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Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Constructing proper Markov semigroups for Arveson systems ☆

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A B S T R A C T

We show that the Markov semigroup obtained by Floricel (2008) in [9] compressing the E_0-semigroup of Skeide (2006) [13], does not consist of endomorphisms. It, therefore, cannot be the tail flow of an E_0-semigroup. As a corollary of our result, Floricel's construction will allow to get examples of proper type III Markov semigroups that are not tensor products of simpler ones, provided we find type III Arveson systems that do not factor into tensor products.

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Algebraically, an **Arveson system** is a family $E^\otimes = (E_t)_{t \in (0,\infty)}$ of infinite-dimensional separable Hilbert spaces E_t with unitaries $u_{t,s} : E_t \otimes E_s \to E_{t+s}$ such that the product $(x_t, y_s) \mapsto x_t y_s := u_{t,s}(x_t \otimes y_s)$ is associative. Technically, the bundle $(E_t)_{t \in (0,\infty)}$ is required to be a Borel bundle isomorphic to the trivial Borel bundle $(0, \infty) \times \mathfrak{H}$ for some Hilbert space $\mathfrak{H}$, and the product is required to be measurable.

Suppose we find a Hilbert space $K (\neq \{0\})$ and an (again measurable) family of unitaries $w_t : E_t \otimes K \to K$ such that the product $(x_t, y) \mapsto x_t y := w_t(x_t \otimes y)$ iterates associatively with the product of the Arveson system. Then the maps $\vartheta_t : a \mapsto w_t(\mathrm{id}_t \otimes a)w_t^*$ on $\mathcal{B}(K)$ (plus $\vartheta_0 := \mathrm{id}_{\mathcal{B}(K)}$) form an E_0-**semigroup** $\vartheta = (\vartheta_t)_{t \in \mathbb{R}_+}$ (that is, a σ-weakly continuous semigroup of normal unital endomorphisms). Such a family $(w_t)_{t \in (0,\infty)}$ has been called a **right dilation** of $E^\otimes$ in Skeide [13] and a **unitary resolution** in Floricel [9]. It is equivalent to the notion of **nondegenerate** or **essential** representation of an Arveson system. Arveson [1] associated with every E_0-semigroup an Arveson system, and it is not difficult to show that the Arveson system of ϑ is $E^\otimes$.

Arveson also proved in [2] that every Arveson system admits a right dilation. Skeide [13] presented an elementary direct construction of a right dilation, and Floricel [9] generalized that further. The first ingredient of the construction in [13], is a right dilation $\check{w}_n : E_n \otimes \check{K} \to \check{K}$ ($n \in \mathbb{N}$) of the **discrete subsystem** $(E_n)_{n \in \mathbb{N}}$ of $E^\otimes$. Such a right dilation can be obtained from any unit vector $\omega_1 \in E_1$ as an inductive limit $\check{K}$ over E_n with respect to the inductive system $E_n \to E_n \omega_1^m \subset E_{n+m}$. It is not difficult to check that the factorization $E_n \otimes E_m \to E_{n+m}$ survives the "limit" $m \to \infty$, giving $\check{w}_n$. Moreover, all $\omega_1^n \in E_n$ end up in the same unit vector $\check{\omega} \in \check{K}$, which fulfills $\omega_1^n \check{\omega} = \check{\omega}$. One may check that $\langle \check{\omega}, \bullet \check{\omega} \rangle$ is an absorbing invariant vector state for the discrete E_0-semigroup $\check{\vartheta}$ on $\mathcal{B}(\check{K})$. In particular, the projections $\check{\vartheta}_n(\check{\omega}\check{\omega}^*)$ increase to the identity. See [1, Appendix] or [8, Section 5] for details.

