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Fuzzy Sets and Functions on Fuzzy Spaces*

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We generally follow the terminology of Azad (*J. Math. Anal. Appl.* **82** (1981), 14–32) and Ming (*J. Math. Anal. Appl.* **76** (1980), 571–599). In addition to the fundamental concepts for fuzzy sets, we emphasize the usefulness of the concepts of fuzzy point–fuzzy elementhood. For fuzzy sets A and B a new characterization is given for relations $A \subset B$ and $A = B$. This knowledge permits us to combine the two definitions of fuzzy point–fuzzy elementhood. In the third section some results are given concerning various special types of fuzzy sets in fuzzy topological spaces, and the fuzzy semi-regular, fuzzy regular spaces defined by Azad. In the last section, the definitions of *H.* almost continuous, *W.* almost open functions Urysohn space which are defined by Hussain, Wilansky, and Noiri, respectively, are extended to fuzzy sets. Furthermore some results are obtained in the functions of the fuzzy topological spaces defined by Azad and those are defined here. © 1987 Academic Press, Inc.

1. BASIC NOTATION AND DEFINITIONS

X always denotes a nonempty set. Fuzzy sets of X will be denoted by capital letters as A, B, C, etc. The value of a fuzzy set A at the element x of X will be denoted by $A(x)$, and fuzzy points will be denoted by p, r, s.

We write $p \in_1 A$, $p \in_2 A$, respectively, when the definitions of a fuzzy point and being an element of a fuzzy set are as given by Srivastava–Lal [3] and Ming [2]. Hence $p \in_1 A$ means p takes its single non-zero value in $(0, 1)$ at the support x_p (the support of p), and $p(x_p) < A(x_p)$, while $p \in_2 A$ means p takes its single non-zero value in $(0, 1]$, and $p(x_p) \leqslant A(x_p)$.

In this article $p \in A$ will stand for either $p \in_1 A$ or $p \in_2 A$. If we say only "fuzzy point p" then p will be considered as in [2] or [3]. Also, in the case $p \in_1 A$ we use the same definitions as given in [2].

Let A and B be fuzzy sets, p a fuzzy point in X. $\kappa(p)$, $\kappa_Q(p)$, $\mathring{A}$, $\bar{A}$, A' will denote, respectively, the neighborhood system of p, the Q-

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neighborhood system of p, the interior of A, the closure of A, and the complement of A. If A is quasi-coincident with B this will be denoted by AqB, and if p is quasi-coincident with A this will be denoted by pqA.

Known results valid for the general case $p \in A$ will not be proved.

2. Fuzzy Sets

Proposition 2.1. *If $A(x)$ is not zero for $x \in X$ then,*

$$A(x) = \sup_{0 < p_\lambda(x) < A(x)} p(x) = \sup_{0 < p_\lambda(x) \leqslant A(x)} p(x).$$

(Here supp $p_\lambda = x$ *for every* λ.)

Proof. Trivial.

Theorem 2.2.

(i) $A \subset B$ *iff* $p \in_1 A$ *implies* $p \in_1 B$ *for every* $p \in_1 X$.

(ii) $A \subset B$ *iff* $p \in_2 A$ *implies* $p \in_2 B$ *for every* $p \in_2 X$.

(iii) $A \subset B$ *iff* $p \in A$ *implies* $p(x_p) \leqslant B(x_p)$ *for every* p *in* X.

Proof. (i) Let $A \subset B$ and $p \in_1 A$. Clearly $p(x_p) < A(x_p) \leqslant B(x_p)$. This gives $p(x_p) < B(x_p)$, so we have $p \in_1 B$.

Conversely suppose $p \in_1 A \Rightarrow p \in_1 B$ but that $A \not\subset B$. Then for some $x \in X$, $B(x) < A(x)$. If p is a fuzzy point with support x, and satisfying $B(x) < p(x) < A(x)$ then $p \in_1 A$ but $p \notin_1 B$, which is a contradiction. This completes the proof of (i).

The proofs of (ii) and (iii) are similar. ∎

Corollary 2.3. $A = B$ *iff* $p \in A \Leftrightarrow p \in B$ *for every* p *in* X.

Theorem 2.4. (i) $p \in A \cap B$ *iff* $p \in A$ *and* $p \in B$.

(ii) $p \in A \cup B$ *iff* $p \in A$ *or* $p \in B$.

Proof. Let us prove that $p \in_1 A \cap B$ iff $p \in_1 A$ and $p \in_1 B$;

$$\begin{aligned} p \in_1 A \cap B &\Leftrightarrow p(x_p) < A \cap B(x_p) \\ &\Leftrightarrow p(x_p) < \inf\{A(x_p), B(x_p)\} \\ &\Leftrightarrow p(x_p) < A(x_p) \quad \text{and} \quad p(x_p) < B(x_p) \\ &\Leftrightarrow p \in_1 A \quad \text{and} \quad p \in_1 B. \end{aligned}$$

The proofs of the other case, and of (ii) can be given in a similar way. ∎

PROPOSITION 2.5. *Let A be a fuzzy set and for $x \in X$, $A(x) = t \neq 0$. If for any λ which satisfies the inequality $0 < \lambda < t$, we choose the fuzzy point p such that $p(x) = 1 - \lambda$, then $p\mathrm{q}A$ because $A(x) + (1 - \lambda) = t - \lambda + 1 > 1$.* ∎

3. Fuzzy Topological Spaces

Throughout this section, X will denote a fuzzy topological space with fuzzy topology τ.

With a small modification the following result is taken from [2, Theorem 4.1′].

