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On the Exterior Boundary-Value Problem for the Time-Harmonic Maxwell Equations

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For the exterior boundary-value problem of electromagnetic reflection at perfect conductors a new integral equation approach is developed. It extends the method introduced by Brakhage and Werner and by Leis for exterior boundary-value problems for the scalar Helmholtz equation to the underlying case of Maxwell's equations. In a unified approach for all frequencies the existence of a solution is established by using the first part of Fredholm's alternative only.

1. INTRODUCTION

Boundary-value problems for the scalar Helmholtz equation

$$\Delta\varphi + \kappa^2\varphi = 0, \qquad \kappa \neq 0, \operatorname{Im}(\kappa) \geqslant 0, \tag{1.1}$$

can be reduced to Fredholm integral equations of the second kind by seeking the solutions in the form of a double-layer potential

$$\varphi(x) = \int_S \mu(y) \frac{\partial \Phi(x, y)}{\partial n(y)}\, ds(y) \tag{1.2}$$

for the Dirichlet problem and a single-layer potential

$$\varphi(x) = \int_S \nu(y)\, \Phi(x, y)\, ds(y) \tag{1.3}$$

for the Neumann problem as described in many textbooks, e.g., [6, 9]. By S we denote the boundary of the domain D in which the boundary-value problem is to be considered and n is the unit normal to S directed into D. By

$$\Phi(x, y) := \frac{1}{4\pi} \frac{e^{i\kappa|x-y|}}{|x-y|} \tag{1.4}$$

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we denote the fundamental solution of (1.1) in three dimensions. The continuous functions μ and ν represent the unknown densities for which the integral equations have to be solved.

For exterior problems where D is unbounded the corresponding homogeneous integral equations have nontrivial solutions if and only if the homogeneous interior Neumann problem in $D_i := \mathbb{R}^3\backslash\bar{D}$ (when dealing with the exterior Dirichlet problem) and the homogeneous interior Dirichlet problem (when dealing with the exterior Neumann problem) have nontrivial solutions. Therefore, since for $\text{Im}(\kappa) > 0$ the homogeneous interior problems only have the trivial solution, the first part of Fredholm's alternative implies that the inhomogeneous integral equations have exactly one solution. Thus, existence for the exterior boundary-value problems is established for $\text{Im}(\kappa) > 0$. However, difficulties arise for real κ because the interior problems have a countable set of positive eigenvalues. Hence, the first part of Fredholm's alternative no longer applies to such values of κ which are eigenvalues to the interior Neumann or Dirichlet problem. For these κ, using the second part of Fredholm's alternative, existence to the exterior problems can still be established, but the solutions to the corresponding integral equations no longer remain unique and the proofs require a detailed study of the interior eigensolutions [6, 7, 11]. Since, after incorporating Sommerfeld's radiation condition in the form[1]

$$\left(\frac{x}{|x|}, \text{grad}\,\varphi\right) - i\kappa\varphi = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty, \tag{1.5}$$

uniformly for all directions $x/|x|$, by Rellich's lemma [14] we always have uniqueness for the exterior problems, the complications at the interior eigenvalues arise from the method of solution and not from the nature of the problem itself. In addition the different behavior of the integral equations at the interior eigenvalues causes complications in the investigations of properties of the solutions to the boundary-value problems such as dependence on the frequency κ. It also leads to considerable numerical difficulties when using the integral equations to obtain approximate solutions to the boundary-value problems. Therefore, various modifications of the original approach have been developed to overcome these difficulties and obtain integral equations which are found to be uniquely solvable for all κ by applying only the first part of Fredholm's alternative.

Werner [18] suggested adding a volume potential

$$\varphi(x) = \int_{D_i} \gamma(y)\, \Phi(x, y)\, dy \tag{1.6}$$

[1] By (a, b), $[a, b]$, and (a, b, c) we denote the scalar product, vector product, and triple scalar product of the vectors a, b, and c, respectively.

with continuous density γ to the surface potentials (1.2) or (1.3) such that the dissipative differential equation

$$\Delta\varphi + (\kappa^2 + i\tau)\varphi = 0 \qquad \text{in } D_i \tag{1.7}$$

is satisfied, where τ is an appropriately chosen continuous real function. Then, for all κ the boundary-value problems are reduced to uniquely solvable systems of two Fredholm integral equations of the second kind for the unknown surface and volume densities.

Werner's approach gave rise to a further modification due to Brakhage and Werner [2] and Leis [8, 9], in which the solutions are sought in the form

$$\varphi(x) = \int_S \mu(y) \left\{\frac{\partial\Phi(x, y)}{\partial n(y)} + i\Phi(x, y)\right\} ds(y). \tag{1.8}$$

Again, integral equations of the second kind are obtained which are uniquely solvable for all κ. Because of the singular behavior of the normal derivative of double-layer potentials in the case of the Neumann problem the integral equation turns out to be singular and a certain regularization technique has to be employed.

In another approach Ursell [16] replaced in potentials (1.2) and (1.3) the fundamental solution Φ, which may be regarded as the free space Green's function, by a Green's function for the exterior of some ball B lying inside D_i and satisfying the dissipative boundary condition

$$\frac{\partial\varphi}{\partial n} + i\varphi = 0 \tag{1.9}$$

on the boundary of B.

Finally Jones [5] proposed a modification in which the free space fundamental solution Φ is altered by adding a suitable finite sum behaving singularly at some point inside D_i. We would like to mention that recently Ursell [17] has given a shortened version of the proof of one of the main results in Jones' work.

Similar problems occur in the treatment of exterior boundary-value problems for the time-harmonic Maxwell equations

$$\operatorname{rot} E - i\kappa H = 0, \qquad \operatorname{rot} H + i\kappa E = 0 \qquad \text{in } D,\ \kappa \neq 0,\ \operatorname{Im}(\kappa) \geqslant 0, \tag{1.10}$$

subject to the radiation condition

$$\left[H, \frac{x}{|x|}\right] - E = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty, \tag{1.11}$$

uniformly for all directions $x/|x|$. The investigation of stationary electromagnetic reflection [13] leads to the electric boundary condition

$$[n, E] = c \text{ on } S, \tag{1.12}$$

where c denotes a given tangential field. Approaching this boundary-value problem by representing the desired solution in the form

$$E(x) = \operatorname{rot} \int_S a(y)\, \Phi(x, y)\, ds(y), \tag{1.13}$$

one obtains a Fredholm integral equation of the second kind for the unknown continuous tangential field a. The corresponding homogeneous equation has nontrivial solutions if and only if the homogeneous interior boundary-value problem for the Maxwell equations in D_i with boundary condition

$$[n, H] = 0 \text{ on } S \tag{1.14}$$

has nontrivial solutions. Again, for $\operatorname{Im}(\kappa) > 0$ this interior problem has only the trivial solution and for real κ there exists a countable number of positive eigenvalues. Hence, in the first case existence follows by means of the first part of Fredholm's alternative while in the latter case the second part of Fredholm's alternative must be applied. Existence proofs by this approach were obtained by Müller [10, 12], Weyl [21], Saunders [15], and Calderón [3].