We rest a moment to clarify some details about tensor products with direct integrals and operations on them. If H_1, H_2 are separable Hilbert spaces, then for $0 \leqslant a < b < \infty$ we will understand by

$$\int_a^b (H_1 \otimes E_\alpha \otimes H_2)\, d\alpha$$

the family of measurable, square integrable sections $X = (x_\alpha)_{\alpha \in (a,b]}$ with $x_\alpha \in H_1 \otimes E_\alpha \otimes H_2$. Since $(E_\alpha)_{\alpha \in (a,b]}$ is Borel isomorphic to $(a,b] \times \mathfrak{H}$, it is clear how this has to be interpreted. In particular, $\int_a^b (H_1 \otimes E_\alpha \otimes H_2)\, d\alpha \cong L^2((a,b], H_1 \otimes \mathfrak{H} \otimes H_2)$ by the Borel isomorphism. It is clear that

$$H_1 \otimes \left(\int_a^b E_\alpha\, d\alpha \right) \otimes H_2 \cong \int_a^b (H_1 \otimes E_\alpha \otimes H_2)\, d\alpha$$

via $x_1 \otimes (y_\alpha)_{\alpha \in (a,b]} \otimes x_2 \mapsto (x_1 \otimes y_\alpha \otimes x_2)_{\alpha \in (a,b]}$, because

$$L^2\big((a,b], H_1 \otimes \mathfrak{H} \otimes H_1\big) \cong L^2(a,b] \otimes H_1 \otimes \mathfrak{H} \otimes H_2 \cong H_1 \otimes L^2\big((a,b], \mathfrak{H}\big) \otimes H_2.$$

Recall that for $x_2 \in H_2$ the operator $\mathrm{id}_1 \otimes x_2^* \in \mathcal{B}(H_1 \otimes H_2, H_1)$ is defined by setting $(\mathrm{id}_1 \otimes x_2^*)(y_1 \otimes y_2) = y_1 \langle x_2, y_2 \rangle$. It is the adjoint of the operator $\mathrm{id}_1 \otimes x_2 : y_1 \mapsto y_1 \otimes x_2$.

Proposition. *Let* $X = (x_\alpha)_{\alpha \in (a,b]} \in \int_a^b E_\alpha\, d\alpha$. *Then the operator* $\mathrm{id}_1 \otimes X^*$ *acts on sections* $Y = (y_\alpha)_{\alpha \in (a,b]} \in \int_a^b (H_1 \otimes E_\alpha)\, d\alpha$ *as*

$$\big(\mathrm{id}_K \otimes X^*\big) Y = \int_a^b \big(\mathrm{id}_1 \otimes x_\alpha^*\big) y_\alpha\, d\alpha$$

in the sense of Bochner integral of H_1-*valued functions.*

Similar statements are true for $\int_a^b (E_\alpha \otimes H_2)\, d\alpha$ and $\int_a^b (H_1 \otimes E_\alpha \otimes H_2)\, d\alpha$.

Corollary. $(\mathrm{id}_1 \otimes X'X^*)Y = ((\int_a^b (\mathrm{id}_1 \otimes x_\beta^*) y_\beta\, d\beta) \otimes x_\alpha')_{\alpha \in (a,b]}$.

Proof of Proposition. Evaluate the operator on the dense set of elements of the form

$$Y = \sum_{i=1}^n \big(h_i \otimes \big(\mathbb{I}_{(a_i,b_i]}(\alpha) y_\alpha' \big) \big)_{\alpha \in (a,b]}$$

($h_i \in H_1$, $Y' \in \int_a^b E_\alpha\, d\alpha$, and the $(a_i, b_i]$ forming a partition of $(a,b]$) and extend it in L^2-norm. $\quad \square$

Note that $\check{K}$ and the right dilation $\check{w}_n$ of the discrete subsystem of $E^\otimes$ to $\check{K}$ coincide with $\widetilde{K}_u$ and $\widetilde{W}_{u,n}$ in [9], if one puts the $s > 0$ in [9, Section 3.1] equal to $s = 1$ and $u \in E_s$ equal to $u = \omega_1$. The vector $\check{\omega}$, in [9] is denoted by $[u]$.

