the graph $J \backslash z_0$ has a 5-coloring such that x is the only vertex colored* 5.

From Lemma 2.2 applied to the boundary of the face $\hat{R}$ of $\hat{J} \backslash v_0$, we deduce that one of $\hat{R}_1, \hat{R}_2$ contains no vertex of G_0 in its interior, and the other contains at most one. Since v_0 has degree five, we may assume from the symmetry between $\hat{R}_1$ and $\hat{R}_2$ that v_0 is adjacent to a_1 and b_1 (and hence $\hat{R}_1$ includes no vertices of G_0 in its interior). We claim that $l = 4$ and v_1 is adjacent to c. To prove the claim suppose to the contrary that either $l \leqslant 3$ or v_1 is not adjacent to c. Then $\deg_{\hat{J}}(v_1) \leqslant 5$. By (1) there exists a coloring of $J \backslash z_0 = \hat{J} \backslash \{v_0, v_1, v_2\}$ such that b_1 is the only vertex colored 5. We give v_2 the color 5, then we color v_1, then we color the unique vertex in the interior of $\hat{R}_2$ if there is one, and finally color v_0. The last three steps are possible, because each vertex being colored sees at most four distinct colors. Thus we obtain a 5-coloring of G_0, a contradiction. This proves our claim that $l = 4$ and v_1 is adjacent to c. It follows that $k = 4$ and $V(G_0) = \{v_0, v_1, v_2, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c\}$. We have $\deg_{G_0}(v_1) = \deg_{G_0}(v_2) = 6$, and since $\deg_J(c) \leqslant \deg_{G_0}(c) - 2$, we deduce that $\deg_{G_0}(c) \geqslant 7$. Thus we have shown that

(2) *if x_1, x_2, x_3, x_4, x_5 are the neighbors of v_0 in G_0 listed in their cyclic order around v_0, the vertex x_1 is not adjacent to x_3 in G_0 and G_{x_1,x_3} has a subgraph isomorphic to L_3 or L_4, then* $\deg_{G_0}(x_1) = \deg_{G_0}(x_3) = 6$ *and* $\deg_{G_0}(x_2) \geqslant 7$.

It also follows that v_1 is not adjacent to a_1 in G_0 and that v_2 is not adjacent to b_1 in G_0. Not both $G_{v_1a_1}$ and $G_{v_2b_1}$ have a subgraph isomorphic to L_3 or L_4 by (2), and so from the symmetry we may assume that $G_{v_1a_1}$ does not. By the optimality of (G_0, v_0) and Lemmas 3.1 and 3.4, it follows that $G_{v_1a_1}$ has a subgraph isomorphic to K_6. Thus $G \backslash \{v_0, v_1, v_2\}$ has a subgraph K isomorphic to K_5. If $v_2 \notin V(K)$, then $V(K) \cup \{z_0\}$ induces a K_6 subgraph in J, a contradiction. Thus $v_2 \in V(K)$, and hence $V(K) = \{v_2, a_2, a_3, a_4, c\}$. Let $i \in \{3, 4\}$. If a_1 is not adjacent to a_i in G_0, then we 5-color G_0 as follows. By (1) there is a 5-coloring of $G_0 \backslash \{v_0, v_1, v_2\}$ such that a_1 and a_i are the only two vertices colored 5. We give v_1 color 5, then color v_2 and finally v_0. Similarly as before, this gives a valid 5-coloring of G_0 a contradiction. Thus, a_1 is adjacent to a_3 and a_4 and hence a_1 is not adjacent to c, for otherwise $\{a_1, a_2, a_3, a_4, v_2, c\}$ induces a K_6 subgraph in G_0.

Since $\deg_{G_0}(v_2) = 6$, it follows from (2) that G_{ca_1} has no subgraph isomorphic to L_3 or L_4. By the optimality of (G_0, v_0) and Lemmas 3.1 and 3.4 it follows that G_{ca_1} has a subgraph isomorphic to K_6. By an analogous argument as above we deduce that $\{v_1, b_1, b_2, b_3, b_4\}$ is the vertex-set of a K_5 subgraph of G_0. The existence of the two K_5 subgraphs implies that $a_2, a_3, a_4, b_2, b_3, b_4$ have K_4 subgraphs in their neighborhoods, and the optimality of (G_0, v_0) implies that $a_2, a_3, a_4, b_1, b_2, b_3$ all have degree at least six in G_0, and hence in J. Thus a_1, b_1, c are the only vertices of J of degree five. Thus, $J = L_3$ and a_1, b_1, c are pairwise adjacent, a contradiction, because we have shown earlier that a_1 is not adjacent to c. □

The results of this section may be summarized as follows.

Lemma 3.6. *Let (G_0, v_0) be an optimal pair, and let v_1, v_2 be an identifiable pair. Then $G_{v_1v_2}$ has a subgraph isomorphic to K_6.*

Proof. Every 5-coloring of $G_{v_1v_2}$ may be extended to a 5-coloring of G_0, and hence $G_{v_1v_2}$ is not 5-colorable. By the choice of G_0 the graph $G_{v_1v_2}$ has a subgraph isomorphic to one of the graphs listed in Theorem 1.3. By Lemmas 3.1, 3.4 and 3.5 that subgraph is K_6, as desired. □

4. Using K_6

Lemma 4.1. *Let (G_0, v_0) be an optimal pair. Then G_0 has at least 10 vertices, and if it has exactly 10, then it has a vertex of degree nine.*

Proof. This follows immediately from Lemma 2.1. □

Lemma 4.2. *Let (G_0, v_0) be an optimal pair. Then there are at least two identifiable pairs.*

Proof. Since G_0 has no subgraph isomorphic to K_6 there is at least one identifiable pair. Suppose for a contradiction that v_1, v_2 is the only identifiable pair. Thus the subgraph of G_0 induced by v_0 and its neighbors is isomorphic to K_6 with one edge deleted. By Lemma 3.6 the graph $G_0 \backslash \{v_0, v_1, v_2\}$ has a subgraph K isomorphic to K_5, and every vertex of K is adjacent to v_1 or v_2. Let t be the number of neighbors of v_0 in $V(K)$. Since v_0 has degree five and its neighbors v_1, v_2 are not in K it follows that $t \leqslant 3$. If $t = 0$, then G_0 has a subgraph isomorphic to L_5 or L_6; if $t = 1$, then G_0 has a subgraph isomorphic to L_1 or L_2; if $t = 2$, then G_0 has a subgraph isomorphic to $K_2 + H_7$; and if $t = 3$, then G_0 has a subgraph isomorphic to $C_3 + C_5$. □

Lemma 4.3. *Let (G_0, v_0) be an optimal pair. Then v_0 has a wheel neighborhood.*

Proof. Let us say that a vertex $v \in V(G_0)$ is a *fan* if its neighbors form a cycle in the order determined by the embedding of G_0. We remark that if v_0 is a fan and v_0 does not have a wheel neighborhood, then the embedding of G_0 can be modified to G_0' so that v_0 will have a wheel neighborhood contradicting condition (vi). Thus it suffices to show that v_0 is a fan. Suppose for a contradiction that there exist non-adjacent vertices $a, b \in N(v_0)$ that are consecutive in the cyclic order of the neighbors of v_0. By condition (iv) in the definition of an optimal pair, the graph $G' = G_0 + ab$ has a subgraph M isomorphic to one of the graphs from Theorem 1.3. Assume, for a contradiction, that $v_0 \notin V(M)$. By optimality condition (i), $G_0 \backslash v_0$ has a 5-coloring c. Since c is not a 5-coloring of M it follows that $c(a) = c(b)$. But then c can be extended to a 5-coloring of G_0, a contradiction. Thus $v_0 \in V(M)$. Since $\deg(v_0) = 5$, we get that $N_{G_0}(v_0) \subseteq V(M)$. Further note that a, b are adjacent in M, because M is not a subgraph of G_0.

First, assume $M = K_6$. Then $V(M) = \{v_0\} \cup N(v_0)$. This implies that there is at most one identifiable pair, contrary to Lemma 4.2. Second, assume $M = L_3$ or L_4. As each is a triangulation, Lemma 2.2 implies that $G_0 = M \backslash ab$. But M is 6-critical, so G_0 has a 5-coloring, a contradiction.

Third, assume that $M = C_3 + C_5$ or $K_2 + H_7$. Because M is one edge short of being a triangulation, there is a unique face in M of length four. As $ab \in E(M)$, the embedding of $M \backslash ab$ has at most two faces of size strictly bigger than three, and if it has two, then they both have size four. Since G_0 has at least 10 vertices by Lemma 4.1, Lemma 2.2 implies that $M \backslash ab$ has a face f of size five whose interior includes a vertex of degree five. However, f is the only face of $M \backslash ab$ of size at least four, and hence it also includes the edge ab, but that is impossible. This completes the case when $M = C_3 + C_5$ or $K_2 + H_7$.

Fourth, suppose that M is either L_5 or L_6, and let the notation be as in the proof of Lemma 3.3. In particular, every face incident with a_2 or b_2 is a triangle. At least one of a_2, b_2, say s, is not equal to v_0 and does not include both a, b in its neighborhood. But then the neighborhoods of s in G and in M are the same, and hence s satisfies conditions (ii)–(iv) in the definition of an optimal pair by Lemma 4.2. But s is a fan in M, and hence has a wheel neighborhood in some embedding of G_0, contrary to condition (vi) in the definition of optimal pair.

If $M = L_1$, then we apply the argument of the previous paragraph to the vertices a_1, b_1, b_4, using the notation of Lemma 3.3. Finally, suppose that M is L_2, and let the notation be again as in the proof of Lemma 3.3. Every face incident with one of the vertices a_3, b_2 is a triangle, and at least one of those vertices, say s, has the property that $s \neq v_0$ and if the neighborhood of s includes both a and b, then a, b are not consecutive in the cyclic ordering around s and $\{a, b\} \cap \{x, y\} \neq \emptyset$ for every pair of distinct non-adjacent vertices $x, y \in N_M(v_0)$. Since s is a fan in M its choice implies that it is a fan in G_0, and hence has a wheel neighborhood in some embedding of G_0. Furthermore, in G_0 there are at most two pairs of non-adjacent vertices in the neighborhood of s, and if there are two, then they are not disjoint. Thus s satisfies conditions (ii)–(iv) in the definition of an optimal pair by Lemma 4.2, contrary to condition (vi) in the definition of an optimal pair. □

A drawing of a graph G in a surface is 2-*cell* if every face of G is homeomorphic to an open disk.