THEOREM 3.1. *Let $p \in X$ and $A \subset X$. If $p \in \bar{A}$ then $A\mathrm{q}M$ for every $M \in \kappa_Q(p)$. If $A\mathrm{q}M$ for every $M \in \kappa_Q(p)$ then $p(x_p) \leqslant A(x_p)$.*

Proof. Clear from [2, Theorem 4.1′]. ∎

THEOREM 3.2. *A is a fuzzy open set iff A is a Q-neighborhood of p for every $p \in X$ which is quasi-coincident with A.*

Proof. Let A be fuzzy open. Clearly if any fuzzy point p is quasi-coincident with A then $A \in \kappa_Q(p)$.

Conversely, let $p \in_1 A$. Then $0 \neq p(x_p) < A(x_p)$ and we may consider the fuzzy point r with support x_p and $r(x_p) = 1 - p(x_p)$. By Proposition 2.5 we have $r\mathrm{q}A$ so by hypothesis $A \in \kappa_Q(r)$. Hence we have $U \in \tau$ with $r\mathrm{q}U$ and $U \subset A$,

$$\begin{aligned} r(x_p) + U(x_p) = 1 - p(x_p) + U(x_p) > 1 &\Rightarrow U(x_p) > p(x_p) \\ &\Rightarrow p \in_1 U \subset A \\ &\Rightarrow p \in_1 \mathring{A}. \end{aligned}$$

Hence $\mathring{A} \subset A$, and A is fuzzy open. ∎

PROPOSITION 3.3. *For a fuzzy topological space X, the following are equivalent:*

(i) *For any distinct fuzzy points $p \in_1 X$, $r \in_1 X$ (i.e., satisfying* $\operatorname{supp} p \neq \operatorname{supp} r$*), there exist fuzzy open sets U and V such that $p \in_1 U$, $r \in_1 V$ and $U \cap V = \varnothing$.*

(ii) *For any distinct fuzzy points $p \in_2 X$, $r \in_2 X$, there exist fuzzy open sets U and U and V such that $p\mathrm{q}U$, $r\mathrm{q}U$, and $U \cap V = \varnothing$.*

(iii) *For any distinct fuzzy points $p \in_1 X$, $r \in_1 X$, there exist fuzzy open sets U and V such that $p\mathrm{q}U$, $r\mathrm{q}V$, and $U \cap V = \varnothing$.*

Proof. The proof is easy but rather long and is omitted. ▮

X is said to be Hausdorff if it satisfies the equivalent condition of the Proposition 3.3 ([3] and [2]).

PROPOSITION 3.4. *Let $A, B \subset X$ and $\mathring{A} \subset B \subset \bar{A}$.*

(i) *If A is a fuzzy semi-open set then so is B.*

(ii) *If A is a fuzzy semi-closed set then so is B.*

Proof. (i) Let A be a fuzzy semi-open set and $\mathring{A} \subset B \subset \bar{A}$. There exists a fuzzy open set U such that $U \subset A \subset \bar{U}$. It follows that $U \subset \mathring{A} \subset A \subset \bar{A} \subset \bar{U}$ and hence $U \subset B \subset \bar{U}$. Thus B is a fuzzy semi-open set.

(ii) The proof is similar to (i). ▮

THEOREM 3.5. *Let $A \subset X$. A is a fuzzy semi-open set iff for every $p \in A$ there exists a fuzzy semi-open set O_p such that $p \in O_p \subset A$.*

Proof. If A is a fuzzy semi-open set then we may take $O_p = A$ for every $p \in A$.

Conversely we have $A = \bigcup_{p \in A} \{p\} \subset \bigcup_{p \in A} O_p \subset A$ and hence $A = \bigcup_{p \in A} O_p$. This shows that A is a fuzzy semi-open set. ▮

It can be easily seen from [2] that A is a fuzzy regular open (regular closed) set iff there exists a fuzzy set B such that $A = \mathring{\bar{B}}$ $(A = \bar{\mathring{B}})$.

PROPOSITION 3.6. *X is a fuzzy semi-regular space iff for every $p \in_1 X$ there exists a neighborhood base of p consisting of fuzzy regular open sets.*

Proof. Let $p \in_1 X$ and $M \in \kappa(p)$. There exists a fuzzy open set T such that $p \in_1 T \subset M$. From the definition of a fuzzy semi-regular space made by Azad [1] and [3, Theorem 2.1] there exists a fuzzy regular open set A such that $p \in_1 A \subset T$. Clearly $A \in \kappa(p)$.

Conversely, let T be a fuzzy open set. Then $T \in \kappa(p)$ for every $p \in_1 T$. Thus there exists a fuzzy regular open set A_p such that $A_p \in \kappa(p)$ and $p \in_1 A_p \subset T$ for every $p \in_1 T$.

From this we get $T = \bigcup A_p$. Hence X is a fuzzy semi-regular space. ▮

THEOREM 3.7. *X is a fuzzy semi-regular space iff for every $p \in X$ there exists a Q-neighborhood base of p consisting of fuzzy regular open sets.*

Proof. Let $p \in X$ and $M \in \kappa_Q(p)$. There exists $T \in \tau$ such that $T \subset M$ and $T \in \kappa_Q(p)$. Since pqT, there exists a regular open set A such that pqA and $A \subset T$ ([1, Definition 7.8; 2, Proposition 2.3]. This implies that $A \in \kappa_Q(p)$.

Conversely we suppose that X is not a fuzzy semi-regular space. This implies that there exists a fuzzy open set T such that T cannot be written as a union of fuzzy regular open sets.

Let $T_0 = \bigcup \{B \mid B \subset T,\ B \text{ a fuzzy regular open set}\}$. We have $T_0 \subset T$ but $T_0 \neq T$. It can be easily shown that $T_0 \neq \varnothing$.