As in the case of the Helmholtz equation it is desirable to develop modifications of (1.13) in order to achieve integral equations for which the first case of Fredholm's alternative applies for all κ. From the four methods mentioned above for the Helmholtz equation so far only Werner's approach using volume potentials has been extended by him to the case of Maxwell's equations [20]. Werner reformulated the problem into the slightly more general form of a boundary-value problem for the vector Helmholtz equation

$$\Delta E + \kappa^2 E = 0 \text{ in } D, \qquad \kappa \neq 0, \operatorname{Im}(\kappa) \geqslant 0, \tag{1.15}$$

with electric boundary conditions

$$[n, E] = c, \qquad \text{on} \quad S, \tag{1.16}$$

$$\operatorname{div} E = \gamma, \tag{1.17}$$

and radiation condition

$$\left[\operatorname{rot} E, \frac{x}{|x|}\right] + \frac{x}{|x|} \operatorname{div} E - i\kappa E = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty, \tag{1.18}$$

uniformly for all directions $x/|x|$. Here, c denotes a given tangential field and γ a given scalar function. In his modified approach Werner adds to (1.13) a volume potential of the form

$$E(x) = \int_{D_i} b(y)\, \Phi(x, y)\, dy \tag{1.19}$$

with a continuous density b such that

$$\Delta E + (\kappa^2 + i\tau)E = 0 \qquad \text{in } D_i \tag{1.20}$$

and in addition, in order to deal with the second boundary condition (1.17), he adds a surface potential

$$E(x) = \int_S \lambda(y)\, n(y)\, \Phi(x, y)\, ds(y) \tag{1.21}$$

with a continuous density λ. Then, a system of three Fredholm integral equations of the second kind is obtained for the one-volume and the two-surface densities which is shown to be uniquely solvable for all κ.

The aim of this paper is to extend the approach due to Brakhage and Werner [2] and Leis [8, 9] from the case of the Helmholtz equation to the case of the Maxwell equations—in other words, to eliminate the volume potential (1.19) in Werner's method. In Section 2 we shall prove a representation theorem for solutions to (1.15) and (1.18). Then, in Section 3 we shall give the precise statement of the boundary-value problem we are investigating and prove uniqueness. In Section 4 we try to find a solution to the boundary-value problem in a form which is motivated by the expressions occurring in the representation theorem and is the sum of four surface potentials with one vector and one scalar density combined in the fashion of (1.8). We obtain a system of two integral equations of the second kind for the two unknown densities. Because of the singular behavior of two of the potentials, it is necessary, for an appropriate discussion of the integral equations, that the integral operators involved be considered in a Banach space of uniformly Hölder continuous functions and vector fields. This difficulty of having to require Hölder continuity rather than merely continuity is related in some way to the singularity in approach (1.8) for the Neumann problem. The investigation of the integral operators is carried out in Section 5 with the result that the Riesz theory for compact operators is applicable. Since the homogeneous integral equation is proved to possess only the trivial solution for all κ, by the Riesz theory existence of a solution to the inhomogeneous integral equation and therefore of the boundary-value problem (1.15) to (1.18) follows. Thus, we have obtained the existence result by showing uniqueness, which actually means that the first part of Fredholm's alternative applies for all κ.

We explicitly want to emphasize that of course we do not obtain any new result on the existence of solutions to the boundary-value problem. Our aim merely is to obtain the existence by a unified approach for all κ. The main advantage of our new method as compared with Werner's method lies in the fact that no volume potential is used. This might make our method more suitable for a numerical implementation. We are also able to weaken slightly Werner's assumptions on the regularity of the boundary and the given boundary data.

We conclude this introduction with the remark that similar investigations can be carried out for magnetic boundary conditions

$$[n, \operatorname{rot} E] = d, \qquad (n, E) = \delta \text{ on } S, \tag{1.22}$$

where d is a given tangential field and δ a given function.

2. A Representation Theorem

Let D be an unbounded connected domain in $\mathbb{R}^3$. The boundary of D, denoted by S, is assumed to consist of a finite number of disjoint, closed, bounded surfaces belonging to the class C^2. The complement of $\bar{D}$ in $\mathbb{R}^3$ is designated by D_i. By n we denote the unit normal to S directed into D.

For convenience we state a representation theorem for solutions of the vector Helmholtz equation

$$\Delta E + \kappa^2 E = 0 \text{ in } D, \qquad \kappa \neq 0, \operatorname{Im}(\kappa) \geqslant 0, \tag{2.1}$$

satisfying the radiation condition

$$\left[\operatorname{rot} E, \frac{x}{|x|}\right] + \frac{x}{|x|} \operatorname{div} E - i\kappa E = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty, \tag{2.2}$$

uniformly for all directions $x/|x|$. This representation is closely related to a representation theorem for solutions of the equation rot rot $E - \kappa^2 E = 0$ given by Wilcox [22]. To simplify notation, for any domain G with boundary ∂G we introduce the linear space of vector fields

$$V(G) := \{E : \bar{G} \to \mathbb{C}^3 \mid E \in C^2(G) \cap C(\bar{G}), \operatorname{div} E, \operatorname{rot} E \in C(\bar{G})\}. \tag{2.3}$$

The representation theorem is based on the first vector Green's theorem

$$\begin{aligned} &\int_G \{(A, \Delta B) + (\operatorname{rot} A, \operatorname{rot} B) + \operatorname{div} A \operatorname{div} B\}\, dx \\ &\qquad = \int_{\partial G} \{(n, A, \operatorname{rot} B) + (n, A) \operatorname{div} B\}\, ds \end{aligned} \tag{2.4}$$

and the second vector Green's theorem

$$\begin{aligned} &\int_G \{(A, \Delta B) - (B, \Delta A)\}\, dx \\ &\quad = \int_{\partial G} \{(n, A, \operatorname{rot} B) + (n, A) \operatorname{div} B - (n, B, \operatorname{rot} A) - (n, B) \operatorname{div} A\}\, ds. \end{aligned} \tag{2.5}$$

Both theorems hold in bounded domains G with a sufficiently smooth boundary, e.g., $\partial G \in C^2$, and for vector fields $A, B \in V(G)$ and are easily obtained from Gauss' theorem. Here n denotes the outward drawn unit normal to ∂G.