Lemma 4.4. *Let (G_0, v_0) be an optimal pair, and let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1 v_2}$ isomorphic to K_6. Then the drawing of J in the Klein bottle is 2-cell.*

Proof. Let $v_0, R_1, R_2, \hat{R}_1, \hat{R}_2$ be as before, and suppose for a contradiction that the drawing of J is not 2-cell. Since K_6 has a unique drawing in the projective plane [13, p. 364], it follows that every face of J is bounded by a triangle, and exactly one face, say F, is homeomorphic to the Möbius strip. If F is not R_1 or R_2, then the boundary of F is a separating triangle of G_0, a contradiction, because no 6-critical graph has a separating triangle. Thus we may assume that $F = R_2$.

Since both R_1 and R_2 are triangles, and they share at least one vertex, there exists a vertex $s \in V(J)$ not incident with R_1 or R_2. Thus in $\hat{J}$ all the faces incident with s are triangles, and hence $\deg_{G_0}(s) = \deg_J(s) = 5$ by Lemma 2.2. Furthermore, if R_1 and R_2 share an edge, then $N_{G_0}(s)$ has a subgraph isomorphic to K_5^-, the complete graph on five vertices with one edge deleted. This implies, by the optimality of (G_0, v_0), that $N_{G_0}(v_0)$ has a subgraph isomorphic to K_5^-, contradicting Lemma 4.2.

So we may assume that R_1 and R_2 have no common edge. Let the facial walk incident with $\hat{R}_1$ be $v_0, v_1, z_1, z_2, v_2, v_0$, and the facial walk incident with $\hat{R}_2$ be $v_0, v_1, z_3, z_4, v_2, v_0$. Notice, from the embedding of J, that the z_i are distinct. Also notice that s is the lone vertex in $\hat{J}$ not incident with either $\hat{R}_1$ or $\hat{R}_2$, and $N_{G_0}(s)$ includes no two disjoint pairs of non-adjacent vertices. This implies, by the optimality of (G_0, v_0), that $N_{G_0}(v_0)$ includes no two disjoint pairs of non-adjacent vertices. We shall refer to this as the DP property.

Let $N(v_0) = \{v_1, v_2, v_3, v_4, v_5\}$. Assume that some neighbor of v_0, say v_3, belongs to $\hat{R}_1$. By Lemma 2.2, v_3 is adjacent to all vertices incident with $\hat{R}_1$. Thus v_4 and v_5 belong to the closure of R_2. In either case, v_3 and v_4 are not adjacent in G_0. Since v_1 and v_2 are also not adjacent, this contradicts the DP property.

Since v_1 is not adjacent to v_2 in G_0 it follows from Lemma 4.3 that at least one of v_3, v_4, v_5 belongs to the closure of $\hat{R}_1$. Thus there remain two cases, depending on whether one or two of those vertices are incident with $\hat{R}_1$. If it is two vertices, then we may assume without loss of generality that $v_3 = z_1$ and $v_4 = z_2$. As z_1 and z_2 are not incident to $\hat{R}_2$, v_3, v_2 and v_4, v_1 are not adjacent in G_0, contrary to the DP property. Thus we may assume that $v_3 = z_1$ and v_4 and v_5 belong to the closure of $\hat{R}_2$. By the DP property v_3, v_4 and v_3, v_5 are adjacent in G_0. Thus, without loss of generality, $v_4 = z_3$ and $v_5 = z_4$. Furthermore, it follows from the DP property that either v_1, v_5 or v_2, v_4 are adjacent in G_0. Thus the subgraph L of $\hat{J}$ consisting of the vertices v_0, v_1, v_2, v_4, v_5 and edges between them that belong to the closure of $\hat{R}_2$ has five vertices and at least eight edges. We can regard L as drawn in the Möbius band with the cycle $v_1 v_0 v_2 v_5 v_4$ forming the boundary of the Möbius band. As such the graph L has at least three faces. Since the sum of the lengths of the faces is at least 11, at most one of them has length at least five. That face of L includes at most one vertex of G_0 by Lemma 2.2, and the other faces of L include none. Thus G_0 has at most nine vertices, contrary to Lemma 4.1. □

Lemma 4.5. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1 v_2}$ isomorphic to K_6. Then some face of J has length six.*

Proof. Let $\tilde{J}$ denote the graph consisting of $\hat{J}$ and edges of G_0 not in $\hat{J}$ from v_1 or v_2 to the boundary of $\hat{R}_1$ or $\hat{R}_2$ that are drawn inside $\hat{R}_1$ or $\hat{R}_2$. Let $\tilde{R}_1$ be the face in $\tilde{J}$ that contains v_0 and is contained in R_1, and similarly for $\tilde{R}_2$. We assume for a contradiction that no face of J has length six. By Lemma 4.4 the embedding of J is 2-cell, and so, by Euler's formula, all faces of J are bounded by triangles, except for either three faces of length four, or one face of length four and one face of length five. Each face of $\tilde{J}$ other than $\tilde{R}_1$ and $\tilde{R}_2$ will be called *special* if it has length at least four. Thus there are at most three special faces, and if there are exactly three, then they have length exactly four.

Let us denote the vertices on the boundary of $\tilde{R}_1$ as v_1, v_0, v_2, $u_1, \ldots, u_k$ in order, and let the vertices on the boundary of $\tilde{R}_2$ be v_2, v_0, v_1, $z_1, \ldots, z_l$ in order. Note that $u_1, u_2, \ldots, u_k$ are pairwise distinct, and similarly for $z_1, z_2, \ldots, z_l$. A special face of length five may include a vertex of G_0 in its interior; such vertex will be called *special*. It follows that there is at most one special vertex. An edge of G_0 is called *special* if it has both ends in $\hat{J} \backslash v_0$, but does not belong to $\hat{J}$, and is not $v_1 z_1$ or $v_2 z_1$ if $l = 1$, and is not $v_1 u_1$ or $v_2 u_1$ if $k = 1$. It follows that every special edge is incident with v_1 or v_2. Furthermore, the multigraph obtained from G_0 by deleting all vertices in the faces $\tilde{R}_1$ and $\tilde{R}_2$ and contracting the edges $v_0 v_1$ and $v_0 v_2$ has J as a spanning subgraph, and each special edge belongs to a face of J of length at least four. It follows that there are at most three special edges. Furthermore, if there is a special vertex, then there is at most one special edge, and each increase of k or l above the value of two decreases the number of special edges by one.

Since R_1 and R_2 have length three, four, or five, we deduce that $k, l \in \{1, 2, 3, 4\}$. The graph $\hat{J} \backslash \{v_0, v_1, v_2\} = J \backslash z_0$ is isomorphic to K_5, and $u_1, u_2, \ldots, u_k$ are its distinct vertices; let $u_{k+1}, \ldots, u_5$ be the remaining vertices of this graph. It follows that if c is a 5-coloring of $\tilde{J}$ and $c(u_i) = c(z_j)$, then $u_i = z_j$. We will refer to this property as *injectivity*. From the symmetry we may assume that $k \geqslant l$. Since J has at most one face of length five, it follows that $l \leqslant 3$. We distinguish three cases depending on the value of l.

Case 1: $l = 1$.

By Lemma 4.3 the vertex v_0 is adjacent to z_1. Also notice then that $v_1 z_1 v_2 u_1 u_2 \ldots u_k$ is a null-homotopic walk W of length at most seven. Since G_0 is 6-critical, the graph $G \backslash v_0$ has a 5-coloring, say c. By Lemma 2.2 applied to the subgraph L of G_0 drawn in the disk bounded by W and the coloring c, the graph L satisfies one of (i)–(vi) of that lemma. We discuss those cases separately.

Case (i): There are eight vertices in $\tilde{J}$ and none in the interior of $\tilde{R}_1$ and $\tilde{R}_2$, and at most one special vertex. Thus $|V(G_0)| \leqslant 9$, contradicting Lemma 4.1.

Case (ii): As before $|V(G_0)| \leqslant 9$, a contradiction, unless there exists a special vertex v_0'. This implies that $|\tilde{R}_1| = |\hat{R}_1| = 6$. Without loss of generality suppose v_0 is adjacent to u_3, v_1, z_1, v_2 and a vertex v_3 which is adjacent to v_0, v_2, u_1, u_2, u_3. Notice that v_0' must have degree five in G_0 and its neighborhood must contain a subgraph isomorphic to $K_5 - P_3$, since four of its neighbors are in $J \backslash z_0$ and thus form a clique. Meanwhile the neighborhood of v_0 is missing the edges $v_1 v_2$, $v_1 v_3$, and $v_2 u_3$. The last one does not belong to $\tilde{J}$, does not lie in $\tilde{R}_1$ (because we have already described the graph therein), and is not special, because all special edges have been accounted for. Thus the pair (G_0, v_0') contradicts the optimality of (G_0, v_0).

Case (iii): The graph $L \backslash W$ consists of three pairwise adjacent vertices, and v_0 is one of them. Let v_3, v_4 be the remaining two. By Lemma 4.3 we may assume, using the symmetry that exchanges v_1, v_4, u_1, u_2 with v_2, v_3, u_k, u_{k-1}, that v_3 has neighbors v_0, v_2, u_1, u_2, v_4 and v_4 is adjacent to

v_1, v_0, v_3, u_2 and either u_3 or u_4. In either case z_1 and u_2 are colored the same, and hence they are equal by injectivity. To be able to treat both cases simultaneously, we swap u_3 and u_4 if necessary; thus we may assume that v_4 is adjacent to u_3. We can do this, because we will no longer use the order of $u_1, u_2, \ldots, u_k$ for the duration of case (iii). The vertex v_1 is adjacent to u_2, u_3, u_4, u_5, for otherwise its color can be changed, in which case the coloring c could be extended to L, contrary to the fact that G_0 has no 5-coloring. Similarly, v_2 is adjacent to u_1, u_2, u_4, u_5. It follows that G_0 has a subgraph isomorphic to L_3, a contradiction. To describe the isomorphism, the vertices corresponding to the top row of vertices in Fig. 2(c) in left-to-right order are u_1, u_4, u_5, u_3, the vertices corresponding to the middle row are $v_3, v_2, u_2 = z_1, v_1, v_4$, and the bottom vertex is v_0. This completes case (iii).