$T_0 \neq T$ implies that there exists an element x of X such that $T_0(x) < T(x)$. We can choose $\varepsilon > 0$ such that $T_0(x) + \varepsilon < T(x)$. If we define p such that $p(x) = 1 - T_0(x) - \varepsilon$ then $p\mathrm{q}T$. Clearly $T \in \kappa_Q(p)$. There exists a fuzzy regular open set A such that $A \subset T$ and $A \in \kappa_Q(p)$. At the same time, $A \subset T_0$. We have

$$A(x) + p(x) \leqslant T_0(x) + 1 - T_0(x) - \varepsilon < 1$$

which is a contradiction. Hence X is a fuzzy semi-regular space. ∎

PROPOSITION 3.8. *X is a fuzzy regular space iff for every $p \in_1 X$ there exists a neighborhood base of p consisting of fuzzy closed sets.*

Proof. For every fuzzy open set T there exist fuzzy open sets U_α, $\alpha \in \Omega$ (Ω is an index set) such that

$$T = \bigcup_\alpha U_\alpha = \bigcup_\alpha \bar{U}_\alpha \qquad \text{and} \qquad U_\alpha \subset \bar{U}_\alpha \quad \text{for every } \alpha.$$

The remainder of the proof is similar to the proof of Proposition 3.6. ∎

THEOREM 3.9. *X is a fuzzy regular space iff for every $p \in X$ there exists a Q-neighborhood base of p consisting of fuzzy closed sets.*

Proof. This is similar to the proof of Theorem 3.7, the set T_0 involved in showing sufficiency in this case being

$$T_0 = \bigcup \{B \subset T \mid B \text{ is a fuzzy open set with } \bar{B} \subset T\}. \quad ∎$$

4. FUNCTION ON FUZZY SPACES

As the light to the knowledge in this section, the following theorem is gathered from some dissertation concerning fuzzy sets. For example, see [1, 4, 5].

THEOREM 4.1. *Let f be a function from X to Y, and I be any index set. The following statements are true:*

(1) *If $A \subset X$ then $f(A)' \subset f(A')$.*

(2) *If $B \subset Y$ then $f^{-1}(B') = f^{-1}(B)'$.*

(3) *If $A_1, A_2 \subset X$ and $A_1 \subset A_2$ then $f(A_1) \subset f(A_2)$.*

(4) *If* $B_1, B_2 \subset Y$ *and* $B_1 \subset B_2$ *then* $f^{-1}(B_1) \subset f(B_2)$.

(5) *If* $A \subset X$ *then* $A \subset f^{-1}(f(A))$.

(6) *If* $B \subset Y$ *then* $f(f^{-1}(B)) \subset B$.

(7) *If* $A_i \subset X$ *for every* $i \in I$ *then* $f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$.

(8) *If* $B_i \subset Y$ *for every* $i \in I$ *then*

$$f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i).$$

(9) *If* f *is one-to-one and* $A \subset X$ *then* $f^{-1}(f(A)) = A$.

(10) *If* f *is onto and* $B \subset Y$ *then* $f(f^{-1}(B)) = B$.

(11) *If* $A, B \subset X$ *then* $f(A \cap B) \subset f(A) \cap f(B)$.

(12) *If* $B_i \subset Y$ *for every* $i \in I$ *then*

$$f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i).$$

(13) *Let* g *be a function from* Y *to* Z. *If* $B \subset Z$ *then* $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$; *If* $A \subset X$ *then* $(g \circ f)(A) = g(f(A))$.

In addition to these properties the following statement is true at the same time.

(14) *If* f *is bijection then for* $A \subset X$, $f(A)' = f(A')$, *because*

$$f(f^{-1}(f(A)')) = f(f^{-1}(f(A))') = f(A').$$

Let f *be a function from* X *to* Y. *Clearly for every* $p \in X$, $f(p)$ *is a fuzzy point in* Y, *and if* supp $p = x_p$ *then* $\operatorname{supp}(f(p)) = f(x_p)$, $f(p)(f(x_p)) = p(x_p)$.

If $p \in Y$ *then* $f^{-1}(p)$ *needs not be a fuzzy point in* X. *If* f *is one-to-one and* $p \in f(X)$ *then* $f^{-1}(p)$ *will be a fuzzy point in* X. *In this case if* supp $p = y_p$ *then* supp $f^{-1}(p) = f^{-1}(y_p)$ and $f^{-1}(p)(f^{-1}(y_p)) = p(y_p)$.

PROPOSITION 4.2. *Let* f *be a function from* X *to* Y *and* $p \in X$.

(1) *If for* $B \subset Y$ *we have* $f(p)\,\mathrm{q}B$ *then* $p\mathrm{q}f^{-1}(B)$.

(2) *If for* $A \subset X$ *we have* $p\mathrm{q}A$ *then* $f(p)\,\mathrm{q}f(A)$.

Proof. (1) Let $f(p)\,\mathrm{q}B$ for $B \subset Y$. Clearly $f(p)(f(x_p)) \subset B(f(x_p)) > 1$. This gives that

$$f^{-1}(B)(x_p) + p(x_p) = B(f(x_p)) + f(p)(f(x_p)) > 1 \Rightarrow p\mathrm{q}f^{-1}(A),$$

[4, Definition 1.1; 2, Definition 2.2′].

(2) Let $p\,\mathrm{q}A$ for $A \subset X$. This gives, $p(x_p) + A(x_p) > 1$. This implies that

$$f(p)(f(x_p)) + f(A)(f(x_p)) = p(x_p) + \sup_{x \in f^{-1}(f(x_p))} A(x)$$
$$\geqslant p(x_p) + A(x_p) > 1.$$

Clearly we have $f(p)\,\mathrm{q}f(A)$. ▮

To the end of this work, X and Y denote fuzzy topological spaces with fuzzy topology τ and ϑ, respectively, and by $f\colon X \to Y$ we denote a function f of a fuzzy space X into a fuzzy space Y.