Put $K := (\int_0^1 E_\alpha\, d\alpha) \otimes \check{K} = \int_0^1 (E_\alpha \otimes \check{K})\, d\alpha$. Both [13] and [9] define right dilations w_t and $W_{u,t}$ ($s = 1$ and $u = \omega_1$ as before), respectively, of $E^\otimes$ to K. We do not know, if the two right dilations coincide, or if the E_0-semigroups ϑ and ρ, respectively, determined by them coincide. However, we know that they coincide for integer $t = n$ and this is enough for our purposes.

Indeed, for $t = n \in \mathbb{N}$ the right dilation w_t defined in [13, Eq. (3.1)][1] acts as

$$w_n : x_n \otimes (y_\alpha \otimes \check{z})_{\alpha \in (0,1]} = (x_n \otimes y_\alpha \otimes \check{z})_{\alpha \in (0,1]} \longmapsto \big((\mathrm{id}_\alpha \otimes \check{w}_n) \big[\big(u_{\alpha,n}^* (x_n y_\alpha) \otimes \check{z} \big) \big] \big)_{\alpha \in (0,1]}.$$

And this is precisely what the definition of $W_{u,t,l}$ in [9, Eq. (3.18)] according to the equation between Eqs. (3.16) and (3.17) in [9] gives for $W_{u,n} := W_{\omega_1, n, 0}$.

Recall that if ϑ is an E_0-semigroup on $\mathcal{B}(K)$ and if $Q \in \mathcal{B}(K)$ is an **increasing projection** (that is, if $\vartheta_t(Q) \geqslant Q$ for all t), then the maps $T_t : QaQ \mapsto Q\vartheta_t(QaQ)Q = Q\vartheta_t(a)Q$ form a **Markov semigroup** T, that is, a semigroup of normal unital CP-maps, on $Q\mathcal{B}(K)Q$. We call T the Markov semigroup obtained from ϑ by **compression** with Q.

---

[1] Well, actually in [13, Eq. (3.1)] a left dilation is defined. By [13, Theorem 3.3], reversing the orders in all tensor products one gets a right dilation. This is, how [13, Eq. (3.1)] must be interpreted here.

We abbreviate $L := \int_0^1 E_\alpha \, d\alpha$, so that $K = L \otimes \check{K}$. Define $Q := \mathrm{id}_L \otimes \check{\omega}\check{\omega}^* \in \mathcal{B}(K)$. We confirm [9, Proposition 4.2]:

Lemma. Q *is increasing for* ϑ.

For integer times $t = n$, the proof will be evident from an intermediate step in the proof of the following result. For arbitrary t we would have to repeat the full definition of w_t from [13], and for the following proof it does not matter if the maps T_t form a Markov semigroup.

Theorem. *The Markov semigroup T does not consist of endomorphisms.*

Proof. Note that $Q \mathcal{B}(K) Q = \mathcal{B}(L) \otimes \check{\omega}\check{\omega}^* \cong \mathcal{B}(L)$. A normal, unital ($T_n$ is Markov!) endomorphism of $\mathcal{B}(L)$ takes non-zero projections to non-zero projections. We shall show that there exists a rank-one projection $a \in Q \mathcal{B}(K) Q$ such that $T_1(a)$ is not a non-zero projection.