THEOREM 2.1. *Let $E \in V(D)$ be a solution of the vector Helmholtz equation* (2.1) *satisfying radiation condition* (2.2). *Then*

$$E(x) = \operatorname{rot} \int_S [n(y), E(y)]\, \Phi(x, y)\, ds(y) - \operatorname{grad} \int_S (n(y), E(y))\, \Phi(x, y)\, ds(y) - \int_S \{[\operatorname{rot} E(y), n(y)] + n(y) \operatorname{div} E(y)\}\, \Phi(x, y)\, ds(y), \qquad x \in D. \tag{2.6}$$

Proof. (I) First we verify that

$$\int_{K_R} |E|^2\, ds = O(1), \qquad R \to \infty, \tag{2.7}$$

where $K_R := \{x \in \mathbb{R}^3 \mid |x| = R\}$. To accomplish this we observe that from the radiation condition (2.2) it follows that

$$\begin{aligned} 0 &= \lim_{R\to\infty} \int_{K_R} |[\operatorname{rot} E, n] + n \operatorname{div} E - i\kappa E|^2\, ds \\ &= \lim_{R\to\infty} \int_{K_R} (|[\operatorname{rot} E, n] + n \operatorname{div} E|^2 + |\kappa|^2 |E|^2 \\ &\quad + 2 \operatorname{Im}\{\kappa(E, [\operatorname{rot} \bar{E}, n] + n \operatorname{div} \bar{E})\})\, ds, \end{aligned} \tag{2.8}$$

where n denotes the outward drawn unit normal to K_R. We take R large enough so that $K_R \subset D$ and apply the first Green's theorem (2.4) to the domain $D_R := \{x \in D \mid |x| < R\}$ to obtain

$$\begin{aligned} &\kappa \int_{K_R} (E, [\operatorname{rot} \bar{E}, n] + n \operatorname{div} \bar{E})\, ds \\ &\quad = \kappa \int_S \{(n, E, \operatorname{rot} \bar{E}) + (n, E) \operatorname{div} \bar{E}\}\, ds \\ &\quad\quad - \bar{\kappa} |\kappa|^2 \int_{D_R} |E|^2\, dx + \kappa \int_{D_R} \{|\operatorname{rot} E|^2 + |\operatorname{div} E|^2\}\, dx. \end{aligned} \tag{2.9}$$

Now we insert the imaginary part of (2.9) into (2.8) and find

$$\begin{aligned} &\lim_{R\to\infty} \Big\{\int_{K_R} (|[\operatorname{rot} E, n] + n \operatorname{div} E|^2 + |\kappa|^2 |E|^2)\, ds \\ &\quad + 2 \operatorname{Im}(\kappa) \int_{D_R} (|\kappa|^2 |E|^2 + |\operatorname{rot} E|^2 + |\operatorname{div} E|^2)\, dx\Big\} \\ &= -2 \operatorname{Im} \Big\{\kappa \int_S ((n, E, \operatorname{rot} \bar{E}) + (n, E) \operatorname{div} \bar{E})\, ds\Big\}. \end{aligned} \tag{2.10}$$

All four terms on the left-hand side of (2.10) are nonnegative since $\mathrm{Im}(\kappa) \geqslant 0$. Therefore, the four terms must be individually bounded for $R \to \infty$, since their sum tends to a finite limit. Hence, (2.7) follows.

(II) To prove the representation theorem we now choose an arbitrary fixed point $x \in D$ and circumscribe it with a sphere $K_\rho(x) := \{y \in \mathbb{R}^3 \mid |x - y| = \rho\}$. We assume the radius R large enough that $x \in D_R$ and the radius ρ small enough that $K_\rho(x) \subset D_R$ and direct the unit normal n to $K_\rho(x)$ into the interior of $K_\rho(x)$. Now we apply the second Green's theorem (2.5) in the domain $\{y \in D_R \mid |x - y| > \rho\}$ to the fields $A(y) := E(y)$ and $B(y) := e\Phi(x, y)$, where e stands for an arbitrary constant vector. Thus, with the help of $(E, \Delta(e\Phi)) - (e\Phi, \Delta E) = 0$ in the domain and

$$(n, E, \operatorname{rot}(e\Phi)) + (n, E) \operatorname{div}(e\Phi) - (n, e\Phi, \operatorname{rot} E) - (n, e\Phi) \operatorname{div} E$$
$$= \left(e, \frac{\partial \Phi}{\partial n} E - (E, \operatorname{grad} \Phi)\, n + (n, E) \operatorname{grad} \Phi - \{[\operatorname{rot} E, n] + n \operatorname{div} E\}\, \Phi\right)$$

on the boundary we attain

$$\Big(e, \int_S \{[[n(y), E(y)], \operatorname{grad}_y \Phi(x, y)] + (n(y), E(y)) \operatorname{grad}_y \Phi(x, y)$$
$$- ([\operatorname{rot} E(y), n(y)] + n(y) \operatorname{div} E(y))\, \Phi(x, y)\}\, ds(y)\Big)$$
$$= \Big(e, \int_{K_\rho(x)+K_R} \Big\{E(y) \frac{\partial \Phi(x, y)}{\partial n(y)} + [E(y), [\operatorname{grad}_y \Phi(x, y), n(y)]]$$
$$- ([\operatorname{rot} E(y), n(y)] + n(y) \operatorname{div} E(y))\, \Phi(x, y)\Big\}\, ds(y)\Big). \tag{2.11}$$