THEOREM 4.3. *If $f\colon X \to Y$ fuzzy open then $f^{-1}(\bar{B}) \subset \overline{f^{-1}(B)}$, for every $B \subset Y$.*

Proof. Let $B \subset Y$ and $p \in f^{-1}(\bar{B})$. First let us show that if $N \in \kappa_Q(p)$ then $f(N) \in \kappa_Q(f(p))$.

Let $N \in \kappa_Q(p)$. Then there exists $U \in \tau$ such that $p\,\mathrm{q}U \subset N$. This implies that $f(p)\,\mathrm{q}f(U) \subset f(N)$ (Proposition 4.2). Since f is fuzzy open function we have $f(U) \in \vartheta$. Thus $f(N) \in \kappa_Q(f(p))$,

$$p \in f^{-1}(\bar{B}) \Rightarrow f(p) \in f(f^{-1}(\bar{B})) \subset \bar{B}.$$

Again let $N \in \kappa_Q(p)$. Since $f(N) \in \kappa_Q(f(p))$ and from Theorem 3.1 there exists $y \in Y$ such that $f(N)(y) + B(y) > 1$. We choose $\varepsilon > 0$ such that $f(N)(y) + B(y) - \varepsilon > 1$. Since $f(N)(y) = \sup_{x \in f^{-1}(y)} N(x)$ there exists $x_0 \in f^{-1}(y)$ such that $f(N)(y) - \varepsilon < N(x_0)$ for this ε. For this x_0,

$$f^{-1}(B)(x_0) = B(f(x_0)) = B(y).$$

We have $N(x_0) + f^{-1}(B)(x_0) > f(N)(y) - \varepsilon + B(y) > 1$. Thus $N\mathrm{q}f^{-1}(B)$. Since this result is true for every $N \in \kappa_Q(p)$, we have $p(x_p) \leqslant \overline{f^{-1}(B)}(x_p)$ (Theorem 3.1). Now we arrive at the result $f^{-1}(\bar{B}) \subset \overline{f^{-1}(B)}$, (Theorem 2.2). ▮

COROLLARY 4.4. *If $f\colon X \to Y$ is a fuzzy open and fuzzy continuous function then $f^{-1}(\bar{B}) = \overline{f^{-1}(B)}$, for every $B \subset Y$.*

Proof. It is clear from [4] and Theorem 4.3.

THEOREM 4.5. *Let $f\colon X \to Y$. The following are equivalent:*

(1) *f is a fuzzy semi-continuous function.*

(2) *For every $p \in X$ and every $M \in \kappa(f(p))$, there exists a fuzzy semi-open set A such that $p \in A$ and $A \subset f^{-1}(M)$.*

(3) *For every* $p \in X$ *and every* $M \in \kappa(f(p))$, *there exists a fuzzy semi-open set* A *such that* $p \in A$ *and* $f(A) \subset M$.

(4) *For every* $p \in X$ *and every* $M \in \kappa_Q(f(p))$, *there exists a fuzzy semi-open set* A *such that* $p\mathrm{q}A$ *and* $A \subset M$.

(5) *For every* $p \in X$ *and every* $M \in \kappa_Q(f(p))$, *there exists a fuzzy semi-open set* A *such that* $p\mathrm{q}A$ *and* $A \subset f^{-1}(M)$.

(6) $f^{-1}(U) \subset \overline{f^{-1}(U)}^{\circ}$, *for every* $U \in \vartheta$.

(7) *For every fuzzy closed set* B *in* Y, $f^{-1}(B)$ *is a fuzzy semi-closed set in* X.

(8) *For every fuzzy closed set* B *in* Y, $f^{-1}(B) \supset \overline{\mathring{f^{-1}(B)}}$.

Proof. (1)$\Rightarrow$(2). Let $p \in X$ and $M \in \kappa(f(p))$. There exists $U \in \vartheta$ such that $f(p) \in U \subset M$. $f^{-1}(U)$ is a fuzzy semi-open set and we have $p \in f^{-1}(U) = A \subset f^{-1}(M)$.

(2)$\Rightarrow$(3) Let $p \in X$ and $M \in \kappa(f(p))$. There exists a fuzzy semi-open set A such that $p \in A$ and $A \subset f^{-1}(M)$. So we have $p \in A$, $f(A) \subset f(f^{-1}(M)) \subset M$.

(3)$\Rightarrow$(1) Let $U \in \vartheta$ and let us take $p \in f^{-1}(U)$. This shows that $f(p) \in f(f^{-1}(U)) \subset U$. Since U is a fuzzy open set we have $U \in \kappa(f(p))$. There exists a fuzzy semi-open set A such that $p \in A$ and $f(A) \subset U$. This shows $p \in A \subset f^{-1}(f(A)) \subset f^{-1}(U)$. From Theorem 3.5, $f^{-1}(U)$ is a fuzzy semi-open set.

(1)$\Rightarrow$(4) Let $p \in X$ and $M \in \kappa_Q(f(p))$. There exists $U \in \vartheta$ such that $f(p)\,\mathrm{q}U \subset M$. $f^{-1}(U)$ is a fuzzy semi-open set and from Proposition 4.2 we have $p\mathrm{q}f^{-1}(U)$. If we take $A = f^{-1}(U)$ then

$$f(A) = f(f^{-1}(U)) \subset U \subset M.$$

(4)$\Rightarrow$(5) Let $p \in X$ and $M \in \kappa_Q(f(p))$. There exists a fuzzy semi-open set A such that $p\mathrm{q}A$ and $f(A) \subset M$. Hence we have $p\mathrm{q}A$ and $A \subset f^{-1}(f(A)) \subset f^{-1}(M)$.

(5)$\Rightarrow$(1) Let us show that $f^{-1}(U)$ is a fuzzy semi-open set for any $U \in \vartheta$.

