JOURNAL OF DIFFERENTIAL EQUATIONS 24, 253–267 (1977)

nth Order Stieltjes Differential Boundary Operators and Stieltjes Differential Boundary Systems

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Received September 2, 1975; revised February 13, 1976

This article discusses linear differential boundary systems, which include nth-order differential boundary relations as a special case, in $\mathscr{L}_n^p[0, 1] \times \mathscr{L}_n^p[0, 1]$, $1 \leqslant p < \infty$. The adjoint relation in $\mathscr{L}_n^q[0, 1] \times \mathscr{L}_n^q[0, 1]$, $1/p + 1/q = 1$, is derived. Green's formula is also found. Self-adjoint relations are found in $\mathscr{L}_n^2[0, 1] \times \mathscr{L}_n^2[0, 1]$, and their connection with Coddington's extensions of symmetric operators on subspaces of $\mathscr{L}_n^2[0, 1]$ is established.

1. INTRODUCTION

Traditionally, discussion of a linear differential problem of nth order was followed by a similar discussion of first-order systems. Further, except under special circumstances [8, p. 205] there was little connection between self-adjoint problems of each type. For example, in the regular case, linear second-order equations were first discussed by Liouville [17], nth-order equations by Birkhoff [3], while systems followed later when discussed by Birkhoff and his colleague Langer [4]. In the singular case, linear second-order equations were first discussed by Weyl [21]; nth-order equations followed in discussions by Kodaira [11] and Levinson [16]; while systems were first discussed later by Brauer [5] and Atkinson [2].

Recently, however, Walker [19, 20] has written two rather remarkable articles which show that an nth order expression, scalar or vector,[1] can be written as a first-order system with three marvelous properties:

1. The adjoint system is found by merely replacing the coefficients by their conjugate transposes. This strengthens the analogy between differential operators and matrices.

2. Self-adjointness is preserved. That is, a self-adjoint nth order expression becomes a self-adjoint vector system of first order.

[1] Some very minor modifications are needed in Walker's paper in the vector case.

ISSN 0022–0396

3. Minimal conditions concerning the differentiability of the coefficients are imposed. This requires the introduction of quasi-derivatives for the functions or vectors to be operated upon, but broadens the generality of the description.

Although the vector–matrix forms are too complicated to exhibit here in any detail, we can state that the first-order equation

$$iq_0(q_0y)' + p_0y = \lambda wy + f$$

becomes

$$iY' = \lambda[q_0^{-1}wq_0^{-1}]Y + [-q_0^{-1}p_0q_0^{-1}]Y + [q_0^{-1}f]$$

upon substituting $Y = q_0y$ and dividing by q_0. The second-order equation

$$-\{(p_0y')' - i[(q_0y)' + (q_0y')]\} + p_1y = \lambda wy + f$$

is equivalent to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y \\ D^{[1]}y \end{pmatrix}'$$

$$= \lambda\begin{pmatrix} -w & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y \\ D^{[1]}y \end{pmatrix} + \begin{pmatrix} p_1 - q_0p_0^{-1}q_0 & iq_0p_0^{-1} \\ -ip_0^{-1}q_0 & p_0^{-1} \end{pmatrix}\begin{pmatrix} y \\ D^{[1]}y \end{pmatrix} + \begin{pmatrix} -f \\ 0 \end{pmatrix}.$$

Higher odd-order systems are slightly different from the first-order representation. Even-order systems also are slightly different from what one would be led to believe by studying the second-order system representation, but are not so different as those of odd order. We refer to Walker's papers [19, 20] for details.

In all cases the vector system which results has the form

$$JY' - BY = \lambda AY + AF,$$

where J is nonsingular, constant, and satisfies $J = -J^*$. If the nth order expression is self-adjoint, then $B = B^*$ and $A = A^*$. Further, except for equations of first order, A consists of $-w$ in the upper left corner and zeros elsewhere.

This paper studies differential boundary relations with the form exhibited by Walker's systems. In so doing we generalize earlier results for systems [12–15], since the matrix A may be singular, and simultaneously we derive for the first time the proper form for nth order differential boundary relations in a system representation. Self-adjoint nth order differential boundary relations fall out automatically merely by making certain assumptions concerning the coefficients.

The connection between nth order differential boundary relations and the self-adjoint extensions of symmetric differential operators is at this point obvious.

2. The Setting and the Problem

Our notation conforms with Walker's [19, 20] rather than the author's earlier work [12–15].