Since on $K_\rho(x)$ it holds that

$$\Phi(x, y) = \frac{e^{i\kappa\rho}}{4\pi\rho}, \qquad \operatorname{grad}_y \Phi(x, y) = \left(\frac{1}{\rho} - i\kappa\right) \frac{e^{i\kappa\rho}}{4\pi\rho}\, n(y),$$

a straightforward calculation shows that

$$\lim_{\rho \to 0} \int_{K_\rho(x)} \{\cdots\}\, ds(y) = E(x). \tag{2.12}$$

We rearrange

$$\int_{K_R} \{\cdots\}\, ds(y)$$
$$= \int_{K_R} \left(\frac{\partial \Phi(x, y)}{\partial n(y)} - i\kappa \Phi(x, y)\right) E(y)\, ds(y)$$
$$- \int_{K_R} \{[\operatorname{rot} E(y), n(y)] + n(y) \operatorname{div} E(y) - i\kappa E(y)\}\, \Phi(x, y)\, ds(y)$$
$$+ \int_{K_R} \left[E(y), \left[\frac{y - x}{|y - x|} - \frac{y}{|y|}, n(y)\right]\right] \left(i\kappa - \frac{1}{|x - y|}\right) \Phi(x, y)\, ds(y)$$
$$=: I_1 + I_2 + I_3$$

and apply Schwarz' inequality to each term. From the radiation condition

$$\frac{\partial \Phi(x, y)}{\partial n(y)} - i\kappa\Phi(x, y) = o\left(\frac{1}{R}\right), \qquad y \in K_R, \tag{2.13}$$

for the fundamental solution and from (2.7) we deduce $I_1 = o(1)$ for $R \to \infty$. The radiation condition (2.2) and

$$\Phi(x, y) = O\left(\frac{1}{R}\right), \qquad y \in K_R, \tag{2.14}$$

yield $I_2 = o(1)$ for $R \to \infty$. Finally, from (2.7), (2.14), and

$$\left| \frac{y - x}{| y - x |} - \frac{y}{| y |} \right| \leqslant 2 \mid x \mid / \mid y \mid = O\left(\frac{1}{R}\right), \qquad y \in K_R, \tag{2.15}$$

we get $I_3 = O(1/R)$ for $R \to \infty$. Hence

$$\lim_{R\to\infty} \int_{K_R} \{\cdots\}\, ds(y) = 0. \tag{2.16}$$

Now summarizing (2.11), (2.12), and (2.16) we arrive at

$$\begin{aligned} E(x) = \int_S \{&[[n(y), E(y)], \operatorname{grad}_y \Phi(x, y)] + (n(y), E(y)) \operatorname{grad}_y \Phi(x, y) \\ &- ([\operatorname{rot} E(y), n(y)] + n(y) \operatorname{div} E(y))\, \Phi(x, y)\}\, ds(y) \end{aligned}$$

which is easily transformed into (2.6) by observing that

$$\operatorname{grad}_x \Phi(x, y) = -\operatorname{grad}_y \Phi(x, y). \tag{2.17}$$

We would like to point out that the third integral I_3 appears because we had to take the center of the large sphere K_R in the origin rather than in the point x. This is due to the fact that the radiation condition (2.2) is imposed with respect to the origin. Wilcox's proof of his representation theorem [22] has to be modified in the same manner.

COROLLARY 2.2. *Let E be as in Theorem* 2.1. *Then*

$$\operatorname{div} E(x) = \int_{\tilde{S}} \left\{ \operatorname{div} E(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \Phi(x, y) \frac{\partial \operatorname{div} E(y)}{\partial n(y)} \right\} ds(y), \qquad x \in D, \tag{2.18}$$

where $\tilde{S}$ denotes a surface parallel to S separating the point x from S. In particular, div E *satisfies the scalar Helmholtz equation* (1.1) *in D and the Sommerfeld radiation condition* (1.5).

Proof. Obviously for any fixed x in (2.6) we can replace the boundary S by a parallel surface $\tilde{S}$ separating x from S by applying the representation theorem to the exterior of $\tilde{S}$. Then taking the divergence we find

$$\operatorname{div} E(x) = \int_{\tilde{S}} \Big\{-(n(y), E(y))\, \Delta_y \Phi(x, y) + (n(y), \operatorname{grad}_y \Phi(x, y), \operatorname{rot} E(y)) + \operatorname{div} E(y) \frac{\partial \Phi(x, y)}{\partial n(y)}\Big\}\, ds(y), \qquad x \in D. \tag{2.19}$$

Since E is two times continuously differentiable on $\tilde{S}$ we can apply Stokes' theorem to obtain

$$\begin{aligned}\int_{\tilde{S}} &(n(y), \operatorname{grad}_y \Phi(x, y), \operatorname{rot} E(y))\, ds(y)\\ &= \int_{\tilde{S}} \{(n(y), \operatorname{rot}_y\{\Phi(x, y) \operatorname{rot} E(y)\}) - (n(y), \operatorname{rot} \operatorname{rot} E(y))\, \Phi(x, y)\}\, \mathrm{ds}(y)\\ &= \int_{\tilde{S}} \{(n(y), \Delta E(y))\, \Phi(x, y) - (n(y), \operatorname{grad} \operatorname{div} E(y))\, \Phi(x, y)\}\, ds(y).\end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20) we get (2.18). This representation clearly implies that $\operatorname{div} E$ solves the scalar Helmholtz equation and satisfies Sommerfeld's radiation condition.

3. Statement of Boundary-Value Problems and Uniqueness

We shall consider the following exterior boundary-value problem from the theory of stationary electromagnetic reflection at perfect electrical conductors[13].

Problem (M). Find two vector fields $E, H \in C^1(D) \cap C(\bar{D})$ satisfying the time-harmonic Maxwell equations

$$\operatorname{rot} E - i\kappa H = 0, \quad \operatorname{rot} H + i\kappa E = 0 \text{ in } D, \quad \kappa \neq 0, \operatorname{Im}(\kappa) \geqslant 0, \tag{3.1}$$

the boundary condition

$$[n, E] = c \text{ on } S, \tag{3.2}$$

and the radiation condition

$$\left[H, \frac{x}{|x|}\right] - E = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty, \tag{3.3}$$

uniformly for all directions $x/|x|$. It is assumed that for the given tangential field $c \in C^{0,\alpha}(S)$, $0 < \alpha < 1$, the surface divergence $\operatorname{Div} c$ exists in the sense of the limit integral definition [13] and is of class $C^{0,\alpha}(S)$.

From the representation theorem for solutions to the Maxwell equations (in bounded domains) [13] we observe that any solution of (3.1) automatically belongs to $C^2(D)$. Therefore, using the vector identity $\operatorname{rot}\operatorname{rot} E = -\Delta E + \operatorname{grad}\operatorname{div} E$, we see that for any solution of the Maxwell equations both of the vector fields E and H are divergence free and satisfy the vector Helmholtz equation $\Delta E + \kappa^2 E = 0$ and $\Delta H + \kappa^2 H = 0$. Therefore, following Werner [20], instead of Problem (M) we shall consider the slightly more general exterior boundary-value

Problem (H). Find a vector field $E \in V(D)$ satisfying the vector Helmholtz equation

$$\Delta E + \kappa^2 E = 0 \text{ in } D, \qquad \kappa \neq 0,\ \operatorname{Im}(\kappa) \geqslant 0, \tag{3.4}$$

the boundary conditions

$$\left.\begin{aligned} [n, E] &= c, \qquad (3.5)\\ \operatorname{div} E &= \gamma, \qquad (3.6)\end{aligned}\right\} \text{ on } S,$$

and the radiation condition

$$\left[\operatorname{rot} E, \frac{x}{|x|}\right] + \frac{x}{|x|}\operatorname{div} E - i\kappa E = o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty, \tag{3.7}$$

uniformly for all directions $x/|x|$. Here, $\gamma \in C^{0,\alpha}(S)$, $0 < \alpha < 1$, is a given function and $c \in C^{0,\alpha}(S)$ is a given tangential field satisfying the same additional property as that required in Problem (M).