Let $U \in \vartheta$ and $p \in_1 f^{-1}(U)$. This implies that $f(p) \in_1 U$ (because $p(x_p) < f^{-1}(U)(x_p)$, $f(x_p)(f(x_p)) = p(x_p)$ and $f^{-1}(U)(x_p) = U(f(x_p))$),

$$f(p) \in_1 U \Rightarrow f(p)(f(x_p)) < U(f(x_p)).$$

If we define the fuzzy point p' being

$$p'(x_p) = 1 - p(x_p),$$

then

$$f(p')(f(x_p)) = p'(x_p) = 1 - p(x_p) = 1 - f(p)(f(x_p)),$$

$f(p')\,qU$ (Proposition 2.5). Since U is a fuzzy open set, we have $U \in \kappa_Q(f(p'))$. Thus there exists a fuzzy semi-open set A such that $p'qA \subset f^{-1}(U)$.

$$\begin{aligned} p'qA &\Rightarrow p'(x_p) + A(x_p) > 1 \\ &\Rightarrow A(x_p) > 1 - p'(x_p) = p(x_p) \\ &\Rightarrow p \in_1 A \subset f^{-1}(U). \end{aligned}$$

From Theorem 3.5, we have $f^{-1}(U)$ is a fuzzy semi-open set.

(1) ⇒ (6) Since $f^{-1}(U)$ is a fuzzy semi-open set for every $U \in \vartheta$ we have $f^{-1}(U) \subset \overline{\mathring{f}^{-1}(U)}$ [1, Theorem 4.2).

(6) ⇒ (1) From [2, Theorem 4.2], since any fuzzy set A which satisfies the relation $A \subset \overline{\mathring{A}}$ in X will be a fuzzy semi-open set, we have that $f^{-1}(U)$ is a fuzzy semi-open set for every $U \in \vartheta$.

(1) ⇒ (7) Let B be a fuzzy closed set in Y. This implies that $B' \in \vartheta$. We have $f^{-1}(B') = f^{-1}(B)'$. $f^{-1}(B')$ is a fuzzy semi-open set so is $f^{-1}(B)'$. Clearly $f^{-1}(B)$ is a fuzzy semi-closed set [1, Theorem 4.2].

(7) ⇒ (8), (8) ⇒ (7), and (7) ⇒ (1) can be easily proved. ∎

THEOREM 4.6. *Let $f: X \to Y$. The following are equivalent:*

(1) *f is a fuzzy weakly continuous function.*

(2) *For every fuzzy closed set B in Y, we have $f^{-1}(B) \supset \overline{f^{-1}(\mathring{B})}$.*

(3) *For every $p \in X$ and $M \in \kappa(f(p))$, $f^{-1}(\bar{M}) \in \kappa(p)$.*

(4) *For every $p \in X$ and $M \in \kappa(f(p))$, there exists $U \in \tau$ such that $p \in U$, $f(U) \subset \bar{M}$.*

(5) *For every $p \in X$ and $M \in \kappa_Q(f(p))$, $f^{-1}(\bar{M}) \in \kappa_Q(p)$.*

(6) *For every $p \in X$ and $M \in \kappa_Q(f(p))$, there exists $U \in \tau$ such that $U \in \kappa_Q(p)$ and $f(U) \subset \bar{M}$.*

(7) *For every $p \in X$ and any fuzzy net $\{p_\alpha\}_{\alpha \in \Phi}$ which is converging to p, if $M \in \kappa_Q(f(p))$ then there exists $\beta \in \Phi$ such that $f(p_\alpha)\,q\bar{M}$ for every $\alpha \geqslant \beta$, where Φ is a directed set.*

(8) *For every $V \in \vartheta$, $\overline{f^{-1}(V)} \subset f^{-1}(\bar{V})$.*

Proof. (1) ⇒ (2) Let B be a fuzzy closed set in $Y. B' \in \vartheta$, $f^{-1}(B') \subset (f^{-1}(\bar{B}'))^0$. This implies that $(f^{-1}(B))' \subset (f^{-1}((\mathring{B})'))^0$, [1, Lemma 3.2],

$$(f^{-1}(B))' \subset ((f^{-1}(\mathring{B}))')^0 = (\overline{f^{-1}(\mathring{B})})' \Rightarrow f^{-1}(B) \supset \overline{f^{-1}(\mathring{B})}.$$

$(2) \Rightarrow (1)$ Proof is similar to $(1) \Rightarrow (2)$.

$(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$ and $(5) \Rightarrow (6)$ can be easily proved.

$(6) \Rightarrow (7)$ Let $p \in X$, $p_\alpha \to p$, and $M \in \kappa_Q(f(p))$. There exists $U \in \tau$ such that $U \in \kappa_Q(p)$ and $f(u) \subset \bar{M}$. Since $p_\alpha \to p$, there exists $\beta \in \Phi$ such that $p_\alpha \mathrm{q} U$ for every $\alpha \geqslant \beta$ [2, Definition 11.4]. From Proposition 4.2 we have $f(p_\alpha) \mathrm{q} f(U) \subset \bar{M}$, for every $\alpha \geqslant \beta$. Clearly $f(p_\alpha) \mathrm{q} \bar{M}$, for every $\alpha \geqslant \beta$.

$(7) \Rightarrow (5)$ We suppose that (7) does not imply (5). There exists at least one $p \in X$ and $M \in \kappa_Q(f(p))$ such that for every fuzzy open set $U \in \kappa_Q(p)$, we have $U \not\subset f^{-1}(\bar{M})$.

Thus, there exists an element x_u of X such that $U(x_u) > f^{-1}(\bar{M})(x_u)$, for every $U \in \kappa_Q(p)$. We can choose $\varepsilon_u > 0$ for every fuzzy open set $U \in \kappa_Q(p)$ such that $U(x_u) > f^{-1}(\bar{M})(x_u) + \varepsilon_u$. Clearly, $U(x_u) + 1 - f^{-1}(\bar{M})(x_u) - \varepsilon_u > 1$ for every fuzzy open set $U \in \kappa_Q(p)$.

