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On the Lattice of Fréchet–Nikodým Topologies

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In this paper we look at the lattice of Fréchet–Nikodým (FN) topologies on an algebra of sets from an algebraic point of view.

Drewnowski, who first studied FN topologies systematically [5], observed that the collection of FN topologies on an algebra forms a complete lattice with 0 and 1. Weber [9] pointed out that the lattice is distributive. Weber [9] and Brook and Traynor [4] independently showed that the exhaustive FN topologies form a complete Boolean algebra and the existence of relative complements is the basis of the Lebesgue decomposition for strongly bounded group-valued measures.

Here we start out by noting that the lattice of FN topologies is a frame or complete Heyting algebra. We characterize the FN topologies which have complements. Then we look at pseudocomplements and show that a countably additive FN topology on a σ-algebra must be regular (equal to its own double pseudocomplement). Unexpectedly, this result implies that, for countably additive group-valued measures, continuity and order-continuity coincide.

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1. PRELIMINARIES

We begin by gathering some results about the lattice of FN topologies. Let $\mathscr{A}$ be an algebra of subsets of a nonempty set X. We assume that $\mathscr{A}$ separates points. A topology on $\mathscr{A}$ is a *Fréchet–Nikodým* (FN) topology if

(1) It makes $\mathscr{A}$ a topological group under symmetric difference, and

(2) For every neighborhood U of $\varnothing$ there is a neighborhood V of $\varnothing$ such that $(A) = \{B \in \mathscr{A} \mid B \subseteq A\} \subseteq U$ whenever A is in V.

Call a neighborhood U of $\varnothing$ *normal* if $(A) \subseteq U$ whenever A is in U. Every

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FN topology has a neighborhood base at $\varnothing$ consisting of normal neighborhoods.

A *submeasure* on $\mathscr{A}$ is a map $\lambda\colon \mathscr{A} \to [0, \infty)$ such that

(1) $\lambda(\varnothing) = 0$,

(2) $\lambda(A) \leqslant \lambda(B)$ whenever $A \subseteq B$ in $\mathscr{A}$,

(3) $\lambda(A \cup B) \leqslant \lambda(A) + \lambda(B)$ for all A and B in $\mathscr{A}$.

If λ is a submeasure on $\mathscr{A}$, then the map $(A, B) \to \lambda(A \Delta B)$ is a semimetric, and the semimetric topology G_λ is an FN topology on $\mathscr{A}$.

The set $\mathrm{FN}(\mathscr{A})$ of all FN topologies on $\mathscr{A}$, ordered by inclusion, is a complete lattice, whose largest element is the discrete topology D and smallest the indiscrete topology 0. The join of a family of FN topologies is the usual supremum topology [10]. We describe the meet of two FN topologies and tell when they are disjoint.

1.1. PROPOSITION. *Let G and H be* FN *topologies on $\mathscr{A}$. Let $\mathscr{M}$ and $\mathscr{N}$ be neighborhood bases at $\varnothing$ for G and H respectively.*

(1) *The collection of all sets $\{A \Delta B \mid A \in U,\ B \in V\}$ where U is in $\mathscr{M}$ and V in $\mathscr{N}$ is a neighborhood base for $G \wedge H$.*

(2) *We have $G \wedge H = 0$ iff for every U in $\mathscr{M}$ and V in $\mathscr{N}$ there is A in $\mathscr{A}$ such that $(A) \subseteq U$ and $(X \setminus A) \subseteq V$.*

Proof. The proof of (1) is straightforward, and (2) follows from (1). See also [8].

Recall [Chapter II, 6] that a *frame* is a complete lattice satisfying the *infinite distributive law*

$$a \wedge \left(\bigvee S\right) = \bigvee \{a \wedge s \mid s \in S\}.$$

1.2. THEOREM. *The set* $\mathrm{FN}(\mathscr{A})$ *is a frame, hence a complete distributive lattice.*

Proof. A straightforward calculation verifies the infinite distributive law in $\mathrm{FN}(\mathscr{A})$. Since a lattice satisfying one (finite) distributive law satisfies both, $\mathrm{FN}(\mathscr{A})$ is a distributive lattice.