For any solution E to Problem (H) with $\operatorname{div} E = 0$ on S from Corollary 2.2 we conclude that $\operatorname{div} E \in C^2(D) \cap C(\bar{D})$ is a solution to the homogeneous exterior Dirichlet problem for the scalar Helmholtz equation satisfying the Sommerfeld radiation condition. Hence, uniqueness for the Dirichlet problem yields $\operatorname{div} E = 0$ in D. But then E and $H := (1/i\kappa)\operatorname{rot} E$ obviously solve Maxwell's equations. Thus, the following equivalence is true.

THEOREM 3.1. *Let E and H be a solution to Problem* (M). *Then E solves the special case of Problem* (H) *where $\gamma = 0$. Conversely, let E be a solution to Problem* (H) *with $\gamma = 0$. Then E and $H := (1/i\kappa)$* rot *E solve Problem (M).*

Uniqueness for Problem (H) can be established by Rellich's lemma [9, 14].

THEOREM 3.2. *Problem* (H) *has no more than one solution.*

Proof. Let E be a solution to Problem (H) with homogeneous boundary conditions

$$[n, E] = 0, \qquad \operatorname{div} E = 0 \text{ on } S. \tag{3.8}$$

Then from (2.10) we deduce

$$\lim_{R\to\infty}\left\{\int_{K_R}(|[\operatorname{rot} E, n] + n \operatorname{div} E|^2 + |\kappa|^2 |E|^2)\, ds + 2\operatorname{Im}(\kappa)\int_{D_R}(|\kappa|^2 |E|^2 + |\operatorname{rot} E|^2 + |\operatorname{div} E|^2)\, dx\right\} = 0. \tag{3.9}$$

In the case $\operatorname{Im}(\kappa) > 0$ we immediately find

$$\int_D |E|^2\, dx = 0,$$

whence $E = 0$ in D follows. In the case $\operatorname{Im}(\kappa) = 0$ from (3.9) we obtain

$$\int_{K_R} |E|^2\, ds = o(1), \qquad R \to \infty. \tag{3.10}$$

Since the components of E satisfy the scalar Helmholtz equation from (3.10) we conclude $E = 0$ in D by Rellich's lemma.

4. Transformation of the Boundary-Value Problem into an Integral Equation

Motivated by the form (2.6) of the representation theorem we try to find a solution to Problem (H) in the form

$$E(x) = \operatorname{rot}\int_S a(y)\,\Phi(x, y)\, ds(y) + i\tau\eta\int_S [n(y), a(y)]\,\Phi(x, y)\, ds(y) - \int_S \lambda(y)\, n(y)\,\Phi(x, y)\, ds(y) + i\tau\eta \operatorname{grad}\int_S \lambda(y)\,\Phi(x, y)\, ds(y), \qquad x \in D. \tag{4.1}$$

The unknown tangential field a and the scalar function λ are assumed to belong to $C^{0,\alpha}(S)$, $0 < \alpha < 1$. We have set

$$\tau := 1 \quad \text{if} \quad \operatorname{Re}(\kappa) \geqslant 0,$$

$$\tau := -1 \quad \text{if} \quad \operatorname{Re}(\kappa) < 0,$$

and by η we denote a positive constant which will be appropriately chosen later (Theorem 5.2).

Obviously $E \in C^2(D)$ solves the differential equation (3.4). To verify that E satisfies the radiation condition (3.7) it suffices to show that for any bounded vector field e defined on the bounded set S the vector fields

$$
\begin{aligned}
E_1(x, y) &:= \operatorname{rot}_x\{e(y)\, \Phi(x, y)\}, \\
E_2(x, y) &:= e(y)\, \Phi(x, y), \\
E_3(x, y) &:= \operatorname{grad}_x \Phi(x, y),
\end{aligned}
$$

fulfill the radiation condition

$$
\begin{aligned}
R_j(x, y) &:= \left[\operatorname{rot}_x E_j(x, y), \frac{x}{|x|}\right] + \frac{x}{|x|} \operatorname{div}_x E_j(x, y) - i\kappa E_j(x, y) \\
&= o\left(\frac{1}{|x|}\right), \qquad |x| \to \infty, \quad j = 1, 2, 3,
\end{aligned}
\tag{4.2}
$$

uniformly for all directions $x/|x|$ and all $y \in S$. By straightforward calculations we find

$$
\begin{aligned}
R_1(x, y) = {} & i\kappa \left[e(y), \operatorname{grad}_x \Phi(x, y) - i\kappa \frac{x}{|x|} \Phi(x, y)\right] \\
& + \left(i\kappa - \frac{1}{|x-y|}\right) \frac{\Phi(x, y)}{|x-y|} \left[e(y), \frac{x}{|x|}\right] \\
& - \left(\kappa^2 + \frac{3i\kappa}{|x-y|} - \frac{3}{|x-y|^2}\right) \\
& \times (e(y), x-y) \frac{\Phi(x, y)}{|x-y|} \left[\frac{x-y}{|x-y|} - \frac{x}{|x|}, \frac{x}{|x|}\right],
\end{aligned}
$$

$$
\begin{aligned}
R_2(x, y) = {} & e(y) \left[\left(\frac{x}{|x|}, \operatorname{grad}_x \Phi(x, y)\right) - i\kappa \Phi(x, y)\right] \\
& + \left[e(y), \left[\frac{x}{|x|}, \operatorname{grad}_x \Phi(x, y)\right]\right],
\end{aligned}
$$

$$
R_3(x, y) = -i\kappa \left(\operatorname{grad}_x \Phi(x, y) - i\kappa \frac{x}{|x|} \Phi(x, y)\right).
$$

Now (4.2) obviously follows from

$$
\begin{aligned}
\operatorname{grad}_x \Phi(x, y) - i\kappa \frac{x}{|x|} \Phi(x, y) &= O\left(\frac{1}{|x|^2}\right), \\
\Phi(x, y) &= O\left(\frac{1}{|x|}\right), \\
\frac{x-y}{|x-y|} - \frac{x}{|x|} &= O\left(\frac{1}{|x|}\right)
\end{aligned}
$$

for $|x| \to \infty$, uniformly for all directions $x/|x|$ and all $y \in S$.