J, A, B are $n \times n$ matrices whose components are bounded and measurable. Further, J is nonsingular, of bounded variation, and regular. A is positive and symmetric; i.e., $y^*Ay \geqslant 0$ for all n-dimensional vectors y, and $A = A^*$.

M, N are $m \times n$ matrices, $m \leqslant 2n$, satisfying $\operatorname{rank}(M : N) = m$, and P, Q are $(2n - m) \times n$ matrices with rank $(P : Q) = 2n - m$, such that $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ is nonsingular. The matrix $\begin{pmatrix} \tilde{M} & \tilde{N} \\ \tilde{P} & \tilde{Q} \end{pmatrix}$ is chosen such that

$$\begin{pmatrix} \tilde{M} & \tilde{N} \\ \tilde{P} & \tilde{Q} \end{pmatrix}^* \begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \begin{pmatrix} -J(0) & 0 \\ 0 & J(1) \end{pmatrix}.$$

Thus $\tilde{M}$ and $\tilde{N}$ are $m \times n$ matrices satisfying rank $(\tilde{M} : \tilde{N}) = m$, and $\tilde{P}$ and $\tilde{Q}$ are $(2n - m) \times n$ matrices satisfying rank $(\tilde{P} : \tilde{Q}) = 2n - m$.

K is a regular $m \times n$ matrix valued function of bounded variation satisfying $d(KJ)(0) = 0$, $d(KJ)(1) = 0$. K_1 is a regular $r \times n$ matrix valued function of bounded variation satisfying $d(K_1J)(0) = 0$, $d(K_1J)(1) = 0$.

Similarly H is a regular $n \times (2n - m)$ valued function of bounded variation satisfying $dH(0) = 0$, $dH(1) = 0$. H_1 is a regular $n \times s$ matrix valued function of bounded variation satisfying $dH_1(0) = 0$, $dH_1(1) = 0$.

The setting we wish to use is the Banach space $\mathscr{L}_n{}^p[0, 1]$, $1 \leqslant p < \infty$, generated by the seminorm

$$\|f\| = \left(\int_0^1 [f^*A^{2/p}f]^{p/2}\, dt\right)^{1/p}.$$

(Since A is positive, it has a spectral resolution $A = \sum_{i=1}^n \lambda_i p_i$, where $\lambda_i \geqslant 0$ are its eigenvalues and p_i are their corresponding projections. $A^{2/p}$ is then given by $A^{2/p} = \sum_{i=1}^n \lambda_i^{2/p} p_i$.)

Two elements f_1 and f_2 are said to be equivalent if $\|f_1 - f_2\| = 0$. Under this equivalence $\|\cdot\|$ becomes a norm, and $\mathscr{L}_n{}^p[0, 1]$ is a Banach space. In the case $A = I$ this is equivalent to the norm used in earlier works [13–15]. Further, when $p = q = 2$, the norms take the usual familiar form.

The dual space is $\mathscr{L}_n{}^q[0, 1]$, $1/p + 1/q = 1$, generated by the norm

$$\|g\| = \left(\int_0^1 [g^*A^{2/q}g]^{q/2}\, dt\right)^{1/q}.$$

Hölder's inequality then quickly follows, since

$$\begin{aligned} \left|\int_0^1 g^*Af\, dt\right| &= \left|\int_0^1 (A^{1/q}g)^*\,(A^{1/p}f)\, dt\right| \\ &\leqslant \left(\int_0^1 \|A^{1/q}g\|_{E_2}^{q/2}\, dt\right)^{1/q} \left(\int_0^1 \|A^{1/p}f\|_{E_2}^{p}\, dt\right)^{1/p} \\ &= \left(\int_0^1 [g^*A^{2/q}g]^{2/q}\, dt\right)^{1/q} \left(\int_0^1 [f^*A^{2/p}f]^{p/2}\, dt\right)^{1/p}. \end{aligned}$$

(Here E_2 denotes the n-dimensional Euclidean vector norm $\| x \| = [\sum_1^n \| x_2 \|^2]^{1/2}$). Continuous linear functionals have the representation

$$\langle f, g \rangle = \int_0^1 g^* A f \, dt,$$

where $f \in \mathscr{L}_n{}^p[0, 1]$, $g \in \mathscr{L}_n{}^q[0, 1]$.[2]

Now let D denote those elements $y \in \mathscr{L}_n{}^p[0, 1]$ satisfying:

1. For each y there is an $s \times 1$ matrix valued constant ψ such that

$$(y + H[Py(0) + Qy(1)] + H_1\psi)$$

is absolutely continuous.