Now a complete lattice is a frame iff it is a Heyting algebra [Chap. I, 6]. Thus $\mathrm{FN}(\mathscr{A})$ is a Heyting algebra.

The set $s(\mathscr{A})$ of all submeasures on $\mathscr{A}$ is also a lattice. If λ and μ are submeasures on $\mathscr{A}$, then

$$(\lambda \vee \mu)(A) = \max\{\lambda(A), \mu(A)\}$$

and

$$(\lambda \wedge \mu)(A) = \inf\{\lambda(B) + \mu(A \backslash B) \mid B \subseteq A,\ B \in \mathscr{A}\}.$$

1.3. PROPOSITION. (1) *The map* $\lambda \to G_\lambda$ *is a lattice homomorphism from* $s(\mathscr{A})$ *into* FN$(\mathscr{A})$.

(2) *If* G *is an* FN *topology on* $\mathscr{A}$, *then* $G = \bigvee\{G_\lambda \mid G_\lambda \subseteq G,\ \lambda \in s(\mathscr{A})\}$.

Proof. The proof of (1) is straightforward. For (2) see [5] or [Theorem 1.2, 3].

Note that 1.3 says that the image of $s(\mathscr{A})$ in FN$(\mathscr{A})$ is a dense sublattice.

For submeasures λ and μ on $\mathscr{A}$, say that λ is *μ-continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\lambda(A) < \varepsilon$ whenever $\mu(A) < \delta$. Call λ and μ *equivalent*, and write $\lambda \sim \mu$, if λ is μ-continuous and μ is λ-continuous. Say λ and μ are *singular*, and write $\lambda \perp \mu$, if for every $\varepsilon > 0$ there is A in $\mathscr{A}$ such that $\lambda(A) < \varepsilon$ and $\mu(X \backslash A) < \varepsilon$. Then λ is μ-continuous iff $G_\lambda \subseteq G_\mu$, $\lambda \sim \mu$ iff $G_\lambda = G_\mu$, and $\lambda \perp \mu$ iff $\lambda \wedge \mu = 0$ iff $G_\lambda \wedge G_\mu = 0$.

Say that a submeasure λ on $\mathscr{A}$ is *exhaustive* if $\lambda(A_n) \to 0$ whenever (A_n) is a disjoint sequence in $\mathscr{A}$. Say that an FN topology G is *exhaustive* if $A_n \to \varnothing$ in G whenever (A_n) is a disjoint sequence in $\mathscr{A}$.

1.4. PROPOSITION. (1) *The set* $E(\mathscr{A})$ *of all exhaustive* FN *topologies on* $\mathscr{A}$ *is a complete ideal in* FN$(\mathscr{A})$.

(2) *The topology* $E = \bigvee\{G \mid G \in E(\mathscr{A})\}$ *is the largest exhaustive* FN *topology on* $\mathscr{A}$ *and* $E(\mathscr{A})$ *is the principal ideal generated by* E *in* FN$(\mathscr{A})$.

Proof. Straightforward.

1.5. THEOREM. *The ideal* $E(\mathscr{A})$ *is a complete Boolean algebra with largest element* E.

Proof. By 1.2 and 1.4, $E(\mathscr{A})$ is a complete distributive lattice with largest element E. If G is in $E(\mathscr{A})$, then by [Theorem 4.2, 7] or [Corollary 2.7, 3] there is a unique H in $E(\mathscr{A})$ such that $G \vee H = E$ and $G \wedge H = 0$. Thus $E(\mathscr{A})$ is complemented.

By [Theorem 2.2, 9] or [Theorem 2.6, 4], $E(\mathscr{A})$ is isomorphic to the Boolean algebra formed by completing $\mathscr{A}$ in the group topology E.

1.6. PROPOSITION. *The set* $I(\mathscr{A}) = \{G_\lambda \mid \lambda \in s(\mathscr{A}),\ \lambda$ *exhaustive*$\}$ *is an ideal in* FN$(\mathscr{A})$.