From the regularity properties of single-layer surface potentials with uniformly Hölder continuous densities [4] we observe that E can be extended into $\bar{D}$ such that E and $\operatorname{div} E$ belong to $C^{0,\alpha}(\bar{D})$. By the jump relations we find that the boundary conditions (3.5) and (3.6) are satisfied provided the densities a and λ are solutions to the integral equations

$$\begin{aligned} \tfrac{1}{2}a(x) &+ \int_S [n(x), \operatorname{rot}_x\{a(y)\,\Phi(x,y)\}]\,ds(y) \\ &+ i\tau\eta \int_S [n(x), [n(y), a(y)]]\,\Phi(x,y)\,ds(y) \\ &- \int_S \lambda(y)\,[n(x), n(y)]\,\Phi(x,y)\,ds(y) \\ &+ i\tau\eta \left[n(x), \operatorname{grad} \int_S \lambda(y)\,\Phi(x,y)\,ds(y)\right] = c(x), \qquad x \in S, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \tfrac{1}{2}\lambda(x) &+ i\tau\eta \operatorname{div} \int_S [n(y), a(y)]\,\Phi(x,y)\,ds(y) \\ &+ \int_S \lambda(y)\,\frac{\partial\Phi(x,y)}{\partial n(y)}\,ds(y) - i\tau\eta\kappa^2 \int_S \lambda(y)\,\Phi(x,y)\,ds(y) = \gamma(x), \qquad x \in S. \end{aligned} \tag{4.4}$$

To make sure that $E \in V(D)$ it remains to be shown that $\operatorname{rot} E \in C(\bar{D})$. In general, for arbitrary densities $a, \gamma \in C^{0,\alpha}(S)$ it can be shown by counterexamples that $\operatorname{rot} E$ is not continuous in $\bar{D}$. However, as we verify presently, if a and λ satisfy (4.3) and (4.4) that means if E satisfies the boundary conditions (3.5) and (3.6) then $\operatorname{rot} E \in C^{0,\alpha}(\bar{D})$.

We can decompose

$$E = \operatorname{rot} A + F + \operatorname{grad} \psi, \tag{4.5}$$

where

$$\begin{aligned} A(x) &:= \int_S a(y)\,\Phi(x,y)\,ds(y), & x \in D, \\ F(x) &:= \int_S \{i\tau\eta[n(y), a(y)] - \lambda(y)\,n(y)\}\,\Phi(x,y)\,ds(y), & x \in D, \\ \psi(x) &:= i\tau\eta \int_S \lambda(y)\,\Phi(x,y)\,ds(y), & x \in D, \end{aligned} \tag{4.6}$$

and obtain

$$\operatorname{rot} E = \operatorname{grad} \operatorname{div} A + \kappa^2 A + \operatorname{rot} F \text{ in } D. \tag{4.7}$$

By virtue of the regularity properties of surface potentials we find that $A, F \in C^{1,\alpha}(\bar{D})$ since $a, \lambda \in C^{0,\alpha}(S)$. Therefore, the proof of $\operatorname{rot} E \in C^{0,\alpha}(\bar{D})$ is accomplished when we show that div $A \in C^{1,\alpha}(\bar{D})$. Since A is a solution to the vector Helmholtz equation (2.1) satisfying radiation condition (2.2) from Corollary 2.2 we have

$$\operatorname{div} A(x) = \int_{\tilde{S}} \left\{ \operatorname{div} A(y) \frac{\partial \Phi(x, y)}{\partial n(y)} - \Phi(x, y) \frac{\partial \operatorname{div} A(y)}{\partial n(y)} \right\} ds(y), \qquad x \in D.$$

Using (4.5) and Stokes' theorem we transform

$$\begin{aligned} \int_{\tilde{S}} \Phi \frac{\partial}{\partial n} \operatorname{div} A \, ds &= \int_{\tilde{S}} \{(n, \operatorname{rot}(E - F)) - \kappa^2(n, A)\} \Phi \, ds \\ &= \int_{\tilde{S}} \{(n, E - F, \operatorname{grad} \Phi) - \kappa^2(n, A) \Phi\} \, ds. \end{aligned}$$

Now we are able to pass to the limit $\tilde{S} \to S$ and using the boundary condition (3.5) we obtain

$$\begin{aligned} \operatorname{div} A(x) = \int_S \Big\{ &\operatorname{div} A(y) \frac{\partial \Phi(x, y)}{\partial n(y)} + ([n(y), F(y)] - c(y), \operatorname{grad}_y \Phi(x, y)) \\ &+ \kappa^2(n(y), A(y)) \Phi(x, y) \Big\} \, ds(y), \qquad x \in D. \end{aligned}$$

By assumption we have Div $c \in C^{0,\alpha}(S)$. Hence, using $\operatorname{Div}\{\Phi([n, F] - c)\} = ([n, F] - c, \operatorname{grad} \Phi) + \Phi \operatorname{Div}\{[n, F] - c\}$ and Gauss' theorem we get

$$\operatorname{div} A(x) = \int_S \left\{ \operatorname{div} A(y) \frac{\partial \Phi(x, y)}{\partial n(y)} + \mu(y) \Phi(x, y) \right\} ds(y), \qquad x \in D, \tag{4.8}$$

where we have set $\mu := \operatorname{Div}(c - [n, F]) + \kappa^2(n, A)$ on S. Now letting x tend to the boundary by the jump relation for double-layer potentials we find that div $A \in C^{0,\alpha}(S)$ solves the integral equation

$$\frac{1}{2} \operatorname{div} A(x) - \int_S \operatorname{div} A(y) \frac{\partial \Phi(x, y)}{\partial n(y)} ds(y) = \int_S \mu(y) \Phi(x, y) \, ds(y), \qquad x \in S. \tag{4.9}$$

The right-hand side of (4.9) belongs to $C^{1,\alpha}(S)$ since for the density we have $\mu \in C^{0,\alpha}(S)$. Thus, since the integral operator in Eq. (4.9) maps $C^{0,\alpha}(S)$ into $C^{1,\alpha}(S)$ [9, p. 42; 19, Lemma 7] we conclude div $A \in C^{1,\alpha}(S)$. But then finally from (4.8) we see div $A \in C^{1,\alpha}(\bar{D})$ because double-layer potentials with densities of class $C^{1,\alpha}(S)$ belong to $C^{1,\alpha}(\bar{D})$ [9, p. 40; 19, Lemma 4]. We now summarize our results in

THEOREM 4.1. *Provided the densities* $a, \lambda \in C^{0,\alpha}(S)$, $0 < \alpha < 1$, *are a solution to the system of integral equations* (4.3) *and* (4.4) *then the vector field* E *defined by* (4.1) *is a solution to Problem* (H).