2. $J(y + H[Py(0) + Qy(1)] + H_1\psi)' - By = Af$, where $f \in \mathscr{L}_n{}^p[0, 1]$.
3. $My(0) + Ny(1) + \int_0^1 d(KJ)y = 0$,

$$\int_0^1 d(K_1 J)y = 0.$$

For all y in D we say that $ly = f$ provided

$$J(y + H[Py(0) + Qy(1)] + H_1\psi)' - By = Af.$$

Note that either ψ or f may not be unique. Further, the domain D may not be dense in $\mathscr{L}_n{}^p[0, 1]$. If $K_1 J$ contains rows which are absolutely continuous, D may be orthogonal to the subspace spanned by their conjugate transposes.

In order to handle these complexities we define a *linear relation* rather than an operator. Let L be the linear relation given by

$$L = \{(y, f): ly = f, y \in D\} \subset \mathscr{L}_n{}^p[0, 1] \times \mathscr{L}_n{}^p[0, 1].$$

It is this linear relation we wish to consider in detail, specifically we wish to determine its adjoint, determine when it is self-adjoint, project it onto subspaces, where it is uniquely and densely defined, and determine spectral resolutions for it where possible. The setting, as indicated, is that of a linear relation. We refer to Arens' paper [1] for details.

[2] Other equivalent norms are possible. If $\| f \|_{E_p}$ denotes $(\sum_{i=1}^n \| f_i \|^p)^{1/p}$, then $\| f \| = (\int_0^1 \| A^{1/p} f \|_{E_p}^p \, dt)^{1/p}$ is (perhaps) even more natural. Hölder's inequality then is

$$\left| \int_0^1 g^* A f \, dt \right| \leqslant \left(\int_0^1 \| A^{1/q} g \|_{E_q}^q \, dt \right)^{1/q} \left(\int_0^1 \| A^{1/p} f \|_{E_p}^p \, dt \right)^{1/p}.$$

Continuous linear functionals have the same form.

3. The Adjoint of L

Just as operators have, so do linear relations have adjoints. They are determined in much the same way as adjoint operators. Further, if a linear relation is generated by a densely defined operator, then the adjoint relation is generated by the adjoint operator.

If $(y, f) \in L$, then $(z, g) \in L^*$ if

$$\langle f, z\rangle - \langle y, g\rangle = 0.$$

As it was used earlier, $\langle f, z\rangle$ denotes a continuous linear functional generated by $z \in \mathscr{L}_n{}^q[0, 1]$ operating on $f \in \mathscr{L}_n{}^p[0, 1]$, $1/p + 1/q = 1$. Clearly $L^* \subset \mathscr{L}_n{}^q[0, 1] \times \mathscr{L}_n{}^q[0, 1]$. In order to find L^*, we introduce our candidate, exhibit Green's formula, then derive the result.

Let D^+ denote those elements $z \in \mathscr{L}_n{}^q[0, 1]$ satisfying:

1. For each z there is an $r \times 1$ matrix valued constant ϕ such that

$$(J^*z + J^*K^* [\tilde{M}z(0) + \tilde{N}z(1)] + J^*K_1{}^*\phi)$$

is absolutely continuous.

2. $-(J^*z + J^*K^* [\tilde{M}z(0) + \tilde{N}z(1)] + J^*K_1{}^*\phi)' - B^*z = Ag$, where $g \in \mathscr{L}_n{}^q[0, 1]$.

3. $\tilde{P}z(0) + \tilde{Q}z(1) + \int_0^1 dH^*(J^*z) = 0,$

$$\int_0^1 dH_1{}^*(J^*z) = 0.$$

For all $z \in D^+$ we say that $l^+z = g$, provided

$$-(J^*z + J^*K^* [\tilde{M}z(0) + \tilde{N}z(1)] + J^*K_1{}^*\phi)' - B * z = Ag.$$

The linear relation L^+ is given by

$$L^+ = \{(z, g)\colon l^+z = g,\ z \in D^+\} \subset \mathscr{L}_n{}^q[0, 1] \times \mathscr{L}_n{}^q[0, 1].$$