Proof. By 1.3(1) and 1.4, $I(\mathscr{A})$ is a sublattice of FN$(\mathscr{A})$. Let G_λ be in $I(\mathscr{A})$ and $H \subseteq G_\lambda$. By [Theorem 2.6, 3] there are submeasures $\lambda_1, \lambda_2 \leqslant \lambda$ such that $\lambda \sim \lambda_1 \vee \lambda_2$, $G_{\lambda_1} \subseteq H$, and $G_{\lambda_2} \wedge H = 0$. Then $H = H \wedge G_\lambda = H \wedge (G_{\lambda_1} \vee G_{\lambda_2}) = (H \wedge G_{\lambda_1}) \vee (H \wedge G_{\lambda_2}) = G_{\lambda_1}$. So $I(\mathscr{A})$ is an ideal.

Last we point out that if $R(\mathscr{A}) = \{G \in E(\mathscr{A}) \mid G \in I(\mathscr{A}) \text{ or } E\setminus G \in I(\mathscr{A})\}$, then $R(\mathscr{A})$ is the Boolean algebra generated by $I(\mathscr{A})$, and $E(\mathscr{A})$ is its Boolean algebra completion. Hence [Sect. 4, 1] $E(\mathscr{A})$ is a maximal ring of quotients of $R(\mathscr{A})$.

2. Complements

Next we describe the FN topologies which have complements in $\mathrm{FN}(\mathscr{A})$. Since $\mathrm{FN}(\mathscr{A})$ is a distributive lattice, complements are unique when they exist.

If G is an FN topology, let Δ_G be the G-closure of $\{\varnothing\}$ in $\mathscr{A}$. Then Δ_G is the intersection of all G-neighborhoods of $\varnothing$, hence an ideal in $\mathscr{A}$. Call the elements of Δ_G *null sets* of G. If λ is a submeasure on $\mathscr{A}$, then Δ_{G_λ} is just the ideal Δ_λ of null sets of λ.

For C in $\mathscr{A}$, define a submeasure λ_C on $\mathscr{A}$ by

$$\lambda_C(A) = \begin{cases} 0 & \text{if } A \subseteq C \\ 1 & \text{otherwise.} \end{cases}$$

Then Δ_λ is the principal ideal (C). The submeasure $\lambda_\varnothing$ generates the discrete topology D. It is easy to see that $\lambda_C \vee \lambda_{X\setminus C} = \lambda_\varnothing$ and $\lambda_C \perp \lambda_{X\setminus C}$. Thus G_{λ_C} and $G_{\lambda_{X\setminus C}}$ are complements. Note that λ_C is exhaustive iff $X\setminus C$ is finite.

2.1. Lemma. *Let G be an* FN *topology on $\mathscr{A}$. Define a submeasure λ on $\mathscr{A}$ by*

$$\lambda(A) = \begin{cases} 0 & \text{if } A \text{ is in } \Delta_G \\ 1 & \text{otherwise.} \end{cases}$$

Then $G \subseteq G_\lambda$, and $G = G_\lambda$ iff Δ_G is G-open.

Proof. Easy.

2.2. Theorem. *An* FN *topology G has a complement in* $\mathrm{FN}(\mathscr{A})$ *iff $G = G_{\lambda_C}$ for some C in $\mathscr{A}$.*

Proof. We know that G_{λ_C} has a complement.

Suppose G has a complement H. Since $G \vee H = D$, there are normal G-neighborhood U and normal H-neighborhood V such that $U \cap V = \{\varnothing\}$. Let A be in U. Let W be a G-neighborhood of $\varnothing$. By 1.1(2) there is B in $\mathscr{A}$ such that $(B) \subseteq W$ and $(X\setminus B) \subseteq V$. Then $A\setminus B$ is in U and V, so $A\setminus B = \varnothing$. Then $A = A \cap B$ is in W. So A is in Δ_G. Thus $U \subseteq \Delta_G$. But

$\Delta_G \subseteq \operatorname{Int} U \subseteq U$, so $\Delta_G = \operatorname{Int} U$. Hence Δ_G is G-open. Similarly Δ_H is H-open.

Since $G \vee H = D$, $\Delta_G \cap \Delta_H = \{\varnothing\}$. Since Δ_G is G-open and Δ_H is H-open, by 1.1(2) there is C in $\mathscr{A}$ such that $(C) \subseteq \Delta_G$ and $(X \setminus C) \subseteq \Delta_H$. Then $\Delta_G = (C)$ and $\Delta_H = (X \setminus C)$. By 2.1, $G = G_{\lambda_C}$ (and $H = G_{\lambda_{X \setminus C}}$). (Theorem 2.2 was proved by the author and Traynor.)