In the subsequent investigation of the integral equations we shall make use of the fact that by (4.1) we can extend the definition of the field E into the interior domain D_i. We shall distinguish by indices + and − the limits obtained by approaching S from inside D and D_i, respectively. Then, again we have E, $\operatorname{div} E \in C^{0,\alpha}(\bar{D}_i)$ and from the jump relations we get

$$[n, E_+] - [n, E_-] = a, \tag{4.10}$$

$$(n, E_+) - (n, E_-) = -i\tau\eta\lambda, \tag{4.11}$$

$$\operatorname{div} E_+ - \operatorname{div} E_- = \lambda \text{ on } S. \tag{4.12}$$

In order to obtain a jump relation for rot E we observe that in D_i we also can decompose by (4.5) with definitions (4.6) extended into D_i. Obviously

$$\Delta \operatorname{div} A + \kappa^2 \operatorname{div} A = 0 \text{ in } D_i$$

and from the jump relations it follows that

$$\operatorname{div} A_+ = \operatorname{div} A_- \text{ on } S. \tag{4.13}$$

Choose a domain $\tilde{D}_i \subset D_i$ such that $S \subset \partial\tilde{D}_i$ is small enough that κ is not an eigenvalue to the interior Dirichlet problem for $\tilde{D}_i$. Then we can represent by a double-layer potential

$$\operatorname{div} A(x) = \int_{\partial\tilde{D}_i} \nu(y) \frac{\partial\Phi(x, y)}{\partial n(y)} \, ds(y), \qquad x \in \tilde{D}_i,$$

the density of which is the unique solution to the integral equation

$$-\frac{1}{2}\nu(x) + \int_{\partial\tilde{D}_i} \nu(y) \frac{\partial\Phi(x, y)}{\partial n(y)} \, ds(y) = \operatorname{div} A(x), \qquad x \in \partial\tilde{D}_i.$$

Since for the boundary data we already know $\operatorname{div} A \in C^{1,\alpha}(S)$ analogously to the investigation of div A in $\bar{D}$ by Eqs. (4.8) and (4.9) we find $\operatorname{div} A \in C^{1,\alpha}(\bar{D}_i)$ [6, p. 14; 19, p. 157]. But then from (4.13) it follows that $[n, \operatorname{grad} \operatorname{div} A_+] = [n, \operatorname{grad} \operatorname{div} A_-]$ on S and finally from (4.7) we conclude $\operatorname{rot} E \in C^{0,\alpha}(\bar{D}_i)$ with the jump

$$[n, \operatorname{rot} E_+] - [n, \operatorname{rot} E_-] = i\tau\eta[n, a] \text{ on } S. \tag{4.14}$$

5. Existence of a Solution to the Integral Equation

To investigate the integral equation we introduce the appropriate function spaces and integral operators. Define for $0 < \alpha < 1$ the Banach spaces

$$X_1^\alpha := \{a\colon S \to \mathbb{C}^3 \mid (n, a) = 0,\ a \in C^{0,\alpha}(S)\},$$

$$X_2^\alpha := \{\lambda\colon S \to \mathbb{C} \mid \lambda \in C^{0,\alpha}(S)\},$$

endowed with the Hölder norms

$$\| a \|_\alpha := \sup_{x \in S} | a(x)| + \sup_{\substack{x,y \in S \\ x \neq y}} \frac{| a(x) - a(y)|}{| x - y |^\alpha},$$

$$\| \lambda \|_\alpha := \sup_{x \in S} | \lambda(x)| + \sup_{\substack{x,y \in S \\ x \neq y}} \frac{| \lambda(x) - \lambda(y)|}{| x - y |^\alpha},$$

and introduce the product space $X^\alpha := X_1^\alpha \times X_2^\alpha$ with norm

$$\left\| \begin{pmatrix} a \\ \lambda \end{pmatrix} \right\|_\alpha := \| a \|_\alpha + \| \lambda \|_\alpha .$$

Define integral operators K_{jl}, L_{jl}: $X_l^\alpha \to X_j^\alpha, j, l = 1, 2$ by

$$(K_{11}a)(x) := -2 \int_S [n(x), \operatorname{rot}_x\{a(y)\, \Phi(x, y)\}]\, ds(y),$$

$$(K_{12}\lambda)(x) := 2 \int_S \lambda(y)\, [n(x), n(y)]\, \Phi(x, y)\, ds(y),$$

$$(K_{21}a)(x) := 0,$$

$$(K_{22}\lambda)(x) := -2 \int_S \lambda(y) \frac{\partial \Phi(x, y)}{\partial n(y)}\, ds(y),$$

$$(L_{11}a)(x) := -2 \int_S [n(x), [n(y), a(y)]]\, \Phi(x, y)\, ds(y),$$

$$(L_{12}\lambda)(x) := -2 \left[n(x), \operatorname{grad} \int_S \lambda(y)\, \Phi(x, y)\, ds(y)\right],$$

$$(L_{21}a)(x) := -2 \operatorname{div} \int_S [n(y), a(y)]\, \Phi(x, y)\, ds(y),$$

$$(L_{22}\lambda)(x) := 2\kappa^2 \int_S \lambda(y)\, \Phi(x, y)\, ds(y), \qquad x \in S.$$

In addition we introduce operators $\mathring{L}_{12}$: $X_2^\alpha \to X_1^\alpha$ and $\mathring{L}_{21} : X_1^\alpha \to X_2^\alpha$ by

$$(\mathring{L}_{12}\lambda)(x) := -2 \left[n(x), \operatorname{grad} \int_S \frac{\lambda(y)}{4\pi \,| x - y |}\, ds(y)\right],$$

$$(\mathring{L}_{21}a)(x) := -2 \operatorname{div} \int_S \frac{[n(y), a(y)]}{4\pi \,| x - y |}\, ds(y), \qquad x \in S.$$

In an obvious notation finally we define operators $K, \mathring{L}$: $X^\alpha \to X^\alpha$ by

$$K := \begin{pmatrix} K_{11} + i\tau\eta L_{11} & K_{12} + i\tau\eta(L_{12} - \mathring{L}_{12}) \\ i\tau\eta(L_{21} - \mathring{L}_{21}) & K_{22} + i\tau\eta L_{22} \end{pmatrix},$$

$$\mathring{L} := \begin{pmatrix} 0 & \mathring{L}_{12} \\ \mathring{L}_{21} & 0 \end{pmatrix}.$$