4.1. Theorem (Green's formula). *Let $(y, ly) \in L$, $(z, l^+z) \in L^+$. Then*

$$\begin{aligned}\int_0^1 [z^*\{J(y + H[Py(0) + Qy(1)] + H_1\psi)' - By\}\\ - \{-(J^*z + J^*K^*[\tilde{M}z(0) + \tilde{N}z(1)] + J^*K_1{}^*\phi)' - B^*z\}^* y]\, dt\\ = [\tilde{M}z(0) + \tilde{N}z(1)]^* \left[My(0) + Ny(1) + \int_0^1 d(KJ)y\right]\\ + \left[\tilde{P}z(0) + \tilde{Q}z(1) + \int_0^1 dH^*(J^*z)\right]^* [Py(0) + Qy(1)]\\ + \left[\int_0^1 dH_1{}^*(J^*z)\right]^* \psi + \phi^* \left[\int_0^1 d(K_1J)y\right].\end{aligned}$$

The proof involves a number of Stieltjes integrations by parts and is similar to that found in [15]. For further reference we recommend [10, 18].

We need a slight variation of l^+z. Let $l^{++}z = g$ if for some ϕ, ϕ_1,

$$-(J^*z + J^*K^*\phi_1 + J^*K_1^*\phi)' - B^*z = Ag, \qquad g \in \mathscr{L}_n{}^q[0, 1].$$

With no boundary constraints, this generates a larger linear relation L^{++}. Green's formula for L and L^{++} is

$$\begin{aligned}\int_0^1 &[z^*(\ell y) - (\ell^{++}z)^* y]\, dt \\ &= [\tilde{M}z(0) + \tilde{N}z(1)]^* \left[My(0) + Ny(1) + \int_0^1 d(KJ)y\right] \\ &\quad + \left[\tilde{P}(0) + \tilde{Q}z(1) + \int_0^1 dH^*(J^*z)\right]^* [Py(0) + Qy(1)] \\ &\quad + \left[\int_0^1 dH_1{}^*(J^*z)\right]^* \psi + \phi^* \left[\int_0^1 d(K_1J)y\right] \\ &\quad + [\phi_1 - \tilde{M}z(0) - \tilde{N}z(1)]^* \left[\int_0^1 d(KJ)y\right].\end{aligned}$$

4.2. THEOREM. *The domain of L^*, D^*, is D^+. $L^* = L^+$.*

Proof. Green's formula shows that $L^+ \subset L^*$. To show the reverse inclusion, temporarily let $y \in D \cap C_0{}^1[0, 1]$. Then

$$0 = \int_0^1 d(KJ)y = -\int_0^1 (KJ)\, y'\, dt,$$

$$0 = \int_0^1 d(K_1J)y = -\int_0^1 (K_1J)\, y'\, dt.$$

Further, ψ may be chosen 0 so that

$$Jy' - By = Af,$$

$ly = f$ and $(y, f) \in L$. Let $(z, g) \in L^*$. Then

$$\langle f, z\rangle - \langle y, g\rangle = 0,$$

or

$$\int_0^1 z^*Af\, dt - \int_0^1 g^*Ay\, dt = 0.$$

Since $Jy' - By = Af$,

$$\int_0^1 z^*[Jy' - By]\, dt - \int_0^1 g^*Ay\, dt = 0.$$

If the terms involving y are isolated and integration by parts is performed on them, the result is

$$\int_0^1 \left\{J^*z + \int_0^t [B^*z + Ag]\, d\xi\right\}^* y' = 0.$$

Since y vanishes at 0 and at 1, y' is orthogonal with weight matrix I to constants C. Further y' is orthogonal with weight matrix I to $(KJ)^*$ and $(K_1J)^*$ and nothing else. (Anything else would imply an extra constraint on D). Thus for appropriate ϕ, ϕ_1,

$$J^*z + \int_0^t [B^*z + Ag]\, d\xi = C - J^*K^*\phi_1 - J^*K_1{}^*\phi.$$

Transposing,

$$J^*z + J^*K^*\phi_1 + J^*K_1{}^*\phi = -\int_0^t [B^*z + Ag]\, d\xi + C.$$

Differentiating,

$$-(J^*z + J^*K^*\phi_1 + J^*K_1{}^*\phi)' - B * z = Ag.$$

This implies $(z, g) \in L^{++}$.