Cases (iv)–(vi): We have $k = 4$. Hence R_1 has length five, and therefore there is at most one special edge. Consequently, one of v_1, v_2 is not adjacent in $\tilde{J}$ to at least two vertices among u_1, u_2, u_3, u_4. Since every face of $\tilde{J}$ except $\tilde{R}_1$ and one other face of length four is bounded by a triangle this implies that in the coloring c, one of v_1, v_2 sees at most three colors. From the symmetry we may assume that v_2 has this property. Thus $c(v_2)$ may be changed to a different color.

By using this fact and examining the cases (iv)–(vi) of Lemma 2.2 we deduce that L is isomorphic to the graph of case (iv). Let the vertices of L be numbered as in Fig. 3(iv). It further follows that $v_2 = x_4$ or $v_2 = x_5$, and so from the symmetry we may assume the former. Since z_1 has a unique neighbor in $L \backslash W$ we deduce that $z_1 = x_3$, $v_1 = x_2$, $u_4 = x_1$ and so on. Notice that x_8 has degree five in G_0 and that its neighborhood is isomorphic to $K_5 - P_3$. Meanwhile, the neighborhood of v_0 is certainly missing the edges v_1v_2 and v_1x_9. Now if $x_3 \neq x_5$ then x_3 is not adjacent to x_9 and $N(v_0)$ is missing at least three edges, a contradiction to the optimality of (G_0, v_0), given the existence of x_8. So $x_3 = x_5$, but then the edges x_3v_2, x_5v_2 are actually the same edge, because $\tilde{J}$ does not have parallel edges. It follows that v_2 has degree at most four in G_0, a contradiction.

Case 2: $l = 2$.

By Lemma 4.3 either v_0 is adjacent to both z_1 and z_2, in which case we define $\bar{v}_0 := v_0$, or there exists a vertex $\bar{v}_0$ in $\tilde{R}_2$ adjacent to v_0, v_1, v_2, z_1, z_2. Let W denote the walk $v_1\bar{v}_0v_2u_1 \ldots u_k$ of length at most seven, and let X denote the set of vertices of G_0 drawn in the open disk bounded by W. We claim that $X \neq \emptyset$. This is clear if $\bar{v}_0 \neq v_0$, and so we may assume that $\bar{v}_0 = v_0$. But then $X = \emptyset$ implies $|V(G_0)| \leqslant 9$, contrary to Lemma 4.1. Thus $X \neq \emptyset$. Let $x \in X$ have the fewest number of neighbors on W. Since G_0 is 6-critical, the graph $G_0 \backslash x$ has a 5-coloring, say c. By Lemma 2.2 applied to the subgraph L of G_0 drawn in the disk bounded by W and the coloring c, the graph L and coloring c satisfy one of (i)–(vi) of that lemma.

Suppose first that L and c satisfy (i). Then $|X| = 1$ by the choice of x. As before $|V(G_0)| \leqslant 9$, contradicting Lemma 4.1, unless there is a special vertex. Hence $k \leqslant 3$. If $k = 3$, then R_1 has length four, and all special faces have been accounted for. In particular, $\tilde{J} = \hat{J}$. The fact that the coloring c cannot be extended to L implies that $\{c(z_1), c(z_2)\} \subseteq \{c(u_1), c(u_2), c(u_3)\}$, and hence $\{z_1, z_2\} \subseteq \{u_1, u_2, u_3\}$ by injectivity. Thus u_1 or u_3 is equal to one of z_1, z_2. Since there are no special edges, either u_1v_2 and z_2v_2, or u_kv_1 and z_1v_1 are the same edge, but then v_1 or v_2 has degree at most four, a contradiction. If $k = 2$ we reach the same conclusion, using the fact that in that case there is at most one special edge. It follows that L and c do not satisfy (i).

Next we dispose of the case $k \leqslant 3$. To that end assume that $k \leqslant 3$. Then W has length at most six. Thus L and c satisfy either (ii) or (iii) of Lemma 2.2, and so W has length exactly six and $k = 3$. In particular, R_1 has length four, and so there is either at most one special vertex, or at most two special edges, and not both. It follows that either $c(v_1)$ or $c(v_2)$ can be changed without affecting the colors of the other vertices of $G_0 \backslash X$. That implies that L and c satisfy (ii). Let v_3 be the unique neighbor of $\bar{v}_0$ in X, and let v_4 be the other vertex of X. From the symmetry we may assume that v_3 is adjacent to $\bar{v}_0, v_1, v_2, u_1, v_4$, and v_4 is adjacent to v_1, v_3, u_1, u_2, u_3. By considering the walk $u_1u_2u_3v_1z_1z_2v_2$ and the subgraph drawn in the disk it bounds, and by applying Lemma 2.2 to this graph and the coloring c we deduce that $|\{c(u_1), c(u_2), c(u_3)\} \cap \{c(z_1), c(z_2)\}| = 1$. That implies $|\{u_1, u_2, u_3\} \cap \{z_1, z_2\}| = 1$ by injectivity, and so we may assume that u_5 is not equal to z_1 or z_2. It follows that the neighborhood of u_5 has a subgraph isomorphic to $K_5 - P_3$. However, the neighborhood of $\bar{v}_0$ is missing v_1v_2 and at least one of the edges v_3z_1 and v_3z_2, contrary to the optimality of (G_0, v_0) if $v_0 = \bar{v}_0$. Similarly, the neighborhood of v_3 is missing v_1v_2 and $\bar{v}_0v_4$, a contradiction if $v_0 = v_3$. This completes the case $k \leqslant 3$.

Thus we may assume that $k = 4$. It follows that R_1 has length five, and hence there is at most one special edge. Let $i \in \{1, 2\}$. If v_i is adjacent to both z_1 and z_2, then one of the edges v_iz_1, v_iz_2 is special. It follows that in G_0, either v_1 is not adjacent to z_2, or v_2 is not adjacent to z_1. But z_2 is the only neighbor of v_1 in $G_0\backslash X$ colored $c(z_2)$, because $G_0\backslash(X \cup \{v_0, v_1, v_2\})$ is isomorphic to $J\backslash z_0$, which, in turn, is isomorphic to K_5. Thus there is a (proper) 5-coloring c_1 of $G_0\backslash X$ obtained by changing the color of at most one of the vertices v_1, v_2 such that either $c_1(v_1) = c_1(z_2)$ or $c_1(v_2) = c_1(z_1)$. Now $c_1(\bar{v}_0)$ can be changed to another color, thus yielding a coloring c_2 of $G_0\backslash X$.

If L and c satisfy one of the cases (iii)–(vi), then one of the colorings c_1, c_2 extends into L, a contradiction. Thus L and c satisfy (ii) of Lemma 2.2. Let $v_3 \in X$ be the unique vertex of X adjacent to $\bar{v}_0$, and let v_4 be the other vertex in X. If both v_3 and v_4 have degree five in G_0, then one of the colorings c_1, c_2 extends into L, a contradiction. Thus one of v_3, v_4 has degree five, and the other has degree six. It follows that v_3 is adjacent to v_1, v_2, and either u_1 or u_4, and so from the symmetry we may assume it is adjacent to u_1. If $c_1(v_1) = c_1(u_1)$, then we can extend one of the colorings c_1, c_2 into L by first coloring v_4 and then v_3. Thus $c_1(v_1) \neq c_1(u_1)$. If v_4 is not adjacent to u_1, then we can extend c_1 or c_2 by giving v_4 the color $c_1(u_1)$, and then coloring v_3. Thus v_4 is adjacent to v_1. If v_4 has degree five, then its neighbors are u_1, u_2, u_3, u_4, v_3, and the neighbors of v_3 are $\bar{v}_0, v_1, v_2, u_1, u_4, v_4$. Let d be a 5-coloring of $G_0\backslash\bar{v}_0$. Since the coloring d cannot be extended to $\bar{v}_0$, it follows that the neighbors of $\bar{v}_0$ receive different colors. Now similarly as in the construction of c_1 above, we can change either the color of v_1, or the color of v_2. The resulting coloring then extends to $\bar{v}_0$, a contradiction. This completes the case when v_4 has degree five, and hence v_4 has degree six. It follows that the neighbors of v_4 are $u_1, u_2, u_3, u_4, v_1, v_3$ and the neighbors of v_3 are $\bar{v}_0, v_1, v_2, u_1, v_4$. Let d_1 be a 5-coloring of the graph $G_0\backslash\{\bar{v}_0, v_3\}$. Since the coloring d_1 does not extend into $\bar{v}_0, v_3$, we deduce that $\{d_1(z_1), d_1(z_2)\} = \{d_1(v_4), d_1(u_1)\}$. By injectivity this implies that $u_1 = z_1$ or $u_1 = z_2$. If $u_1 = z_2$, then one of the edges v_2u_1, v_2z_2 is special, because they cannot be the same edge, given that v_2 has degree at least five in G_0. Thus all special edges have been accounted for, and so z_1 is not adjacent to u_1. Thus $d_1(v_1)$ can be changed to $d_1(u_1)$, and the new coloring extends to all of G_0, a contradiction. Thus $u_1 = z_1$. It follows that G_0 is isomorphic to L_3. First of all, the vertex v_1 is not adjacent to both u_2 and u_3, for otherwise the vertices $v_1, v_4, u_1, u_2, u_3, u_4$ form a K_6 subgraph in G_0. If v_1 is adjacent to neither u_2 nor u_3, then v_2 is adjacent to these vertices, and an isomorphism between G_0 and L_3 is given by mapping the vertices in the top row in Fig. 2(c), in left-to-right order, to u_4, u_2, u_3, u_5, the middle row to $v_1, v_4, u_1 = z_1, v_2, \bar{v}_0$ and the bottom vertex to v_3. If v_1 is adjacent to exactly one of u_2, u_3, then due to the symmetry in the forthcoming argument we may assume that v_1 is adjacent to u_3, and hence v_2 is adjacent to u_2. Then an isomorphism is given by mapping the top row to v_4, u_4, u_3, u_2, the middle row to $v_3, v_1, u_1 = z_1, u_5, v_2$, and mapping the bottom vertex to $\bar{v}_0$. This completes the case $l = 2$.

Case 3: $l = 3$.

Lemma 4.3 implies that v_0 has at most one neighbor among $\{z_1, z_2, z_3, u_1, u_2, \dots, u_k\}$, and such neighbor must be u_1, u_k, z_1, or z_3.