Then, the system of integral equations (4.3) and (4.4), abbreviated, reads

$$\binom{a}{\lambda} - K\binom{a}{\lambda} - i\tau\eta\mathring{L}\binom{a}{\lambda} = 2\binom{c}{\gamma}. \tag{5.1}$$

THEOREM 5.1. *The operator $K: X^\alpha \to X^\alpha$ is compact and the operator $\mathring{L}$: $X^\alpha \to X^\alpha$ is bounded.*

Proof. We decompose $K_{11} = K_{11}' + K_{11}''$, where

$$(K_{11}'a)(x) := 2\int_S a(y)\frac{\partial\Phi(x, y)}{\partial n(x)}\,ds(y),$$

$$(K_{11}''a)(x) := -2\int_S (n(x), a(y))\,\mathrm{grad}_x\,\Phi(x, y)\,ds(y), \qquad x \in S.$$

Since $K_{11}'a$ is the normal derivative of a single-layer potential [4, p. 62; 19, Lemma 6], for any $\beta \in (0, 1)$ there exists a positive number k_{11}' (depending on β) such that

$$|(K_{11}'a)(x)| \leqslant k_{11}' \| a \|_\infty, \qquad x \in S,$$

and

$$|(K_{11}'a)(x) - (K_{11}'a)(y)| \leqslant k_{11}' \| a \|_\infty \, | x - y |^\beta, \qquad x, y \in S.$$

Using the fact that

$$|(n(x), a(y))| = |(n(x) - n(y), a(y))| \leqslant C \, | x - y | \, \| a \|_\infty, \qquad x, y \in S,$$

with some constant C, by the same lengthy procedure as carried out by Günter [4, p. 47] for the double-layer potential we find that for any $\beta \in (0, 1)$ there exists a positive number k_{11}'' (depending on β) such that

$$|(K_{11}''a)(x)| \leqslant k_{11}'' \| a \|_\infty, \qquad x \in S,$$

and

$$|(K_{11}''a)(x) - (K_{11}''a)(y)| \leqslant k_{11}'' \| a \|_\infty \, | x - y |^\beta, \qquad x, y \in S.$$

Hence, $K_{11}: X_1^\alpha \to X_1^\beta$ is bounded for $0 < \alpha, \beta < 1$.

Analogously, by the properties of single- and double-layer potentials with continuous densities [4, p. 46; 19, Lemmas 1 and 6] we deduce that the operators $K_{jl}: X_l^\alpha \to X_j^\beta$, $j, l = 1, 2$, and $L_{jj}: X_j^\alpha \to X_j^\beta$, $j = 1, 2$ are bounded for $0 < \alpha, \beta < 1$. By expanding

$$\Phi(x, y) - \frac{1}{4\pi \, | x - y |} = \frac{1}{4\pi} \sum_{m=0}^{\infty} \frac{(i\kappa)^{m+1}}{(m+1)!} \, | x - y |^m$$

we find that the operators $L_{12} - \mathring{L}_{12}: X_2^\alpha \to X_1^\beta$ and $L_{21} - \mathring{L}_{21}: X_1^\alpha \to X_2^\beta$ are bounded for $0 < \alpha, \beta < 1$. Now compactness of K follows from the property that the embedding operators

$$I_j^{\alpha,\beta}: X_j^\beta \to X_j^\alpha, \qquad j = 1, 2,$$

are compact provided $\beta > \alpha$ [1; 19, Lemma 14].

Finally, from the properties of single-layer potentials with uniformly Hölder continuous densities [4, p. 66; 19, Lemma 3] we see that the operators $\mathring{L}_{12}: X_2^\alpha \to X_1^\alpha$, $\mathring{L}_{21}: X_1^\alpha \to X_2^\alpha$ are bounded for $0 < \alpha < 1$. Hence, $\mathring{L}$ is bounded.

THEOREM 5.2. *For all κ with $\kappa \neq 0$, $\mathrm{Im}(\kappa) \geqslant 0$ the integral equation* (5.1) *has a unique solution provided $\eta < \| \mathring{L} \|_\alpha^{-1}$.*

Proof. (I) First we prove that the operator $M := I - i\tau\eta\mathring{L} - K$ is injective, where $I: X^\alpha \to X^\alpha$ denotes the identity operator. Let $\binom{a}{\lambda}$ be a solution to the homogeneous equation and define the vector field E by (4.1). Then, by Theorem 4.1, the field E solves the homogeneous Problem (H) and thus by the uniqueness theorem, Theorem 3.2, we have $E = 0$ in D. Now, from (4.10), (4.11), (4.12) and (4.14) we get

$$\left.\begin{aligned} (n, \bar{E}_-, \operatorname{rot} E_-) &= i\tau\eta\, | a |^2, \\ (n, \bar{E}_-) \operatorname{div} E_- &= i\tau\eta\, | \lambda |^2, \end{aligned}\right\} \quad \text{on} \quad S.$$

Therefore, by the first Green's theorem (2.4), we obtain

$$i\tau\eta \int_S (| a |^2 + | \lambda |^2)\, ds = \int_{D_i} (| \operatorname{rot} E |^2 + | \operatorname{div} E |^2 - \kappa^2 | E |^2)\, dx.$$

The imaginary part of this equation reads

$$\eta \int_S (| a |^2 + | \lambda |^2)\, ds = -2\tau \,\mathrm{Re}(\kappa)\, \mathrm{Im}(\kappa) \int_{D_i} | E |^2\, dx.$$

From this, since $\eta > 0$ and $\tau \,\mathrm{Re}(\kappa)\, \mathrm{Im}(\kappa) \geqslant 0$, we obtain

$$\int_S (| a |^2 + | \lambda |^2)\, ds = 0,$$

whence $a = 0$ and $\lambda = 0$ follow.

(II) For $\eta < \| \mathring{L} \|_\alpha^{-1}$ obviously the inverse $(I - i\tau\eta\mathring{L})^{-1}$ exists and is bounded. Hence, by the Riesz theory for compact operators $I - i\tau\eta\mathring{L} - K$ is surjective since it is injective.

Summarizing from Theorems 5.2, 4.1, and 3.2 by a unified approach for all κ with $\kappa \neq 0$ and $\mathrm{Im}(\kappa) \geqslant 0$ we have the existence result

TROBLEM 5.3. *Problem* (H) *has a unique solution.*

Finally, by the equivalence theorem, Theorem 3.1, we get

THEOREM 5.4. *Problem* (M) *has a unique solution.*

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