If y is now permitted to be an arbitrary element in D, Green's formula for L and L^{++} shows

$$0 = \left[\tilde{P}z(0) + \tilde{Q}z(1) + \int_0^1 dH_1{}^*(J^*z)\right]^* [Py(0) + Qy(1)]$$

$$+ \left[\int_0^1 dH_1{}^*(J^*z)\right]^* \psi$$

$$+ [\phi_1 - \tilde{M}z(0) - \tilde{N}z(1)]^* \left[\int_0^1 d(KJ)y\right].$$

$Py(0) + Qy(1)$ varies over C^{2n-m}, for if not, a linear combination of its rows would vanish, putting an extra constraint on D. Likewise ψ varies over C^s. Finally if a linear combination of rows of KJ were constant, then so would the same linear combination of components of $\int_0^1 d(KJ)y$ be zero. Its coefficient from $\phi_1 - \tilde{M}z(0) - \tilde{N}z(1)$ would be arbitrary. But then the corresponding product within

$$(J^*z + J^*K[\tilde{M}z(0) + \tilde{N}z(1)] + J^*K_1{}^*\phi)'$$

would vanish. So effectively

$$\tilde{P}z(0) + \tilde{Q}z(1) + \int_0^1 dH^*(J^*z) = 0,$$

$$\int_0^1 dH_1{}^*(J^*z) = 0,$$

and

$$\phi_1 = \tilde{M}z(0) + \tilde{N}z(1).$$

Throughout the preceding argument ψ was free. If, however, $\psi = \psi_0 + Q \int_0^1 d(K_1J)y$, then a minor variation in the argument and a minor change in Green's formula shows that $\phi = \phi_0 - Q^* \int_0^1 dH_1^*(J * z)$. These integrals are introduced in papers by Coddington [6] and Zimmerberg [22].

Note further there is nothing to stop a portion of H_1 or (K_1J) from being identical with H or (KJ). Hence integrals involving H and (KJ) may also appear in the differential equations.

4. Self-Adjoint Differential Boundary Relations

In this section we restrict our attention to the Hilbert space $\mathscr{L}_n^2[0, 1]$, generated by the inner product

$$\langle y, z \rangle = \int_0^1 z^*Ay\, dt,$$

and characterize those linear relations which are self-adjoint, i.e., those linear relations L which satisfy $L = L^*$.

5.1. Theorem. *The linear relation L is self-adjoint if and only if:*

(1) $J = -J^*, J' + B - B^* = 0$;

(2) $m = n, r = s$;

(3) $d(KJ) = d(CH^*)J^*, d(CH^*) = d(KJ)J^{*-1}$, *where* $C = MJ(0)^{-1}P^* - NJ(1)^{-1}Q^*$;

(4) $MJ(0)^{-1} M^* - NJ(1)^{-1}N^* = 0$;

(5) $dH[PJ(0)^{-1} P^* - QJ(1)^{-1} Q^*] = 0$;

(6) $d(K_1J) = d(EH_1^*)J^*, d(EH_1^*) = d(K_1J)J^{*-1}$, *where E is a nonsingular $r \times r$ matrix under which* $dH_1[E * \phi + \psi] = 0$.

Proof. We note that the assumptions concerning $M, N, P \cdot Q, \tilde{M}, \tilde{N}, \tilde{P} \cdot \tilde{Q}$ result in the equations

$$\begin{aligned} -MJ(0)^{-1} \tilde{M}^* + NJ(1)^{-1}\tilde{N}^* &= I, \\ -PJ(0)^{-1} \tilde{M}^* + QJ(1)^{-1}\tilde{N}^* &= 0, \\ -MJ(0)^{-1}\tilde{P}^* + NJ(1)^{-1}\tilde{Q}^* &= 0, \\ -PJ(0)^{-1} \tilde{P}^* + QJ(1)^{-1}\tilde{Q}^* &= I. \end{aligned}$$

Assume that L is self-adjoint. By restricting y to be absolutely continuous and vanish at 0 and 1, a comparison of the differential equations shows

$$Jy' - By = -(J^*y)' - B^*y.$$

If y is locally constant, we find J^* is differentiable and $0 = [-J^{*\prime} + B - B^*] = 0$. A reinsertion of this into the previous equation establishes $J = -J^*$.