We claim that either v_0 is adjacent to z_1 or z_3, or $k = 3$ and v_0 is adjacent to u_1 or u_3. To prove this claim let us assume that v_0 has no neighbor among $\{z_1, z_2, z_3\}$. Let C be the cycle $v_1z_1z_2z_3v_2v_0$, and let X denote the set of vertices of G_0 drawn in the open disk bounded by C. We have $X \neq \emptyset$ by Lemma 4.3. Let c be a coloring of $G\backslash X$, and let L denote the subgraph of G_0 consisting of all vertices and edges drawn in the closed disk bounded by C. By Lemma 2.2 the graph L satisfies one of the conditions (i), (ii), (iii) of that lemma. The vertices z_1 and z_3 are adjacent, because the graph obtained from $\hat{J}$ by deleting v_0, v_1, v_2 and the vertices drawn in the faces $\tilde{R}_1$ or $\tilde{R}_2$ is isomorphic to K_5. We may also assume, by the symmetry between v_1 and v_2, that v_1 is adjacent to z_2. We claim that we may assume that the neighborhood of v_0 is a 5-cycle. This is clear if v_0 has no neighbor in $\{u_1, u_2, u_3, u_4\}$, and so we may assume that v_0 is adjacent to u_1. Then we may assume that $k = 4$, for otherwise the claim we are proving holds. Thus there is no special edge. By Lemma 4.3 there exists a vertex inside $\tilde{R}_1$ adjacent to v_0, v_1, u_1. Since there is no special edge the vertex v_1 is not adjacent to u_1, and u_1 is not adjacent to z_1, because v_2 has degree at least five in G_0. It follows that the neighborhood of v_0 is indeed a 5-cycle. If $|X| \geqslant 2$, then there exists a vertex in X whose neighborhood has a subgraph that is a 5-cycle plus at least one additional edge, namely z_1z_3 or v_1z_2. That contradicts the optimality of (G_0, v_0). Thus $|X| = 1$. Let x denote the unique element of X, and let us assume first that $k = 4$. Then there are no special edges, and so v_1 is not adjacent to z_3 and v_2 is not adjacent to z_1. Let C' denote the cycle $v_1xv_2u_1u_2u_3u_4$, and let X' be the set of vertices of G_0 drawn in the open disk bounded by C'. Then $G_0 \backslash (X' \cup \{x\})$ has a 5-coloring c' such that $c'(v_1) = c'(z_3)$ and $c'(v_2) = c'(z_1)$. Then c' can be extended to x in at least two different ways. By Lemmas 2.2 and 4.3 the coloring c' can be extended to all of G_0, unless (up to symmetry between v_1 and v_2) v_0 is adjacent to u_1, there exists a vertex y adjacent to u_1, u_2, u_3, u_4 and $c'(v_1) = c'(u_5)$. But v_1 is not adjacent to u_1 (because v_2 is and there are no special edges), and hence the color of v_1 can be changed to $c'(u_1)$, and the resulting coloring extends to all of G_0, a contradiction. This completes the case $k = 4$. Thus $k = 3$, and so there is at most one special edge. Let c'' be a 5-coloring of $G_0 \backslash X'$. It follows that the color of at least one of the vertices v_1, v_2 can be changed to a different color, without affecting the colors of the other vertices of $G \backslash X'$. It follows from Lemma 2.2 that $|X'| \leqslant 2$. That, in turn, implies that v_0 is adjacent to u_1 or u_3, and hence proves our claim from the beginning of this paragraph.

Thus we may assume that v_0 is adjacent to z_3. By Lemma 4.3 there exists a vertex v_3 adjacent to v_0, v_1, z_1, z_2, z_3 and a vertex v_4 in $\tilde{R}_1$ that is adjacent to v_0, v_1, v_2. The neighborhood of v_3 includes the edge z_1z_3, and so by the optimality of (G_0, v_0) the neighborhood of v_0 includes the edge v_4z_3. Thus $z_3 \in \{u_1, u_2, u_3, u_4\}$. Assume first that $k = 4$. Then there are no special edges, and hence $z_3 \neq u_4$. Next we deduce that $z_3 \neq u_1$, for otherwise v_2u_1 and v_2z_3 are the same edge, which implies (given that $z_3 = u_1$ is adjacent to v_4) that v_2 has degree at most three, a contradiction. Thus $z_3 \in \{u_2, u_3\}$. Let Y consist of v_0 and all vertices in $\tilde{R}_1$ or $\tilde{R}_2$. Since z_3 is adjacent to v_4 we deduce that $|Y| \leqslant 4$. Since there are no special edges, z_3 is not adjacent to v_1, and v_2 is not adjacent to u_4. Thus $G_0 \backslash Y$ has a coloring d such that $d(v_1) = d(z_3)$ and $d(v_2) = d(u_4)$. Since $z_3 \in \{u_2, u_3\}$ this coloring can be extended to the vertices drawn in $\tilde{R}_1$, and since $d(v_1) = d(z_3)$ it can be further extended to v_0 and v_3, a contradiction.

Thus $k = 3$. Let W denote the walk $v_1v_3z_3v_2u_1u_2u_3$, and let d' be a 5-coloring of $G_0 \backslash (Y - \{v_3\})$. We now apply Lemma 2.2 to the graph drawn in the closed disk bounded by W and coloring d', and note that either the color of each of v_1, v_2 can be changed to a different color, independently of each other and independently of the colors of other vertices, except possibly v_3, or the color of one of v_1, v_2 can be changed to two different values. In either case, one of the resulting colorings extends to G_0, a contradiction. □

Lemma 4.6. *Let (G_0, v_0) be an optimal pair, let v_1, v_2 be an identifiable pair, and let J be a subgraph of $G_{v_1v_2}$ isomorphic to K_6. Then the drawing of J in the Klein bottle does not have a facial walk of length six.*

Proof. Suppose for a contradiction that there exists a subgraph J of $G_{v_1v_2}$ isomorphic to K_6 such that the drawing of J in the Klein bottle has a face F_0 bounded by a walk W of length six. Let the vertices of J be $z_1, z_2, \dots, z_6$. Since K_7 cannot be embedded in the Klein bottle, it follows that W has a repeated vertex. If W has exactly one repeated vertex, then (since J is simple) we may assume that the vertices on W are $z_6, z_2, z_4, z_6, z_3, z_5$, in order. There exists a closed curve ϕ passing through z_6 and otherwise confined to F_0 such that there is an edge of J on either side of ϕ in a neighborhood of z_6. The curve ϕ cannot be separating, because $G_0 \backslash z_0$ is connected, and it cannot be 2-sided, because $G_0 \backslash z_0$ is not planar. It follows that ϕ is 1-sided. By Euler's formula every face of J other than F_0 is bounded by a triangle. It follows that the triangles $z_4z_5z_6$, $z_1z_6z_3$, and $z_1z_6z_2$ bound faces of J. Furthermore, either $z_3z_5z_2$ or $z_3z_5z_4$ is a face, but since J is simple we deduce that it is the former. It follows that $z_1z_3z_4$, $z_2z_3z_4$, $z_1z_2z_5$ and $z_1z_4z_5$ are faces of J, and those are all the faces of J. The drawing of J is depicted in Fig. 6, where diagonally opposite vertices and edges are identified, and the asterisk indicates another cross-cap.

Similarly, if W has at least two repeated vertices, then it has exactly two, and we may assume that the vertices of W are $z_6z_5z_4z_6z_2z_4$. Similarly as in the previous paragraph, the embedding is now uniquely determined, and is depicted in Fig. 7.

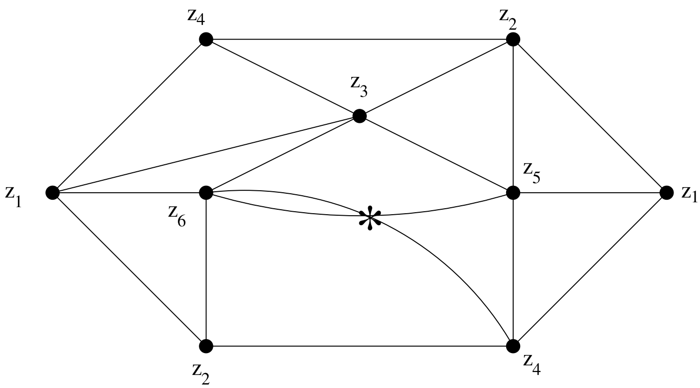

Fig. 6. An embedding of K_6 with a facial walk on five vertices.

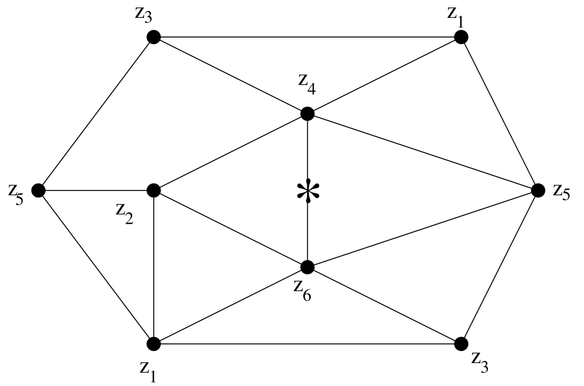

Fig. 7. An embedding of K_6 with a facial walk on four vertices.

In either case let R_1 and R_2 be the hinges of J, and let F_{ijk} denote the facial triangle incident with z_i, z_j, z_k if it exists. We should note that specifying the hinges does not uniquely determine the graph $\hat{J}$, because the face F_0 has multiple incidences with some vertices. For instance, if W has five vertices, $z_0 = z_6$, and $R_1 = F_0$, then it is not clear whether the split occurs in the "angle" between the edges z_3z_6 and z_4z_6, or in the angle between z_5z_6 and z_2z_6. To overcome this ambiguity we will write $R_1 = F_{364}$ in the former case, and $R_1 = F_{265}$ in the latter case. Notice that this is just a notational device; there is no face bounded by $z_3z_6z_5$ or $z_2z_6z_4$. We proceed in a series of claims.

(1) *Not both R_1 and R_2 are bounded by triangles.*

To prove (1) suppose for a contradiction that R_1 and R_2 are both facial triangles. Let us recall that z_0 is the vertex of $G_{v_1v_2}$ that results from the identification of v_1 and v_2. Suppose first that R_1 and R_2 are consecutive in the cyclic order around z_0. Then v_0 and one of v_1 or v_2 is in the interior of a 4-cycle in G_0, contrary to Lemma 2.2. Similarly, if the cyclic order around z_0 has R_1 followed by a facial triangle, followed by R_2, then there would be two vertices in the interior of a 5-cycle in G_0, contrary to Lemma 2.2. In addition, if the cyclic order has R_1, followed by two facial triangles, followed by R_2, then there are two vertices inside a 6-cycle. Hence, we are in either case (ii) or (iii) of Lemma 2.2. However, the boundary has five vertices that form a clique. So 5-color all but the interior of this 6-walk (using that G_0 is 5-critical); the boundary must have five colors, contrary to Lemma 2.2. We conclude that R_1 and R_2 must have F_0 in between them in the cyclic order around z_0, on both sides. In particular, W has five vertices.