Let us define the fuzzy points in the following way.

$p_u(x_u) = 1 - f^{-1}(\bar{M})(x_u) - \varepsilon_u$, for every fuzzy open set $U \in \kappa_Q(p)$.

This follows that $U(x_u) + p_u(x_u) > 1$ for every fuzzy open set $U \in \kappa_Q(p)$.

If we denote the family of fuzzy open sets which belong to $\kappa_Q(p)$ by $\mathscr{L}$, then we can easily see from [2, Proposition 2.2] that $\mathscr{L}$ is a directed set (the relation $\leqslant$ in $\mathscr{L}$ is in the meaning of $U \subset V \Leftrightarrow U \geqslant V$).

The net $\{p_u\}_{U \in \mathscr{L}}$ which is chosen in the above way converges to p.

For every $U \in \mathscr{L}$,

$$\begin{aligned} f(p_u)(f(x_u)) &= p_u(x_u) \\ &= 1 - f^{-1}(\bar{M})(x_u) - \varepsilon_u \\ &= 1 - \bar{M}(f(x_u)) - \varepsilon_u. \end{aligned}$$

We have $f(p_u)(f(x_u)) + \bar{M}(f(x_u)) = 1 - \varepsilon_u < 1$, which contradicts (7).

Thus (7) implies (5).

$(1) \Rightarrow (5)$ Let $p \in X$ and $M \in \kappa_Q(f(p))$. There exists $U \in \vartheta$ such that $U \in \kappa_Q(f(p))$. Since, $p \mathrm{q} f^{-1}(U) \subset \overset{\circ}{f^{-1}}(\bar{M}) \subset f^{-1}(\bar{M})$ and $\overset{\circ}{f^{-1}}(\bar{M}) \in \tau$, we have $f^{-1}(\bar{M}) \in \kappa_Q(p)$.

$(5) \Rightarrow (1)$ Let us show that $f^{-1}(U) \subset \overset{\circ}{f^{-1}}(\bar{U})$ for $U \in \vartheta$. Let $U \in \vartheta$ and $p \in_1 f^{-1}(U) \Rightarrow f(p) \in_1 U$. If we define $p'(x_p) = 1 - p(x_p)$ then $f(p')(f(x_p)) = 1 - f(p)(f(x_p))$. Since $f(p)(f(x_p)) < U(f(x_p))$, we have $f(p') \mathrm{q} U$ (Proposition 2.5). Thus $U \in \kappa_Q(f(p'))$. This implies that $f^{-1}(\bar{U}) \in \kappa_Q(p')$. There exists $T \in \tau$ such that $p' \mathrm{q} T \subset f^{-1}(\bar{U})$. From here we write, $p'(x_p) + T(x_p) = 1 - p(x_p) + T(x_p) > 1$. This gives that

$$\begin{aligned} T(x_p) > p(x_p) &\Rightarrow p \underset{1}{\in} T \subset f^{-1}(\bar{U}) \\ &\Rightarrow p \underset{1}{\in} \overset{\circ}{f^{-1}}(\bar{U}). \end{aligned}$$

Hence we obtain $f^{-1}(U) \subset \overset{\circ}{f^{-1}}(\bar{U})$.

(2) ⇒ (8) Let $V \in \vartheta$. $\bar{V}$ is a fuzzy closed set and $V \subset \mathring{\bar{V}}$. Hence $f^{-1}(\bar{V}) \supset f^{-1}(\mathring{\bar{V}}) \supset \overline{f^{-1}(V)}$.

(8) ⇒ (2) Let B be a fuzzy closed set in Y. This gives that $\mathring{B} \in \vartheta$ and $\bar{\mathring{B}} \subset B$. Hence $\overline{f^{-1}(\mathring{B})} \subset f^{-1}(\bar{\mathring{B}}) \subset f^{-1}(B)$. ▮

DEFINITION 4.7. Let (X, τ) be a fuzzy topological space. If for any distinct fuzzy points p, r in X such that $p(x_p) < 1$, $r(x_r) < 1$, there exists fuzzy open sets U and V such that $p(x_p) < U(x_p)$, $r(x_p) < V(x_r)$ and $\bar{U} \cap \bar{V} = \varnothing$ then we say that X is a fuzzy Urysohn space.

PROPOSITION 4.8. *For fuzzy topological space X, the following are equivalent*:

(1) *X is a fuzzy Urysohn space.*

(2) *For any distinct fuzzy points $p \in_1 X$, $r \in_1 X$, there exist fuzzy open sets U and V such that $p \in_1 U$, $r \in_1 V$, and $\bar{U} \cap \bar{V} = \varnothing$.*

(3) *For any distinct fuzzy points $p \in X$, $r \in X$, there exist fuzzy open sets U and V such that $p\mathrm{q}U$, $r\mathrm{q}V$, and $\bar{U} \cap \bar{V} = \varnothing$.*

Proof. The proof is easy but long.

THEOREM 4.9. *Let Y be a fuzzy Urysohn space. If $f\colon X \to Y$ is fuzzy weakly continuous and one-to-one then X is a fuzzy Hausdorff space.*

Proof. Omitted.

THEOREM 4.10. *Let $f_1\colon X \to Y$, $f_2\colon X \to Y$ be fuzzy weakly continuous functions, let Y be a Urysohn space, and $A = \bigcup \{p \in X \mid f_1(p) = f_2(p)\}$. Then, A is a fuzzy closed set.*

Proof. Omitted.