Since the boundary conditions are equivalent, there must exist nonsingular matrices C and E such that

$$My(0) + Ny(1) + \int_0^1 d(KJ)y = C\left[\tilde{P}y(0) + \tilde{Q}y(1) + \int_0^1 dH^*(J^*y)\right],$$

$$\int_0^1 d(K_1J)y = E\int_0^1 dH_1{}^*(J^*y).$$

This implies $M = C\tilde{P}$, $N = C\tilde{Q}$, and

$$\int_0^1 d(KJ)y = C\int_0^1 dH^*(J^*y).$$

Multiplying $M^* = \tilde{P}^*C^*$ by $-MJ(0)^{-1}$ and $N^* = \tilde{Q}^*C^*$ by $NJ(1)^{-1}$ and adding results in

$$-MJ(0)^{-1}M^* + NJ(1)^{-1}N^* = 0.$$

Multiplying M^* by $-PJ(0)$, N^* by $QJ(1)^{-1}$, and adding results in

$$C = MJ(0)^{-1}P^* - NJ(1)^{-1}Q^*.$$

In

$$\int_0^1 d(KJ)y = \int_0^1 d(CH^*)(J^*y)$$

we vary y to conclude that for arbitrary measurable sets $S \subset [0, 1]$

$$\int_S d(KJ)\, J^{*-1} = \int_S d(CH^*).$$

This implies that (KJ) is absolutely continuous with respect to (CH^*). The Radon–Nikodym theorem then implies that

$$\int_S d(KJ) = \int_S d(CH^*)\, F_0,$$

where F_0 is the Radon–Nikodym derivative of (KJ) with respect to CH^*. Likewise

$$\int_S d(CH^*) = \int_S d(KS)F_1,$$

where F_1 is the Radon–Nikodym derivative of (CH^*) with respect to (KJ). An elementary substitution shows $F_0 = J^*$, $F_1 = J^{*-1}$ and

$$\int_S d(KJ) = \int_S d(CH^*)\, J^*,$$

$$\int_S d(CH^*) = \int_S d(KJ)\, J^{*-1}.$$

We will drop $\int_S$ and merely write $d(-)$ in the future.

Similarly

$$d(EH_1^*) = d(K_1J)J^{*-1},$$
$$d(K_1J) = d(EH_1^*)J^*.$$

Finally, comparing the differential equations in general,

$$Jdy + JdH[Py(0) + Qy(1)] + JdH_1\psi - Bydt$$
$$= -J^{*\prime}\, dt\, y - J^*\, dy - d(KJ)^*\, [\tilde{M}y(0) + \tilde{N}y(1)] - d(K_1J)^*\, \phi - B^*Y\ dt.$$

By permitting y to vary, this implies upon canceling

$$dH[C^*\tilde{M} - P] = 0, \qquad dH[C^*\tilde{N} - Q] = 0, \qquad dH_1[E^*\phi + \psi] = 0.$$

Multiplying the first by $-J(0)^{*-1}P^*$, the second by $J(1)^{*-1}Q^*$, and adding results in

$$dH[PJ(0)^{-1}\, P^* - QJ(1)^{-1}Q^*] = 0.$$

Conversely, suppose conditions 1–6 of the theorem hold. Then from the equations noted at the beginning of the proof and from condition 4, we find

$$(MJ(0)^{-1}\ NJ(1)^{-1}) \begin{pmatrix} \tilde{P}^* \\ -\tilde{Q}^* \end{pmatrix} = 0,$$

and

$$(MJ(0)^{-1}\ NJ(1)^{-1}) \begin{pmatrix} M^* \\ -N^* \end{pmatrix} = 0.$$

Employing an argument analogous to that of [8, pp. 289–291] we find there exists a nonsingular matrix T such that $M = T\tilde{P}$, $N = T\tilde{Q}$. A further simplification shows $T = C$.

Similarly, modulo multiplication by dH,

$$(PJ(0)^{-1}\ QJ(1)^{-1}) \begin{pmatrix} P^* \\ -Q^* \end{pmatrix} = 0$$

and

$$(PJ(0)^{-1}\ QJ(1)^{-1}) \begin{pmatrix} \tilde{M}^* \\ \tilde{N}^* \end{pmatrix} = 0.$$