Thus the only case remaining is that $z_0 = z_6$, where J is embedded with a facial 6-walk on five vertices. Suppose without loss of generality that $R_1 = F_{126}$ and $R_2 = F_{456}$, and that v_2 is adjacent to z_1, z_3 and z_4. Then the faces of the subgraph induced by $v_1, v_2, z_1, z_2, z_3, z_4, z_5$ are all triangles but perhaps for two six-cycles: $v_1, z_2, z_1, v_2, z_4, z_5$ and $v_1, z_5, z_3, v_2, z_4, z_2$. Since v_0 is adjacent to v_1 and v_2 it follows from Lemma 2.2 that the only vertex in G_0 in the interior of the first six-cycle is v_0. Hence there must be at least two vertices in the interior of the other six-cycle, else $|V(G_0)| \leqslant 9$, a contradiction. Thus we are in either case (ii) or (iii) of Lemma 2.2. Note that the disk bounded by the second cycle includes no chord. So v_1 is not adjacent to z_3. Now if v_1 is not adjacent to z_1, we color G_0 as follows. Let the color of z_i be i. Color v_1 with color 1. Then color v_0 and v_2, and extend the coloring to the interior of the second six-cycle by Lemma 2.2. Hence we may assume that v_1 is adjacent to z_1. But then v_0 is adjacent to z_1, z_4, z_5 while v_1 is not adjacent to z_4. Now v_1 may be colored either 3 or 4. One of these options extends to the interior of the second six-cycle after we color v_1, v_0, v_2 in that order. This proves claim (1).

In light of (1) we may assume that $R_1 = F_0$.

(2) *If R_2 is bounded by a triangle, then it is not consecutive with F_0 in the cyclic order around z_0 in J.*

To prove (2) suppose for a contradiction that R_2 is bounded by a triangle and that it is consecutive with F_0 in the cyclic order around z_0 in J. It follows that one of v_1, v_2 has degree two in $\hat{J}$, and so we may assume that it is v_1 and that its neighbors are v_0 and z_j. Consider the subgraph $\hat{J} \backslash \{v_0, v_1\}$. All of its faces are triangles but for a 7-walk. We 5-color this subgraph, which is isomorphic to K_6 minus an edge. Thus v_2 must receive the same color as z_j. Since this subgraph only has six vertices, the interior of the 7-walk must be as in case (v) or (vi) of Lemma 2.2, for otherwise there would be at most nine vertices in G_0, contrary to Lemma 4.1. Consider the edge $z_0 z_j$ in J, which must be on the boundary of F_0. Now the vertex or vertices not on the boundary of F_0 must be on the boundary of R_2, for otherwise the 7-walk would only have four colors and we could extend the 5-coloring to its interior, a contradiction. Since R_2 is a facial triangle this means that either z_0 or z_j is z_6 and that W has five vertices. However, then the color of z_0 and z_j appears three times on the boundary of the 7-walk. So the 5-coloring may also be extended, a contradiction. This proves (2).

By an *s-vertex* we mean a vertex $s \in V(G_0)$ of degree five such that $N_{G_0}(s)$ has a subgraph isomorphic to $K_5 - P_3$. If G_0 has an s-vertex, then the optimality of (G_0, v_0) implies that $N_{G_0}(v_0)$ does not include two disjoint pairs of non-adjacent vertices.

(3) Let R_2 be bounded by a triangle; then $\hat{R}_2$ is bounded by a pentagon, say $v_0 v_1 r_1 r_2 v_2$. Assume further that G_0 has an s-vertex. Then either
 (a) $\hat{R}_2$ includes a unique vertex v of G, and v is adjacent to v_0, r_1, r_2 and all neighbors of v_0 other than v, or
 (b) v_0 is adjacent to r_1, r_2, and r_1, r_2 are adjacent to the neighbor of v_0 other than v_1, v_2, r_1, r_2, or
 (c) v_0, v_1, v_2 are all adjacent to r_i for some $i \in \{1, 2\}$, and r_i is adjacent to the two neighbors of v_0 other than v_1, v_2, r_i.

To prove (3) we first notice that $\hat{R}_2$ includes at most one vertex of G_0 by Lemma 2.2. If it includes exactly one vertex, then (a) holds by the existence of an s-vertex, and the optimality of (G_0, v_0). If $\hat{R}_2$ includes no vertex of G_0, then by Lemma 4.3 either v_0 is adjacent to both r_1 and r_2, or v_0, v_1, v_2 are all adjacent to r_i for some $i \in \{1, 2\}$. We deduce from the existence of an s-vertex and the optimality of (G_0, v_0) that either (b) or (c) holds. This proves (3).

(4) *The walk W has five vertices.*

To prove (4) we suppose for a contradiction that W has four vertices. Suppose first that $z_0 = z_2$. Then by (2) and the symmetry we may assume that $R_2 = F_{125}$. It follows that z_3 is an s-vertex, and so we may apply (3). But (a) does not hold, because in that case v_0 has four neighbors in $\hat{R}_1$ or on its boundary, and not all of them can be adjacent to the neighbor of v_0 in $\hat{R}_2$. If (b) holds, then v_0 is adjacent to z_1 and z_5, and v is adjacent to z_1, where v is the neighbor of v_0 other than v_1, v_2, z_1, z_5. Now $v \neq z_5$, because otherwise both $\hat{R}_1$ and $\hat{R}_2$ include an edge joining v_0 and z_5, contrary to the fact that G_0 is simple. Since v is adjacent to z_1 we deduce that $v = z_4$ or $v = z_6$. In either case Lemma 4.3 implies that v_1 or v_2 has degree at most four, a contradiction.

Thus we may assume that (c) holds, and so v_0, v_1, v_2 are all adjacent to z_1 or z_5. In the former case we can change notation so that $R_2 = F_{126}$, contrary to (2). Thus v_0, v_1, v_2 are all adjacent to z_5. Let v_1 be adjacent to z_3, z_4, z_5; then v_2 is adjacent to z_1, z_5, z_6. Let the vertices v_2, z_5, v_1, v_4, v_5 form the wheel neighborhood of v_0, in order. Since an s-vertex exists, the optimality of (G_0, v_0) implies that either v_1 is adjacent to v_5, or v_2 is adjacent to v_4, or both. We may assume from the symmetry that v_1 is adjacent to v_5. Since v_5 is adjacent to z_5 by (c), we deduce that $v_5 = z_4$ or $v_5 = z_6$, because $v_5 \neq z_5$ for the same reason as above. If $v_5 = z_6$, then v_2z_6 and v_2v_5 are the same edge, and it follows from Lemma 4.3 that v_2 has degree at most four. Thus $v_5 = z_4$. It follows that v_2 is adjacent to z_4, and hence the neighborhood of z_1 has a subgraph isomorphic to K_5^-, contrary to Lemma 4.2. This completes the case $z_0 = z_2$.

Thus by symmetry we may assume that $z_0 = z_4$. Again by symmetry we may assume that $R_1 = F_{246}$ and R_2 is either F_{134} or F_{145}. Assume first that $R_2 = F_{145}$. Let v_1 be adjacent to z_1, z_2, z_3. Then z_3 is an s-vertex, and so we may use (3). If (a) holds, and v is as in (a), then it is not possible for v to be adjacent to all neighbors of v_0 other than v, a contradiction. If (b) holds, then v_2 is not adjacent to z_1, and hence v_1 is adjacent to z_5, by the optimality of (G_0, v_0), because an s-vertex exists. Thus the neighborhood of z_3 in G_0 has a subgraph isomorphic to K_5^-, contrary to Lemma 4.2. Thus (c) holds. If v_0, v_1, v_2 are adjacent to z_5, then $N_{G_0}(z_3)$ has a subgraph isomorphic to K_5^-, contrary to Lemma 4.2. Hence v_0, v_1, v_2 are adjacent to z_1. By (c) the vertex z_1 is adjacent to v_4, v_5, the two neighbors of v_0 other than v_1, v_2, z_1. It follows that $\{v_4, v_5\} \subseteq \{z_2, z_5, z_6\}$. However, if v_0 is adjacent to z_2, then v_1 would be of degree at most four in G_0, a contradiction. Thus v_0 is adjacent to z_5 and z_6; hence v_1 is adjacent to z_5 by Lemma 4.3. Now the graph has eight vertices and perhaps one more inside the 5-cycle $v_1z_2z_6v_2z_5$. Hence G_0 has at most nine vertices, contrary to Lemma 4.1. This completes the case $R_2 = F_{145}$.

We may therefore assume that $R_2 = F_{246}$. From the symmetry we may assume that v_1 is adjacent to z_2 and z_3. If $\hat{R}_2$ includes an edge incident with v_1 or v_2, then Lemma 4.3 implies that v_0, v_1, v_2 are all adjacent to z_1 or z_3. Then we may change our notation so that either $R_2 = F_{145}$ or $R_2 = F_{234}$. In the former case we get a contradiction by the result of the previous paragraph, and in the latter case we get a contradiction by (2). Thus $\hat{R}_2$ includes no edge incident with v_1 or v_2. Hence either v_0 is adjacent to z_1 and z_3, or v_0 is adjacent to an internal vertex v_3 of degree five which is adjacent to z_1 and z_3. In either case there is a vertex of degree five in G_0 adjacent to v_1, z_3, z_1, and v_2. For this vertex, z_3, v_2 is an identifiable pair. Note that $G_{z_3v_2}$ is not 5-colorable. We 5-color the vertices z_1, z_2, $v_2 = z_3$, z_5, z_6 so that each gets a unique color. Then this coloring extends to $G_{z_3v_2}$, unless we are in case (ii) of Lemma 2.2 for the following walk on six vertices: $z_5, v_2 = z_3, z_6, z_2, v_2 = z_3, z_6$ in $G_{z_3v_2}[\{z_1, z_2, v_2 = z_3, z_4, z_5, z_6\}]$. This implies that there are two adjacent vertices w_1 and w_2 such that, in G_0, w_1 is adjacent to z_2, z_6, v_2, and z_5, while w_2 is adjacent to z_6, z_5, z_2, and one of v_2, z_3. But then the subgraph induced by the eight vertices: $z_1, z_2, z_3, z_5, z_6, v_2, w_1, w_2$ has all facial triangles except for perhaps one 5-cycle. Yet there can be at most one vertex in the interior of that 5-cycle. Thus G_0 has at most nine vertices, a contradiction. This proves (4).