COROLLARY 4.11. *Let $f_1\colon X \to Y$, $f_2\colon X \to Y$ be fuzzy weakly continuous functions, Y be a fuzzy Urysohn space and $A \subset X$. If $\bar{A} = X$ and $f_1(p) = f_2(p)$ for every $p \in A$ then $f_1 = f_2$.*

THEOREM 4.12. *Let $f\colon X \to Y$. The following are equivalent*:

(1) *f is fuzzy almost continuous.*

(2) *For every $V \in \vartheta$, $f^{-1}(V) \subset (f^{-1}(\mathring{\bar{V}}))^0$.*

(3) *For every fuzzy regular closed set A in Y, $f^{-1}(A)$ is a fuzzy closed set.*

(4) *For every fuzzy closed set B in Y, $\overline{f^{-1}(\bar{\mathring{B}})} \subset f^{-1}(B)$.*

(5) *For every* $p \in X$ *and every* $M \in \kappa(f(p))$, $f^{-1}(\mathring{M}) \in \kappa(p)$.

(6) *For every* $p \in X$ *and every* $M \in \kappa(f(p))$, *there exists* $U \in \tau$ *such that* $p \in U$ *and* $f(U) \subset \mathring{M}$.

(7) *For every* $p \in X$ *and every* $M \in \kappa_Q(f(p))$, *there exists* $U \in \tau$ *such that* $p \mathrm{q} U$ *and* $f(U) \subset \mathring{M}$.

(8) For every $p \in X$ *and every* $M \in \kappa_Q(f(p))$, $f^{-1}(\mathring{M}) \in \kappa_Q(p)$.

(9) *For every* $p \in X$, *if any net* $\{p_\alpha\}_{\alpha \in \Phi}$ *converges to* p, *then for every* $M \in \kappa_Q(f(p))$, *there exists* $\beta \in \Phi$ *such that* $f(p_\alpha) \mathrm{q} \mathring{M}$ *for every* $\alpha \geqslant \beta$.

Proof. (1) $\Leftrightarrow$ (2), (3) $\Leftrightarrow$ (4) are proved by Azad [1]. The others can be proved in a similar way to the proof of Theorem 4.6. ∎

DEFINITION 4.13. Let $f: X \to Y$ be a function. f is called,

(a) Fuzzy H. almost continuous, if for every fuzzy open set U in Y, $f^{-1}(U) \subset \overline{f^{-1}(U)}^{\circ}$ (in short f, fuzzy H.a.c.).

(b) Fuzzy W. almost open, if for every fuzzy open set U in Y, $f^{-1}(\bar{U}) \subset \overline{f^{-1}(U)}$ (in short f, fuzzy W.a.o.).

Remark 4.14. For the function $f: X \to Y$, the following statements are valid:

f, fuzzy continuous $\Rightarrow$ f, fuzzy H.a.c.,
f, fuzzy H.a.c. $\not\Rightarrow$ f, fuzzy weakly continuous,
f, fuzzy almost continuous $\not\Rightarrow$ f, fuzzy H.a.c.,
f, fuzzy H.a.c. $\not\Rightarrow$ f, fuzzy semi-continuous,
f, fuzzy semi-continuous $\not\Rightarrow$ f, fuzzy H.a.c.,
f, fuzzy open $\Rightarrow$ f, fuzzy W.a.o.,
f, fuzzy W.a.o. $\not\Rightarrow$ f, fuzzy semi-open.

EXAMPLE 4.15. Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$ and $T_1 \subset X$, $T_2 \subset X$, $T_3 \subset X$, $U_1 \subset Y$, $U_2 \subset Y$, $U_3 \subset Y$ be defined as follows:

$$
\begin{array}{lll}
T_1(a) = 0, & T_1(b) = 0,3, & T_1(c) = 0,2, \\
T_2(a) = 0,9, & T_2(b) = 0,6, & T_2(c) = 0,7, \\
T_3(a) = 0,2, & T_3(b) = 0,3, & T_3(c) = 0,2, \\
U_1(x) = 0, & U_1(y) = 0,4, & U_1(z) = 0,2, \\
U_2(x) = 0, & U_2(y) = 0,8, & U_2(z) = 0,2, \\
U_3(x) = 0, & U_3(y) = 0,6, & U_3(z) = 0,2,
\end{array}
$$

(a) Let $\tau=\{X, \varnothing, T_1, T_2\}$, $\vartheta=\{Y, \varnothing, U_1\}$.

If we define $f\colon Y\to X$ satisfying $f(x)=a$, $f(y)=b$, $f(z)=c$ then f is fuzzy H.a.c. but not fuzzy weakly continuous.

(b) Let $\tau=\{X, \varnothing, T_3, T_2\}$, $\vartheta=\{Y, \varnothing, U_3\}$.

If we define $f\colon X\to Y$ satisfying $f(a)=x$, $f(b)=y$, $f(c)=z$ then f is fuzzy almost continuous but not fuzzy H.a.c.

(c) If we define τ and ϑ as in (a) and f as in (b) then f is fuzzy W.a.o. but not fuzzy semi-open.

(d) Let $\tau=\{X, \varnothing, T_1, T_2\}$, $\vartheta=\{Y, \varnothing, U_2\}$.

If define f as in (b) then f is fuzzy H.a.c. but not fuzzy semi-continuous.

COROLLARY 4.16. *$f\colon X\to Y$ is fuzzy* W.a.o. *and fuzzy weakly continuous iff $f^{-1}(\bar{V})=\overline{f^{-1}(V)}$ for every fuzzy open set V in Y.*

Proof. Clear from Definition 4.13 and Theorem 4.6.

THEOREM 4.17. *Let $f\colon X\to Y$. The following are equivalent:*

(1) *f is fuzzy* H.a.c.