Fix a unit vector $X = (x_\alpha)_{\alpha \in (0,1]} \in L$, and define the rank-one projection $a := XX^* \otimes \check{\omega}\check{\omega}^* \in Q \mathcal{B}(H) Q$. The norm of the positive operator $T_n(a) \in \mathcal{B}(L) \otimes \check{\omega}\check{\omega}^*$ is the supremum of the matrix elements $\langle (Y \otimes \check{\omega}), T_n(a)(Y \otimes \check{\omega}) \rangle$ over all unit vectors $Y = (y_\alpha)_{\alpha \in (0,1]} \in L$. First, we observe that $Q(Y \otimes \check{\omega}) = Y \otimes \check{\omega}$. Next, we compute

$$
\begin{aligned}
w_n^*(Y \otimes \check{\omega}) &= \left(\left(u_{n,\alpha}^* u_{\alpha,n} \otimes \mathrm{id}_{\check{K}} \right) \left(\mathrm{id}_\alpha \otimes \check{w}_n^* \right) (y_\alpha \otimes \check{\omega}) \right)_{\alpha \in (0,1]} \\
&= \left(\left(u_{n,\alpha}^* u_{\alpha,n} \otimes \mathrm{id}_{\check{K}} \right) (y_\alpha \otimes \check{\omega}_n \otimes \check{\omega}) \right)_{\alpha \in (0,1]} = \left(u_{n,\alpha}^* \left(y_\alpha \omega_1^n \right) \otimes \check{\omega} \right)_{\alpha \in (0,1]}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\left\langle (Y \otimes \check{\omega}), \left(Q \, \vartheta_n(a) \, Q \right)(Y \otimes \check{\omega}) \right\rangle &= \left\langle w_n^*(Y \otimes \check{\omega}), \left(\mathrm{id}_n \otimes XX^* \otimes \check{\omega}\check{\omega}^* \right) w_n^*(Y \otimes \check{\omega}) \right\rangle \\
&= \left\langle \left(u_{n,\alpha}^* \left(y_\alpha \omega_1^n \right) \right)_{\alpha \in (0,1]}, \left(\mathrm{id}_n \otimes XX^* \right) \left(u_{n,\alpha}^* \left(y_\alpha \omega_1^n \right) \right)_{\alpha \in (0,1]} \right\rangle \\
&= \left\langle \int_0^1 \left(\mathrm{id}_n \otimes x_\beta^* \right) u_{n,\beta}^* \left(y_\beta \omega_1^n \right) d\beta, \int_0^1 \left(\mathrm{id}_n \otimes x_\gamma^* \right) u_{n,\gamma}^* \left(y_\gamma \omega_1^n \right) d\gamma \right\rangle \\
&= \left\| \int_0^1 \left(\mathrm{id}_n \otimes x_\gamma^* \right) u_{n,\gamma}^* \left(y_\gamma \omega_1^n \right) d\gamma \right\|^2. \tag{$\dagger$}
\end{aligned}
$$

(At this point, replacing in the first two lines XX^* with id_L, we see that, indeed, $\vartheta_n(Q) \geqslant Q$; that proves the preceding Lemma for integer times: T_n is, indeed, Markov.)

We put $n = 1$ and we shall find a unit vector X such that ($\dagger$) is not bigger than a constant $M^2 < 1$ no matter what unit vector Y we choose. Note that there exists a unit vector $z_1 \in E_1$ such that the square root of ($\dagger$) is given by

$$
\left\langle z_1, \int_0^1 \left(\mathrm{id}_1 \otimes x_\gamma^* \right) u_{1,\gamma}^* (y_\gamma \omega_1) \, d\gamma \right\rangle = \int_0^1 \langle z_1 x_\gamma, y_\gamma \omega_1 \rangle \, d\gamma.
$$

Choose a **measurable ONB** for $E^\otimes$. By this, we mean a family $(e^m)_{m \in \mathbb{N}}$ of measurable sections $e^m = (e_\alpha^m)_{\alpha \in (0,1]}$ such that for each α the family $(e_\alpha^m)_{m \in \mathbb{N}}$ is an ONB for E_α. (Such a measurable ONB exists, because $E^\otimes$ is isomorphic to a trivial bundle.) Then the vectors $f_{1-\alpha}^m := (\mathrm{id}_{1-\alpha} \otimes e_\alpha^{m*}) \omega_1 \in E_{1-\alpha}$, $\alpha \in (0,1)$, depend measurably on α (all Hilbert spaces are separable) and fulfill $\omega_1 = \sum_m f_{1-\alpha}^m e_\alpha^m$ for each $\alpha \in (0,1)$. For the integration the point $\alpha = 1$ does not count because $\{1\}$ has measure 0. By *dominated convergence*, we find