(5) $z_0 \neq z_2, z_3$.

We may assume to a contradiction that $z_0 = z_2$ since the case where $z_0 = z_3$ is symmetric. By (2) $R_2 = F_{125}$ or F_{235}. Suppose first that some edge of G_0 is incident with v_1 or v_2 and lies inside $\hat{R}_2$. Then v_0, v_1, and v_2 are all adjacent to z_5, for otherwise we may change our notation so that $\hat{R}_2 = F_{126}$, contrary to (2). Let v_4 and v_5 be neighbors of v_0 such that the cyclic order around v_0 is v_1, z_5, v_2, v_5, v_4. Now notice that z_1 is degree five in G_0 and $N_{G_0}(z_1)$ has a subgraph isomorphic to $K_5 - P_3$. Since $N_{G_0}(z_0)$ is missing the edge v_1v_2, one of the edges v_1v_5, v_2v_4 must be present or z_1 would contradict the choice of v_0. This implies that v_1 and v_2 are both adjacent to v_j for some $j \in \{4, 5\}$. Thus the edges v_1v_j, v_2v_j must go to a repeated vertex on the boundary of R_1 or v_0 would be in a four-cycle in G_0, a contradiction. Thus $v_j = z_6$ and the edge v_2z_6 is already present. The edge v_1z_6

then implies that z_4 is degree five in G_0 and that $N_{G_0}(z_4)$ has a subgraph isomorphic to K_5^-, contrary to Lemma 4.2. Thus $\hat{R}_2$ includes no edge of G_0 incident with v_1 or v_2.

Now suppose that $R_2 = F_{125}$. We may assume that v_1 is adjacent to z_3, z_4, z_5. Then either the cyclic order around v_0 is v_1, z_5, z_1, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_5, z_1, v_2, v_0. In either case, z_1v_1 is an identifiable pair for a vertex of degree five in G_0. Note that $G_{v_1z_1}$ is not 5-colorable. We 5-color the vertices $v_1 = z_1$, z_3, z_4, z_5, z_6 of $G_{v_1z_1}$ such that each gets a unique color. Since this coloring does not extend to $G_{v_1z_1}$ we deduce from Lemma 2.2 applied to the walk z_6, $v_1 = z_1$, z_4, z_6, z_3, z_5 on six vertices that case (i) of that lemma holds. That implies there exists a vertex w_1 in G_0 that is adjacent to v_1, z_4, z_6, z_3 and z_5. Let $H := G[\{z_1, z_3, z_4, z_5, z_6, v_1, w_1\}]$. The edge w_1z_6 may be embedded in two different ways. In one way of embedding the edge the graph H has all faces bounded by triangles, except for one bounded by a 4-cycle and one bounded by a 5-cycle. But then G_0 has at most eight vertices by Lemma 2.2, contrary to Lemma 4.1. It follows that the edge w_1z_6 is embedded in such a way that all faces of H are bounded by triangles, except for one face bounded by the walk $z_6z_1z_5v_1w_1z_5$ of length six. Since G_0 has at least ten vertices by Lemma 4.1, we must be in case (iii) of Lemma 2.2 when applied to said walk. This can happen in two ways. In the first case there are pairwise adjacent vertices $a, b, c \in V(G_0)$ such that a is adjacent to z_1, z_5, z_6, the vertex b is adjacent to z_5, v_1, w_1 and c is adjacent to w_1, z_5, z_6. Now G_0 is isomorphic to L_4 by an isomorphism that maps z_3 and z_4 to the top two vertices in Fig. 2(d) (in left-to-right order), z_6 and w_1 to the vertices in the second row, z_5 to the unique vertex of degree nine, and z_1, a, c, b, v_1 to the last row of vertices in that figure. In the second case there are pairwise adjacent vertices $a, b, c \in V(G_0)$ such that a is adjacent to z_1, z_5, v_1, the vertex b is adjacent to z_5, v_1, w_1 and c is adjacent to z_1, z_5, z_6. Now G_0 is isomorphic to L_3 by an isomorphism that maps the top row of vertices in Fig. 2(c) to z_6, z_3, z_4, w_1 (again in left-to-right order), the middle row to c, z_1, z_5, v_1, b and the bottom vertex to a. Since either case leads to a contradiction, this completes the case $R_2 = F_{125}$.

It follows that $R_2 = F_{235}$. We may assume that v_2 is adjacent to z_1, z_5, z_6. Then either the cyclic order around v_0 is v_1, z_3, z_5, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_3, z_5, v_2, v_0. Note that z_1 is degree five in G_0 and $N_{G_0}(z_1)$ has a subgraph isomorphic to $K_5 - P_3$. Thus in either case, v_2z_3 is an identifiable pair for a vertex of degree five in G_0, for otherwise $N_{G_0}(z_1)$ has a subgraph isomorphic to K_5^-, a contradiction. Note that $G_{v_2z_3}$ is not 5-colorable. We 5-color the vertices z_1, $v_2 = z_3$, z_4, z_5, z_6 of $G_{v_2z_3}$ such that each gets a unique color. Since this coloring does not extend to $G_{v_2z_3}$, we deduce that the 6-walk $z_6v_2 = z_3z_4z_6v_2 = z_3z_5$ satisfies (ii) of Lemma 2.2. Thus, in G_0, there exists two adjacent vertices w_1 and w_2 such that w_1 is adjacent to z_4, z_6, z_3, and z_5, while w_2 is adjacent to z_4, z_5, z_6 and v_2. But then w_1 is degree five in G_0 and $N_{G_0}(w_1)$ has a subgraph isomorphic to K_5^-, a contradiction. This proves (5).

(6) $z_0 \neq z_4, z_5$.

To prove (6) we may assume for a contradiction that $z_0 = z_4$ since the case where $z_0 = z_5$ is symmetric. Thus $R_2 = F_{134}$ or F_{145} by (2). Assume first that $R_2 = F_{145}$, and that $\hat{R}_2$ includes no edges incident with v_1 or v_2. Then either the cyclic order around v_0 is v_1, z_1, z_5, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_1, z_5, v_2, v_0. If the edge v_1z_5 is present, then in the subgraph of G_0 induced by z_1, z_2, z_3, z_5, z_6 and v_2, there is only one face that is not bounded by a triangle or 4-cycle—the following walk on six vertices: z_5, $z_3z_6z_5z_1v_2$. Thus there are at most nine vertices in G_0 by Lemma 2.2, contrary to Lemma 4.1. Hence, in either case v_1z_5 is an identifiable pair for a vertex of degree five in G_0. Note that $G_{v_1z_5}$ is not 5-colorable. We 5-color the vertices z_1, z_2, z_3, $v_1 = z_5$, z_6 of $G_{v_1z_5}$ such that each gets a unique color. Since this 5-coloring does not extend to a 5-coloring of $G_{v_1z_5}$ we deduce that case (ii) of Lemma 2.2 holds for the following walk on six vertices: z_6, z_2, $v_1 = z_5$, z_6, z_3, $v_1 = z_5$. Thus, in G_0, there are two adjacent vertices w_1 and w_2 such that w_1 is adjacent to z_2, z_6, z_5, and z_3, while w_2 is adjacent to z_2, z_6, z_3 and v_1. But then w_1 is degree five in G_0 and $N_{G_0}(w_1)$ has a subgraph isomorphic to K_5^-, contrary to Lemma 4.2. This completes the case when $R_2 = F_{145}$ and $\hat{R}_2$ includes no edges incident with v_1 or v_2.

For the next case assume that $R_2 = F_{134}$, and again that $\hat{R}_2$ includes no edges incident with v_1 or v_2. Then either the cyclic order around v_0 is v_1, z_3, z_1, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_3, z_1, v_2, v_0. Next we dispose of the case that v_2 is adjacent to z_3. In that case we consider the subgraph of G_0 induced by z_1, z_2, z_3, z_5, z_6 and v_2. There is only one face that is not bounded by a triangle or 4-cycle—the following walk on seven vertices: $z_5z_3v_2z_1z_3z_2z_6$. We 5-color the subgraph as follows: $c(z_i) = i$ for $i = 1, 2, 3, 5$, $c(z_6) = 4$, and $c(v_2) = 2$ and apply Lemma 2.2. By Lemma 4.1 cases (v) or (vi) of Lemma 2.2 hold. Since z_2 and v_2 have the same color and z_3 is a repeated vertex it follows from Lemma 2.2 that G_0 has four vertices a, b, c, d such that d is adjacent to z_2, z_3, z_5, z_6, the vertices a, b, c form a triangle and either a is adjacent to z_1, v_2, z_3, the vertex b is adjacent to z_1, z_2, z_3, and c is adjacent to z_2, z_3, d (case (v) of Lemma 2.2), or a is adjacent to z_1, v_2, z_3, the vertex b is adjacent to v_2, z_3, d, and c is adjacent to z_2, z_3, d (case (vi) of Lemma 2.2). In the former case d is an s-vertex, and yet $v_0 = a$, c is not adjacent to z_1 and b is not adjacent to v_2, contrary to the optimality of (G_0, v_0). In the latter case G_0 is isomorphic to L_3 by a mapping that sends the top row of vertices in Fig. 2(c) to z_1, z_6, z_5, z_2 (in left-to-right order), the middle row to a, v_2, z_3, d, c and the bottom vertex to b, a contradiction. Thus v_2 is not adjacent to z_3, and hence v_2z_3 is an identifiable pair for a vertex of degree five in G_0. Note that $G_{v_2z_3}$ is not 5-colorable. We 5-color the vertices z_1, z_2, $v_2 = z_3$, z_5, z_6 of $G_{v_2z_3}$ such that each gets a unique color. Since this coloring not extend to $G_{v_2z_3}$ we deduce that case (ii) of Lemma 2.2 holds for the following 6-walk: z_6, z_2, $z_3 = v_2$, z_6, $z_3 = v_2$, z_5. However this would imply that there are two internal vertices w_1 and w_2, both adjacent to z_2 and both adjacent to z_5. But then one of them is not adjacent to $z_3 = v_2$, a contradiction. This completes both cases when $\hat{R}_2$ includes no edges incident with v_1 or v_2.