Consequently, modulo dH, there exists a nonsingular matrix T_1 such that $P = T_1\tilde{M}$, $Q = T_1\tilde{N}$. A further simplification shows $T_1 = -C^*$ and that $dH[P + C^*\tilde{M}] = 0, dH[Q + C^*\tilde{N}] = 0$.
Hence

$$\begin{aligned}\{J(y &+ H[Py(0) + Qy(1) + H_1\psi)' - By\}\, dt\\ &= Jdy + JdH[Py(0) + Qy(1)] + JdH_1 - By\, dt\\ &= Jdy + dJy - B^*y\, dt - JdH[C^*\tilde{M}y(0) + C^*\tilde{N}y(1)] - JdH_1E^*\phi\\ &= -d(J^*y) - d(KJ)^*[\tilde{M}y(0) + \tilde{N}y(1)] - d(K_1J)^*\phi - B^*y\, dt\\ &= \{-(J^*y + J^*K^*[\tilde{M}y(0) + \tilde{N}y(1)] + J^*K_1^*(\phi)' - B^*y\}\, dt.\end{aligned}$$

Further,

$$E\int_0^1 dH_1^*(J^*y) = \int_0^1 d(K_1J)\, J^{*-1}(J^*y) = \int_0^1 d(K_1J)y,$$

$$C\left[\tilde{P}y(0) + \tilde{Q}y(1) + \int_0^1 dH^*(Jy)\right] = My(0) + Ny(1) + \int_0^1 d(KJ)y,$$

and L is self-adjoint.

If

$$\begin{aligned}\psi &= \psi_0 + Q\int_0^1 d(K_1J)y,\\ &= \psi_0 + QE\int_0^1 dH_1^*(J^*y),\end{aligned}$$

then it is easy to see $dH_1[E^*\phi_0 + \psi_0] = 0$ and $dH_1[E^*Q^* - QE]\int_0^1 dH_1^*(Jy) = 0$. Consequently the differential equation generating a self-adjoint linear relation in this case is

$$J\left(y + H[Py(0) + Qy(1)] + H_1QE\int_0^1 dH_1^*(J^*y) + H_1\psi_0\right)' - By = Af,$$

where $QE = E^*Q^*$. It is precisely this equation which is considered by Coddington [6] and Zimmerberg [22]. Differences in notation, however, almost preclude exhibiting the connection in detail.

In closing this section we would like to give three examples. First, if $J = (1/i)I$, $B = iP$, $A = I$, and M, N, P, Q are replaced by $(1/i)A$, $(1/i)B$, C, D, then L is the same as the linear relation M discussed in [15]. Second, if H, H_1, K, K_1 are all zero, then Theorem 5.1 is the classical result. Third, if

$$J = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \quad A = \begin{pmatrix} -w & 0\\ 0 & 0\end{pmatrix}, \quad B = \begin{pmatrix} p_1 & 0\\ 0 & 1\end{pmatrix},$$

$$H = -K = \begin{pmatrix} 0 & H(t-1/2)\\ -H(t-1/2) & 0\end{pmatrix}, \quad H_1 = K_1 = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix},$$

$$M = \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix},$$

then L is equivalent to the self-adjoint second-order problem

$$-y'' + \delta'(t - \tfrac{1}{2})\, y'(0) + \delta(t - \tfrac{1}{2})\, y(1) + p_1 y = -wf,$$
$$y(0) + y(\tfrac{1}{2}) = 0,$$
$$Dy(1) + Dy(\tfrac{1}{2}) = 0,$$

where $Dy = -y'(t) + \delta(t - \tfrac{1}{2})\, y'(0)$, $H(t - \tfrac{1}{2})$ is the Heaviside function jumping from 0 to 1 at $t = \tfrac{1}{2}$, and $\delta(t - \tfrac{1}{2})$, $\delta'(t - \tfrac{1}{2})$ are its first and second generalized derivatives.

5. Self-Adjoint Operators on Subspaces of $\mathscr{L}_n^2[0, 1]$

Let the columns of H_1 be suitably rearranged such that $H_1 = (H_c H_s)$, where the first s_1 form a maximal $n \times s_1$ absolutely continuous matrix H_c satisfying $JH_c' = AH_0$ with the columns of H_0 in $\mathscr{L}_n^p[0, 1]$. Let $\mathscr{H}_1$ denote the subspace of $\mathscr{L}_n^p[0, 1]$ spanned by the columns of H_0.

Likewise let the rows of K_1 be suitably rearranged such that $K_1 = \binom{K_c}{K_s}$, where the first r_1 when multiplied by J forms a maximal $r_1 \times n$ absolutely continuous matrix $K_c J$ satisfying $(K_c J)' = K_0 A$ with the columns of $K_0{}^*$ in $\mathscr{L}_n^q[0, 1]$. Let $\mathscr{H}_1{}^*$ denote the subspace of $\mathscr{L}_n^q[0, 1]$ spanned by the columns of $K_0{}^*$.