$$
\begin{aligned}
\int_0^1 \langle z_1 x_\gamma, y_\gamma \omega_1 \rangle \, d\gamma &= \int_0^1 \left\langle z_1 x_\gamma, y_\gamma \left(\sum_m f_{1-\gamma}^m e_\gamma^m \right) \right\rangle d\gamma \\
&= \sum_m \int_0^1 \langle z_1 x_\gamma, y_\gamma f_{1-\gamma}^m e_\gamma^m \rangle \, d\gamma = \sum_m \int_0^1 \langle z_1, y_\gamma f_{1-\gamma}^m \rangle \langle x_\gamma, e_\gamma^m \rangle \, d\gamma.
\end{aligned}
$$

Observe that $\| f_\alpha^m \| \leqslant 1$ for all $\alpha \in (0,1)$, $m \in \mathbb{N}$. There exists an m such that $\int_0^1 \| f_{1-\gamma}^m \|^2 \, d\gamma < 1$. (Indeed, if this integral is 1 for a certain m_0, then it is 0 for all other $m \neq m_0$.) Choose $X = e^m$ for that m, so that $\langle x_\gamma, e_\gamma^m \rangle = 1$ for all γ, and put $M := \sqrt{\int_0^1 \| f_{1-\gamma}^m \|^2 \, d\gamma}$. Then

$$\left\| \int_0^1 \langle z_1 x_\gamma, y_\gamma \omega_1 \rangle \, d\gamma \right\| = \left\| \int_0^1 \langle z_1, y_\gamma f^m_{1-\gamma} \rangle d\gamma \right\| = \left\| \left\langle z_1, \int_0^1 y_\gamma f^m_{1-\gamma} \, d\gamma \right\rangle \right\| \leqslant \left\| \int_0^1 y_\gamma f^m_{1-\gamma} \, d\gamma \right\|$$

$$\leqslant \int_0^1 \| f^m_{1-\gamma} \| \, \| y_\gamma \| \, d\gamma \leqslant \sqrt{\int_0^1 \| f^m_{1-\gamma} \|^2 \, d\gamma} \sqrt{\int_0^1 \| y_\gamma \|^2 \, d\gamma} = M \| Y \| = M < 1.$$

The constant M is independent of the choice of the unit vector Y. In conclusion, for $X = e^m$ we have $\| T_1(a) \| \leqslant M^2 < 1$. Therefore, $T_1(a)$ cannot be a non-zero projection. So, T_1 is not an endomorphism. $\square$

Observation. The Arveson system in the theorem is arbitrary. Since the ϑ constructed in [13] and the ρ constructed in [9] coincide (for the choice of the parameters in ρ as specified before) for integer $t = n \in \mathbb{N}_0$, also the compressed Markov maps T_t coincide at least for integer $t = n$. As the theorem says T_1 is not an endomorphism, it follows that [9, Theorem 4.4] is false. (We believe that the error is in Lemma 4.1. Check it for $t = s = 1$, applying both sides to $\omega_1 x_1 \otimes (Y \otimes \breve{z})$ when x_1 is taken from a unit $x^\otimes$ and ω_1 is taken from another unit $\omega^\otimes$; see the computations below.)