We continue the proof of (6). We have just shown that $\hat{R}_2$ includes an edge incident with v_1 or v_2. Then v_0, v_1, v_2 are all adjacent to z_1, z_3 or z_5. However, if they are all adjacent to z_3, then we can change notation so that $R_2 = F_{234}$, contrary to (2), and if they are all adjacent to z_5, then we can change notation so that $R_2 = F_{456}$, again contrary to (2). Thus v_0, v_1, v_2 are all adjacent to z_1. We may assume that the notation is chosen so that v_1 is adjacent to z_2 and z_3 while v_2 is adjacent to z_5 and z_6. Let v_4 and v_5 be neighbors of v_0 numbered so that the cyclic order around v_0 is v_2, z_1, v_1, v_4, v_5.

Next we claim that v_1 is not adjacent to z_6. Suppose it were. The triangle $z_2v_1z_6$ is null-homotopic in G_0 by Lemma 2.2 applied to the 4-cycle $z_1z_5z_6v_1$. Now consider the subgraph induced by the vertices z_1, z_2, z_3, z_5, z_6, and v_1. All of its faces are triangles but for the 7-walk $z_1z_5z_6z_3z_5z_6v_1$. We 5-color these vertices as follows: $c(z_i) = i$ for $i = 1, 3, 5$, $c(z_6) = 4$, and $c(v_1) = 5$. Now we must be in case (v) or (vi) of Lemma 2.2, for otherwise $|V(G_0)| \leqslant 9$, contrary to Lemma 4.1. Yet, since the fifth color would appear three times on the boundary, we can extend this coloring to all of G_0, a contradiction. Thus v_1 is not adjacent to z_6.

Now we claim that $v_4, v_5 \notin \{z_1, z_2, \dots, z_6\}$. To prove this claim we suppose the contrary. Then v_0 is adjacent to z_2, z_3, z_5 or z_6. If v_0 is adjacent to z_2, then v_1 has degree at most four in G_0. If v_0 is adjacent to z_6, then either v_2 is degree four in G_0, a contradiction, or v_1 is adjacent to z_6, a contrary to the previous paragraph. If $v_4 = z_3$, then the 5-cycle $v_1z_3z_6z_5z_1$ has the vertices v_0 and v_2 in its interior, contrary to Lemma 2.2. Let us assume that $v_5 = z_3$. Then v_2 is degree five and $N(v_2)$ is missing at most the edges v_0z_5 and v_0z_6. Yet these edges must not be present, for otherwise $N(v_2)$ has a subgraph isomorphic to K_5^-, contrary to Lemma 4.2. Hence $v_4 \notin \{z_1, z_2, \dots, z_6\}$, but then it is not adjacent to z_1. Thus $N_{G_0}(v_0)$ includes two disjoint edges. However, $N_{G_0}(v_2)$ has a subgraph isomorphic to $K_5 - P_3$, contradicting the optimality of (G, v_0). Thus we may assume that v_0 is adjacent to z_5. This implies, by Lemma 4.3, that $v_4 = z_5$, because v_2 is already adjacent to z_5 and $v_5 = z_5$ would imply the existence of another edge from v_2 to z_5, not homotopic to the existing one. Then the subgraph of G_0 induced by z_1, z_2, z_3, z_5, z_6, and v_1 has only one face—a six-walk—that can have vertices in its interior. But then there are at most nine vertices in G_0 by Lemma 2.2, contrary to Lemma 4.1. This proves our claim that $v_4, v_5 \notin \{z_1, z_2, \dots, z_6\}$.

Continuing with the proof of (6), we note that v_2 is not adjacent to v_4, for otherwise v_5 is of degree four in G_0, a contradiction. Similarly v_1 is not adjacent to v_5. Since z_1 is not adjacent to v_4 or v_5, the neighborhood of v_0 in G_0 is a cycle of length five. The vertex v_2 is not adjacent to z_2, for otherwise the 4-cycle $z_2v_2v_0v_1$ includes the vertices v_4 and v_5 in its interior, contrary to

Lemma 2.2. Furthermore, the vertex v_4 is not adjacent to z_2, for otherwise the neighborhood of v_1 in G_0 has a subgraph isomorphic to a 5-cycle plus one edge, contrary to the optimality of (G_0, v_0). We now consider the graph $G_{v_2v_4}$. It has a subgraph H isomorphic to K_6, and the new vertex w of H obtained by identifying v_2 and v_4 belongs to H. Let Δ denote the open disk bounded by the walk $z_1z_5z_6z_3z_5z_6z_2z_3$ of $G_{v_2v_4}$. Since w belongs to Δ, all vertices of H belong to the closure of Δ. However, $z_2 \notin V(H)$, because z_2 is not adjacent to v_2 or v_4 in G_0. Since v_1 is not adjacent to z_6 as shown two paragraphs ago, we deduce that not both z_6 and v_1 belong to H. That implies that $z_1 \notin V(H)$, because at most six neighbors of z_1 in $G_{v_2v_4}$ (including $z_2 \notin V(H)$) belong to the closure of Δ. If $v_1 \notin V(H)$, then no edge incident with one of the two occurrences of z_3 on the boundary of Δ belongs to H. Thus regardless of which of v_1, z_6 does not belong to H, there is a planar graph H' obtained from H by splitting at most two vertices, and a drawing of H' in the unit disk with vertices p, q, r, s drawn on the boundary in order such that H is obtained from H' by identifying p with r, and q with s. It follows that H can be made planar by deleting one vertex, contrary to the fact that it is isomorphic to K_6. This proves (6).

Since $R_1 = F_0$ it follows that $z_0 \neq z_1$. Thus $z_0 = z_6$ by (5) and (6).

(7) *We may assume that $R_2 \neq F_{136}$ and $R_2 \neq F_{126}$.*

To prove (7) we may assume for a contradiction by symmetry that $R_2 = F_{136}$. Then by (2) we have $R_1 = F_{264}$. We may assume that v_1 and v_2 are numbered so that v_1 is adjacent to z_1 and z_2. We may assume that $\hat{R}_2$ includes no edge incident with v_1 or v_2; for if it includes the edge v_2z_1, then we can change notation so that $R_2 = F_{126}$, contrary to (2), and if it includes the edge v_1z_3, then we can change notation and reduce to the case when $R_2 = F_0$, which is handled below. Then either the cyclic order around v_0 is v_1, z_1, z_3, v_2, and an unspecified vertex v_3, or v_0 is adjacent to a vertex v_3 of degree five with cyclic order: v_1, z_1, z_3, v_2, v_0. In either case, z_1, v_2 is an identifiable pair for a vertex of degree five in G_0. Note that $G_{v_2z_1}$ is not 5-colorable. We 5-color the vertices $z_1 = v_2, z_2, z_3, z_4, z_5$ of $G_{v_2z_1}$ such that each gets a unique color. Since this coloring does not extend to the rest of $G_{v_2z_1}$ we deduce that case (i) of Lemma 2.2 holds for the following 6-walk on five vertices: z_1v_2, z_2, z_4, z_1v_2, z_3, z_5. This implies that there exists a vertex w_1 in G_0 such that w_1 is adjacent to z_2, z_4, v_2, z_3 and z_5 in G_0. In the subgraph of G_0 induced by those six vertices and z_1, all the faces are triangles but for the face bounded by the cycle $z_1z_3v_2z_5w_1z_2$. Since G_0 must have at least ten vertices, we must be in case (iii) of Lemma 2.2. Now 5-color the subgraph induced by those six vertices and z_4 such that $c(z_i) = i$ for $i = 1, 2, 3, 5$, $c(w_1) = 1$, and $c(v_2) = 2$. The above-mentioned cycle is colored using four colors, and hence the 5-coloring may be extended to G_0, a contradiction. This proves (7).

In light of (7) we may assume that both R_1 and R_2 are equal to F_0. Thus we may assume that $R_1 = F_{264}$ and $R_2 = F_{365}$. We may assume that v_1 and v_2 are numbered so that v_1 is adjacent to z_1, z_2 and z_3. Let the remaining neighbors of v_0 be v_3, v_4, v_5 numbered so that the cyclic order around v_0 is v_1, v_3, v_2, v_5, v_4. This specifies the cyclic order uniquely up to reversal, and so we may assume by symmetry that the cyclic order around v_1 (of a subset of the neighbors of v_1) is $z_1, z_3, v_3, v_0, v_4, z_2$, where possibly $v_3 = z_3$ and $z_2 = v_4$.

(8) *The vertex v_1 is not adjacent to z_4 or z_5.*

To prove (8) we note that z_1 has degree five in G_0 and that its neighborhood has a subgraph isomorphic to $K_5 - P_3$. If v_1 was adjacent to z_4 or z_5, then the neighborhood of z_1 would have a subgraph isomorphic to K_5^-, contrary to Lemma 4.2 and the optimality of (G_0, v_0). This proves (8).

Since z_1 has degree five in G_0 and its neighborhood has a subgraph isomorphic to $K_5 - P_3$, we deduce from the optimality of (G_0, v_0) and Lemma 4.2 that the neighborhood of v_0 is isomorphic to $K_5 - P_3$. It follows that

(9) *the vertex v_3 is adjacent to v_4 or v_5*

and

(10) *either v_1 is adjacent to v_5, or v_2 is adjacent to v_4, and not both.*
(11) *The vertex v_2 is adjacent to v_4.*

To prove (11) suppose for a contradiction that v_2 and v_4 are not adjacent. We will consider $G_{v_2v_4}$ and its new vertex w formed by identifying v_2 and v_4. Let us note that all faces of the subgraph of $G_{v_2v_4}$ induced by z_1, z_2, z_3, z_4, z_5, v_1, w are bounded by triangles except for a face bounded by the 8-walk $W_1 = v_1wz_5z_3v_1wz_4z_2$. Let D_1 be the open disk bounded by W_1, let $W_0 = v_1v_4v_5v_2z_5z_3v_1v_3v_2z_4z_2$ be a corresponding walk in G_0, and let D_0 be the open disk bounded by W_0. By Lemma 3.6 the graph $G_{v_2v_4}$ has a subgraph H isomorphic to K_6. Since G has no K_6 subgraph it follows that $w \in V(H)$. If $z_1 \in V(H)$, then, since z_1 has degree five in G_0, all neighbors of z_1 belong to $V(H)$, contrary to (8). Thus all vertices of H belong to W_1 or D_1, and by Lemma 4.3 each vertex of $H\backslash w$ (when regarded as a vertex of G_0) belongs to W_0 or D_0. Assume for a moment that all but possibly one vertex of H belong to W_1. Then z_4 or z_5 belongs to $V(H)$, and so $v_1 \notin V(H)$ by (8). Thus exactly one vertex of H, say w_1, belongs to D_1 and $V(H) = \{w, w_1, z_2, z_3, z_4, z_5\}$. It follows that $v_4 \notin \{w_1, z_2, z_3, z_4, z_5\}$. Thus v_4 is not adjacent to z_3 in G_0, because the edge z_3v_4 would have to lie in D_0, where it would have to cross the path $z_4w_1z_5$. But w is adjacent to z_3 in H, and so v_2 is adjacent to z_3 in G_0. It follows that the 4-cycle $v_1v_0v_2z_3$ is null-homotopic, for otherwise the edge v_2z_3 and path $z_2w_1z_5$ would cross in D_0. We deduce from Lemma 2.2 applied to the 4-cycle $v_1v_0v_2z_3$ that $v_3 = z_3$. But v_3 is adjacent to v_4 by (9), and yet z_3 is not adjacent to v_4, a contradiction. This completes the case when at most one vertex of H belongs to D.