The system describing L can be written in the form

$$J(y + H[Py(0) + Qy(1)] + H_s\psi_s)' - By = Af - AH_0\psi_c\,,$$

where $\psi = \binom{\psi_c}{\psi_s}$,

$$My(0) + Ny(1) - \int_0^1 d(KJ)y = 0,$$
$$\int_0^1 d(K_s J)y = 0,$$
$$\int_0^1 K_0 Ay\, dt = 0.$$

If $ly = f$ is written as $l_s y + H_0\psi_c$, then $l_s y = f - H_0\psi_c$. Consequently L is defined on $D \subset \mathscr{H}_1^{*\perp}$, but is uniquely defined only in $\mathscr{L}_n^p[0, 1]/\mathscr{H}_1$.

Similarly the system describing L^* is

$$-(J^*z + J^*K^*\, [\tilde{M}z(0) + \tilde{N}z(1)] + J^*K_s{}^*\phi_s)' - B^*z = Ag + AK_0{}^*\phi_c,$$

where $\phi = \binom{\phi_s}{\phi_s}$,

$$\tilde{P}z(0) + \tilde{Q}z(1) + \int_0^1 dH^*(J^*z) = 0,$$

$$\int_0^1 dH_s^*(J^*z) = 0,$$

$$\int_0^1 H_0^*Az\,dt = 0.$$

If $l^+z = g$ is written as $l_s^+z - K_0^*\phi_c$, then $l_s^+z = g + K_0^*\phi_c$. Consequently L^* is defined on $D^* \subset \mathscr{H}_1^\perp$, but is uniquely defined only in $\mathscr{L}_n^q[0, 1]/\mathscr{K}_1^*$.

If the extra integral terms used by Coddington [6] and Zimmerberg [22] are present, then the same situation results with the integral terms within the parentheses to be differentiated.

The difficulties with uniqueness are considerably simplified if $p = q = 2$, $\mathscr{H}_1 = \mathscr{K}_1^*$ and the linear relation L is self-adjoint. L is then defined in $D \subset \mathscr{H}_1^\perp$ and is uniquely defined in $\mathscr{L}_n^2[0, 1]/\mathscr{H}_1 \simeq \mathscr{H}_1^\perp$. Therefore if ψ_c is determined by projecting l onto $\mathscr{H}_1^\perp$, a unique operator is defined which has domain dense in $\mathscr{H}_1^\perp$ and range also in $\mathscr{H}_1^\perp$.

We assume without loss of generality that the columns of H_0 are mutually orthonormal. If $ly = l_s y + H_0\psi_c$ is orthogonal to $\mathscr{H}_1$, then

$$\langle l_s y + H_0\psi_c, H_0\rangle = 0,$$

and

$$\psi_c = -\langle l_s y, H_0\rangle.$$

Hence the operator l projected onto $\mathscr{H}_1$ is given by

$$ly = l_s y - H_0\langle l_s y, H_0\rangle.$$

It is this expression with the integral term present in l_s which is determined by Coddington [6].

An inspection shows

6.1. THEOREM. *The linear relation L is self-adjoint if and only if the projected operator l is self-adjoint.*

7. EIGENFUNCTION EXPANSIONS

There are several variations of spectral resolutions or eigenfunction expansions in the literature. First the author in the regular case has discussed four situations. In [12], assuming $J = I$, $A = I$, H and K are absolutely continuous, and $H_1 = 0$, $K_1 = 0$, a nonself-adjoint eigenfunction series expansion was developed.

Likewise in [12] if $J = iI$, $A = I$, H and K are absolutely continuous, and $H_1 = 0$, $K_1 = 0$, a self-adjoint eigenfunction series expansion was found. In [14] the first was extended to permit H and K to be singular Stieltjes measures, but either H or K was required to be continuous. The second was also extended to permit H, K, H_1, K_1 to be singular Stieltjes measures.

Coddington [6] derived both regular and singular self-adjoint spectral resolutions for both nth order and vector problems under the assumption that H, K, H_1, K_1 were absolutely continuous. These results overlap with and generalize the self-adjoint expansion in [14]. Dijksma and de Snoo [9] extended Coddington's expansions to include pointwise convergence. Most recently Coddington and Dijksma [7] have derived expansions with self-adjoint case which permit H, K, H_1, K_1 to be arbitrary Stieltjes measures of bounded variation.

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