For whom who wishes to have more concrete examples, we mention that it is possible to obtain simpler and calculable examples when the Arveson system $E^\otimes$ is spatial. In that case, we would choose a unital unit $\omega^\otimes = (\omega_t)_{t \in (0, \infty)}$ and for ω_1 really the member at $t = 1$ of that unit. With this choice, the part $u^*_{n,\alpha}(y_\alpha \omega^n_1)$ in $w^*_n(Y \otimes \breve{\omega})$ may be computed as $y_\alpha \omega_{n-\alpha} \otimes \omega_\alpha$. When computing $(\mathrm{id}_n \otimes X^*)(y_\alpha \omega_{n-\alpha} \otimes \omega_\alpha)_{\alpha \in (0,1]}$ this gives $\int_0^1 y_\alpha \omega_{n-\alpha} \langle x_\alpha, \omega_\alpha \rangle \, d\alpha$. Taking $x_\alpha = \omega_\alpha$ and for y_α pieces from an independent unit, $\langle Y, X \rangle \langle X, Y \rangle$ and (†) can be computed. This works for an arbitrary spatial Arveson system of index not smaller than 1. Of course, it also works for type I systems, that is, for Fock spaces (with $\omega^\otimes$ the vacuum unit). Here, everything may be computed explicitly in terms of exponential vectors.

Remark. We should note that there is a simple theoretical argument, why a type III E_0-semigroup ϑ (that is, the Arveson system of ϑ is type III) can never be compressed to an automorphism semigroup T. (See the proof of the proposition below for the following terminology.) In fact, the Arveson system of ϑ contains the Arveson system of the minimal dilation of T, and the minimal dilation of an E_0-semigroup (that is, in particular, of an automorphism semigroup) T is T itself. But the Arveson system of an automorphism semigroup would be the "trivial" one, $(\mathbb{C})_{t \in (0,\infty)}$.[2] And the "trivial" Arveson system, like every Arveson system containing it, has a unit. This is not possible if ϑ is type III.

But our theorem is much more far-reaching. It tells that, no matter from which Arveson system $E^\otimes$ we start, T is **proper** in the sense that it is not even an endomorphism semigroup. In the remainder, we explain briefly why this promises to provide the first examples of nontrivial type III Markov semigroups.

A **type III** or **nonspatial** Markov semigroup is a Markov semigroup with type III Arveson system. (This property is equivalent to the property that the semigroup has no *units* in the sense of Arveson [3, Definition 2.1]; see Bhat, Liebscher and Skeide [6]. It should not be confused with Powers' definition [11], which is more restrictive.) Of course, every type III E_0-semigroup is also an example for a type III Markov semigroup. By a **nontrivial** type III Markov semigroup we understand a proper type III Markov semigroup that is not the tensor product of a type III E_0-semigroup and a proper spatial Markov semigroup.

So far, there are no known examples of such nontrivial type III Markov semigroups. With some basic knowledge about minimal dilation and Arveson system of a Markov semigroup, our theorem allows to show that for certain type III Arveson systems, Floricel's Markov semigroup, necessarily type III, is nontrivial. The prerequisites are collected in the following proposition and its proof. Observe that with Q also the projection $Q_t := \vartheta_t(Q)$ is increasing for ϑ. For $\alpha \geqslant 0$, we denote by T^α the Markov semigroup on $Q_\alpha \mathcal{B}(K) Q_\alpha$ obtained by compressing ϑ with Q_α. Observe that with T, also T^α is proper. (This follows from $\vartheta_\alpha \circ T_t = T^\alpha_t \circ \vartheta_\alpha$. So, if T_t does not factor on $a_1 a_2$ ($a_i \in Q \mathcal{B}(K) Q$), then T^α_t does not factor on $\vartheta_\alpha(a_1) \vartheta_\alpha(a_2)$ ($\vartheta_\alpha(a_i) \in Q_\alpha \mathcal{B}(K) Q_\alpha$).)