Thus at least two vertices of H, say w_1 and w_2 belong to of D_1. Since W_1 has exactly two repeated vertices, the argument used at the end of the proof of (6) shows that w_1 and w_2 are the only two vertices of H in D_1. Also, it follows that w, v_1, the two repeated vertices of W_0, belong to H. Since v_1 is in H, (8) implies that $z_4, z_5 \notin V(H)$. It follows that $z_2, z_3 \in V(H)$, and consequently $v_4 \notin \{z_2, z_3\}$. Thus each of w_1, w_2 is adjacent in G_0 to v_1, z_2, z_3 and to v_2 or v_4. It follows from considering the drawing of G_0 inside D_0 that one of w_1, w_2, say w_1, is adjacent to v_2 and the 4-cycle $v_1v_0v_2w_1$ is null-homotopic. By Lemma 2.2 applied to this 4-cycle we deduce that $w_1 = v_3$. Thus the edge v_3v_4 belongs to D_0. But $w_2 \neq v_4$, because v_4 is not a vertex of H, and yet the edge v_3v_4 intersects the path $z_3w_2z_2$ inside D_0, a contradiction. This proves (11).

(12) *The vertex v_5 is adjacent to v_1.*

We prove (12) similarly as the previous claim. Suppose for a contradiction that v_1 and v_5 are not adjacent, and consider $G_{v_1v_5}$ and its new vertex w. The subgraph of $G_{v_1v_5}$ induced by z_1, z_2, z_3, z_4, z_5, w, v_2 has all faces bounded by triangles except for one bounded by the 8-walk $W_1 = wv_2z_5z_3wv_2z_4z_2$. Let D_1 be the open disk bounded by W_1, and let W_0, D_0 be as in (11). Similarly as in the proof of (11) the graph $G_{v_1v_5}$ has a subgraph H isomorphic to K_6 with $w \in V(H)$. We claim that $z_4 \notin V(H)$. Indeed, if z_4 is in H, then it is adjacent to w in H; but z_4 is not adjacent in G_0 to v_1 by (8), and hence z_4 is adjacent to v_5 in G_0. Yet v_2 is adjacent to v_4 by (10). Since $v_4 \notin \{z_4, z_5\}$ by (8), the edges v_2v_4 and z_4v_5 must cross inside D_0, a contradiction. This proves our claim that $z_4 \notin V(H)$. It follows that $z_1 \notin V(H)$, because z_1 has degree five in $G_{v_1v_5}$, and z_4 is one of its neighbors.

If D_1 includes at most one vertex of H, then $w, v_2, z_2, z_3, z_5 \in V(H)$, and exactly one vertex of H, say w_1, belongs to D_1. Thus w_1 is adjacent to z_2 and z_5 in G_0, and that implies that the edges v_3v_4 and v_3v_5 do not lie in D_1. Therefore $v_3, v_4, v_5 \in \{z_2, z_3, z_4, z_5\}$, but that is impossible, given the existence of w_1. This completes the case that D_1 includes at most one vertex of H. Thus, similarly as in (11), it follows that D_1 includes exactly two vertices of H, say w_1 and w_2. Now $V(H)$ includes w, v_2 and exactly two of $\{z_2, z_3, z_5\}$. But it cannot include z_5 and z_3, because otherwise for some $j \in \{1, 2\}$ the paths $z_5w_jv_2$ and $z_3w_{3-j}v_2$ cross inside D_0. Thus $V(H)$ includes z_2 and z_i for some $i \in \{3, 5\}$. Choose $j \in \{1, 2\}$ such that $w_j \neq v_3$. Then the path $z_2w_jz_i$ is not disjoint from the edges v_3v_4, v_3v_5 (because they cross inside D_0), and so it follows that $i = 3$ and $v_3 = z_3$. Since there is no crossing in D_0 and w_1 and w_2 are adjacent to z_2 and z_3, they are not both adjacent to v_5. Thus we may assume that w_1 is adjacent to v_1. This argument shows, in fact, that the cycle $v_1v_0v_2w_1$ is null-homotopic, and so it follows from Lemma 2.2 that $v_3 = w_1$, a contradiction, because w_1 lies in D_1 and $v_3 = z_3$ does not. This proves (12).

Now claims (10), (11), and (12) are contradictory. This completes the proof of Lemma 4.6. □

Proof of Theorem 1.3. It follows by direct inspection that none of the graphs listed in Theorem 1.3 is 5-colorable. Conversely, let G_0 be a graph drawn in the Klein bottle that is not 5-colorable. We may assume, by taking a subgraph of G_0, that G_0 is 6-critical. Then G_0 has minimum degree at least five. By Lemma 2.3 the graph G_0 has a vertex of degree exactly five, and so we may select a vertex v_0 of G_0 such that (G_0, v_0) is an optimal pair. If there is no identifiable pair, then G_0 has a K_6 subgraph, as desired. Thus we may select an identifiable pair v_1, v_2. Let $G' := G_{v_1 v_2}$. By Lemma 3.6 the graph G' has a subgraph H isomorphic to K_6. By Lemma 4.4 the drawing of H is 2-cell, and by Lemma 4.5 some face of H has length six, contrary to Lemma 4.6. □

References

[1] M. Albertson, J. Hutchinson, The three excluded cases of Dirac's map-color theorem, in: Second International Conference on Combinatorial Mathematics, New York, 1978, in: Ann. New York Acad. Sci., vol. 319, New York Acad. Sci., New York, 1979, pp. 7–17.

[2] K. Appel, W. Haken, Every planar map is four colorable, Part I: discharging, Illinois J. Math. 21 (1977) 429–490.

[3] K. Appel, W. Haken, Every planar map is four colorable, Contemp. Math. 98 (1989).

[4] K. Appel, W. Haken, J. Koch, Every planar map is four colorable, Part II: reducibility, Illinois J. Math. 21 (1977) 491–567.

[5] G.A. Dirac, Map color theorems, Canad. J. Math. 4 (1952) 480–490.

[6] G.A. Dirac, The coloring of maps, J. London Math. Soc. 28 (1953) 476–480.

[7] D. Eppstein, Subgraph isomorphism in planar graphs and related problems, J. Graph Algorithms Appl. 3 (1999) 1–27.

[8] S. Fisk, The nonexistence of colorings, J. Combin. Theory Ser. B 24 (1978) 247–248.

[9] T. Gallai, Kritische Graphen I, II, Publ. Math. Inst. Hungar. Acad. Sci. 8 (1963) 165–192, 373–395.

[10] M.R. Garey, D.S. Johnson, Computers and Intractability. A Guide to the Theory of NP-Completeness, W.H. Freeman, San Francisco, 1979.

[11] P.J. Heawood, Map-color theorem, Quart. J. Pure Appl. Math. 24 (1890) 332–338.

[12] K. Kawarabayashi, D. Král', J. Kynčl, B. Lidický, 6-Critical graphs on the Klein bottle, SIAM J. Discrete Math. 23 (2009) 372–383.

[13] W. Kocay, Donald L. Kreher, Graph Algorithms and Optimization, CRC Press, New York, 2004.

[14] D. Král', B. Mohar, A. Nakamoto, O. Pangrác, Y. Suzuki, Coloring Eulerian triangulations on the Klein bottle, Graphs Combin., http://dx.doi.org/10.1007/s00373-011-1063-9, in press.

[15] W. Mader, $3n - 5$ Edges do force a subdivision of K_5, manuscript.

[16] W.S. Massey, A Basic Course in Algebraic Topology, Springer-Verlag, New York, 1991.

[17] B. Mohar, A linear time algorithm for embedding graphs in an arbitrary surface, SIAM J. Discrete Math. 12 (1999) 6–26.

[18] B. Mohar, Triangulations and the Hajós conjecture, Electron. J. Combin. 12 (2005), N15.

[19] B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, MD, 2001.

[20] G. Ringel, Map Color Theorem, Springer-Verlag, Berlin, 1974.

[21] N. Robertson, D.P. Sanders, P.D. Seymour, R. Thomas, The four-colour theorem, J. Combin. Theory Ser. B 70 (1997) 2–44.

[22] V. Rödl, J. Zich, Triangulations and the Hajós conjecture, J. Graph Theory 59 (2008) 293–325.

[23] N. Sasanuma, Chromatic numbers of 6-regular graphs on the Klein bottle, Australas. J. Combin. 45 (2009) 73–85.

[24] C. Thomassen, Five-coloring graphs on the torus, J. Combin. Theory Ser. B 62 (1994) 11–33.

[25] C. Thomassen, Every planar graph is 5-choosable, J. Combin. Theory Ser. B 62 (1994) 180–181.

[26] C. Thomassen, Color-critical graphs on a fixed surface, J. Combin. Theory Ser. B 70 (1997) 67–100.

[27] C. Thomassen, Some remarks on Hajós' conjecture, J. Combin. Theory Ser. B 93 (2005) 95–105.

[28] B. Toft, On critical subgraphs of colour-critical graphs, Discrete Math. 7 (1974) 377–392.

[29] C. Yerger, Color-critical graphs on surfaces, PhD dissertation, Georgia Institute of Technology, 2010.