(2) *For every fuzzy closed set F in Y, $f^{-1}(F)\supset\overline{\mathring{f}^{-1}(F)}$.*

(3) *For every $p\in X$ and every $M\in\kappa(f(p))$, $\overline{f^{-1}(M)}\in\kappa(p)$.*

(4) *For every $p\in X$ and every $M\in\kappa_Q(f(p))$, $\overline{f^{-1}(M)}\in\kappa_Q(p)$.*

(5) *For every $p\in X$, if any net $\{p_\alpha\}_{\alpha\in\Phi}$ converges to p then for every $M\in\kappa_Q(f(p))$ there exists $\beta\in\Phi$ such that $p_\alpha q\overline{f^{-1}(M)}$ for every $\alpha\geqslant\beta$.*

(6) *For every fuzzy open set T in X, $f(\bar{T})\subset\overline{f(T)}$.*

Proof. (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Leftrightarrow$(5) can be proved in a similar way to that of Theorem 4.6.

(2)$\Rightarrow$(6) Let $T\in\tau$. Clearly $\mathring{\bar{T}}=\bar{T}$. $\overline{f(T)}$ is a fuzzy closed set in Y. We have

$$f^{-1}(\overline{f(T)})\supset\overline{(f^{-1}(\overline{f(T)}))^0}\supset\overline{(f^{-1}(f(T)))^0}\supset\mathring{\bar{T}}\supset\bar{T}.$$

This implies that $f(f^{-1}(\overline{f(T)}))\supset f(\bar{T})$

Hence $\overline{f(T)}\supset f(\bar{T})$.

(6)$\Rightarrow$(2) Let F be a fuzzy closed set in Y. Clearly $\mathring{f}^{-1}(F)\in\tau$. We have

$$f(\overline{(f^{-1}(F))^0})\subset\overline{f((f^{-1}(F))^0)}\subset\overline{f(f^{-1}(F))}\subset\bar{F}=F.$$

This implies that

$$\overline{(f^{-1}(F))^0}\subset f^{-1}(f(\overline{(f^{-1}(F))^0}))\subset f^{-1}(F). \quad \blacksquare$$

THEOREM 4.18. *If $f: X \to Y$ is fuzzy* W.a.o. *and fuzzy weakly continuous then f is fuzzy almost continuous.*

Proof. Let A be a fuzzy regular closed set in Y. Clearly $A = \bar{\mathring{A}}$. Since $\mathring{A}$ is a fuzzy open set and by Corollary 4.16, we have $f^{-1}(\bar{\mathring{A}}) = \overline{f^{-1}(\mathring{A})}$. Hence $f^{-1}(A) = \overline{f^{-1}(\mathring{A})}$. This shows that $f^{-1}(A)$ is a fuzzy closed set. Thus f is a fuzzy almost continuous function, (Theorem 4.12). ∎

COROLLARY 4.19. *Let $f: X \to Y$. f is fuzzy* W.a.o. *and fuzzy almost continuous iff $f^{-1}(\bar{V}) = \overline{f^{-1}(V)}$ for every fuzzy open set V in Y.*

Proof. Necessity is clear from [1, Remark 8.2] and Corollary 4.16. Sufficiency is clear from Corollary 4.16 and Theorem 4.18. ∎

THEOREM 4.20. *If $f: X \to Y$ is fuzzy* W.a.o. *and fuzzy weakly continuous then, f is fuzzy* H.a.c.

Proof. Let $U \in \vartheta$. We have $f^{-1}(U) \subset f^{-1}(\bar{U}) \subset \overline{f^{-1}(U)}$ (Theorem 4.6 and Definition 4.13). Hence f is fuzzy H.a.c. ∎

PROPOSITION 4.21. *If $f: X \to Y$ is fuzzy* W.a.o. *and fuzzy weakly continuous then for every fuzzy regular open (fuzzy regular closed) set A in Y, $f^{-1}(A)$ is fuzzy regular open (fuzzy regular closed) set in X.*

Proof. Easy. ∎

REFERENCES

1. K. K. AZAD, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, *J. Math. Anal. Appl.* **82** (1981), 14–32.
2. P. P. MING AND L. Y. MING, Fuzzy topology. I. Neighborhood structure of a fuzzy point and Moore Smith convergence, *J. Math. Anal. Appl.* **76** (1980), 571–599.
3. R. SRIVASTAVA, S. N. LAL, AND SRIVASTAVA, Fuzzy Hausdorff topological spaces, *J. Math. Anal. Appl.* **81** (1981), 497–506.
4. P. P. MING AND L. Y. MING, Fuzzy topology II. Product and quotient spaces, *J. Math. Anal. Appl.* **77** (1980), 20–37.
5. K. K. AZAD, Fuzzy Hausdorff spaces and fuzzy perfect mappings, *J. Math. Anal. Appl.* **82** (1981), 297–305.
6. N. BOURBAKI, "General Topology," Part 1, Hermann–Addison–Wesley, Reading, MA, 1966.
7. N. LEVINE, Semi-open sets and semi-continuity in topological spaces. *Amer. Math. Monthly* **70** (1963), 36–41.
8. N. LEVINE, A decomposition of continuity in topological spaces, *Amer. Math. Monthly* **68** (1961), 44–46.

9. P. E. LONG AND D. A. CARNAHAN, Comparing almost-continuous functions, *Proc. Amer. Math. Soc.* **38** (1973), 413–418.
10. P. E. LONG AND E. E. MC GEHEE, Properties of almost continuous functions, *Proc. Amer. Math. Soc.* **27** (1970), 175–180.
11. M. K. SINGAL AND A. R. SINGAL, Almost-continuous mappings, *Yokohama Math. J.* **16** (1968), 63–73.
12. T. NOIRI, On weakly continuous mappings, *Proc. Amer. Math. Soc.* **46** (1974), 120–124.
13. T. NOIRI, Semi-continuity and weak-continuity, *Czechoslovak Math. J.* **31** (1979), 314–321.
14. T. NOIRI, Properties of *S*-closed spaces, *Acta Math. Acad. Sci. Hungar.* **35** (1980), 431–436.