Proposition. *Let (ϑ, Q) be a dilation of a Markov semigroup T.*

1. *If $E^\otimes$ is an Arveson system that does not factor into the tensor product of two Arveson systems, then, for each $\alpha > 0$, T^α is a proper Markov semigroup that does not factor into the tensor product of two Markov semigroups.*
2. *If $E^\otimes$ is an Arveson system that has no subsystem factoring into the tensor product of two Arveson systems, then T is a proper Markov semigroup that does not factor into the tensor product of Markov semigroups.*

Proof. The dilation (ϑ, Q) of T is **minimal** if the smallest subspace of K invariant for $\vartheta_t(a)$ ($t \in \mathbb{R}_+, a \in Q \mathcal{B}(K) Q$) and containing $Q K$ is K. By Bhat [4, Theorem 4.7], every (normal) Markov semigroup T on $\mathcal{B}(H)$ admits a *minimal dilation* and

---

[2] Recall that, in these notes like Arveson in [1], we did exclude the one-dimensional case. In fact, our Theorem is false in the one-dimensional case, and our proof breaks down once we have only one element in our measurable ONB.

that minimal dilation is unique up to suitable unitary equivalence. Bhat [4, Section 6] defines the ***Arveson system of*** T as the Arveson system of the unique minimal dilation. (This Arveson system can be constructed directly as explained in Skeide [12] following the construction of Bhat and Skeide [8], or in Bhat and Mukherjee [7] following notions of Arveson [3]. But the statement we need here, really, is that the Arveson system of T is that of the minimal dilation.)

There are two easy to verify consequences. Firstly, every dilation can be compressed to the smallest invariant subspace containing QK (as above) to obtain the minimal dilation; see Bhat [5, Section 3]. (See also Shalit and Solel [14, Theorem 5.12] for a similar result in more general circumstances.) In either way to construct the Arveson system of that dilation (Arveson's [1] and Bhat's [5]) it is easy to see that the projection onto that subspace gives rise to a projection morphism of that Arveson system onto a subsystem that is the Arveson system of the minimal dilation: The Arveson system of every dilation contains the Arveson system of the minimal dilation. Secondly, given two Markov semigroups, the tensor product of their minimal dilations is the minimal dilation of their tensor product; this is mentioned in [4] between Theorems 6.3 and 6.4.

Putting these two statements together, immediately proves 2. (The Arveson system of T is a subsystem of $E^\otimes$. If $E^\otimes$ has no subsystem that factors, then the minimal dilation does not factor, thus, neither does T.)

Statement 1 follows the same way from the following two theorems. [5, Theorem 3.7]: If (ϑ, Q) is a ***primary dilation*** (that is, if $Q_t \uparrow \mathrm{id}_K$), then, for all $\alpha > 0$, the dilation (ϑ, Q_α) of T^α is minimal. [5, Theorem 3.6(ii)] (reformulated for our needs): If (ϑ, K) is not primary, then it has a corner containing Q (hence, Q_t) that is a primary dilation with the same Arveson system as ϑ, to which the former theorem can be applied. (Recall that, by footnote 2, ϑ does not consist of automorphisms and [5, Theorem 3.6(i)] does not apply. Anyway, without (the not very difficult direct) proof we communicate that the dilation (ϑ, Q) of T as in our theorem, actually, is primary. For Floricel's dilation this statement is contained in [9, Proposition 4.2], and since Q is increasing, it is sufficient to know it only for integer times $t = n$, for which we clarified equality with [9].) □

Supplement. If we specify that the Arveson (sub)system does not factor into certain types, then the Markov semigroup does not factor into these types either.

Corollary. *If $E^\otimes$ is a type III Arveson system that does not factor into the tensor product of a type III system and a spatial system, then the semigroups T^α ($\alpha > 0$) derived from Floricel's dilation are nontrivial type III Markov semigroups. If $E^\otimes$ has even no subsystems factoring in that way, then Floricel's Markov semigroup itself is nontrivial type III.*

Existence of such Arveson systems is, however, an open question. (Good candidates are *generalized CCR-flows* from Izumi and Srinivasan [10] with one-dimensional multiplicity space.)

Acknowledgments

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