2.3. Corollary. *The only* T_2 FN *topology which has a complement is the discrete topology.*

Proof. If G is T_2, then $\Delta_G = \{\varnothing\}$.

Note that 2.3 gives a quick proof of [Theorem 4.5, 3].

2.4. Corollary. *The map* $C \to G_{\lambda_{X \setminus C}}$ *is an isomorphism of* $\mathscr{A}$ *onto the Boolean algebra of all* FN *topologies which have complements.*

3. Pseudocomplements and Order-Continuity

Now we turn from complements to pseudocomplements. For G in $\mathrm{FN}(\mathscr{A})$, set $G^c = \bigvee \{H \in \mathrm{FN}(\mathscr{A}) \mid G \wedge H = 0\}$. Call G^c the *pseudocomplement* of G. By the infinite distributive law $G \wedge G^c = 0$. If G and H are complements, then $G^c = H$. If $G \subseteq H$, then $H^c \subseteq G^c$. One of DeMorgan's laws holds: $(G \vee H)^c = G^c \wedge H^c$.

Note that in any Heyting algebra we may define a pseudocomplement with the properties above [Chap. I, 6].

We look at the relationship between G and its double pseudocomplement G^{cc}. Always $G \subseteq G^{cc}$.

3.1. Proposition. *Let G be an* FN *topology on* $\mathscr{A}$ *and E the largest exhaustive* FN *topology. Then*

(1) $E \subseteq G \vee G^c$.

(2) $G^{cc} \wedge E = G \wedge E$.

Proof. (1) Set $H = E \setminus (G \wedge E)$. Then $H \subseteq G^c$, so $E = (G \wedge E) \vee H \subseteq G \vee G^c$.

(2) By (1), $E = E \wedge (G \vee G^c) = (G \wedge E) \vee (G^c \wedge E)$. Replacing G by G^c, we have $(G^c \wedge E) \vee (G^{cc} \wedge E) = E$. So $G \wedge E = E \setminus (G^c \wedge E) = G^{cc} \wedge E$.

There are surprising connections between pseudocomplements and order-continuity in $\mathrm{FN}(\mathscr{A})$. For FN topologies G and H, say that H is *G-order-continuous* if $B_\alpha \to \varnothing$ in H whenever (B_α) is a decreasing net in $\mathscr{A}$ such that $B_\alpha \to \varnothing$ in G.

For each decreasing net (B_α) in $\mathscr{A}$, define a submeasure $\lambda_{(B_\alpha)}$ on $\mathscr{A}$ by

$$\lambda_{(B_\alpha)}(A)=\begin{cases}0 & \text{if } A\cap B_{\alpha_0}=\varnothing \text{ for some } \alpha_0\\ 1 & \text{otherwise.}\end{cases}$$

3.2. LEMMA. *Let G be an* FN *topology on $\mathscr{A}$. If (B_α) is a decreasing net in $\mathscr{A}$ such that $B_\alpha\to\varnothing$ in G, then $G_{\lambda_{(B_\alpha)}}\subseteq G^c$.*

Proof. Let U be a normal G-neighborhood of $\varnothing$. Find α_0 such that B_α is in U for all $\alpha\geqslant\alpha_0$. Then the ideal $(B_{\alpha_0})\subseteq U$ and $\lambda_{(B_\alpha)}(X\setminus B_{\alpha_0})=0$. By 1.1(2), $G\wedge G_{\lambda_{(B_\alpha)}}=0$. Then $G_{\lambda_{(B_\alpha)}}\subseteq G^c$.

3.3. PROPOSITION. *If G is an* FN *topology on $\mathscr{A}$, then G^{cc} is G-order-continuous.*

Proof. Let (B_α) be a decreasing net in $\mathscr{A}$ such that $B_\alpha\to\varnothing$ in G. By 3.2, $G_{\lambda_{(B_\alpha)}}\wedge G^{cc}=0$. Let V be a normal G^{cc}-neighborhood of $\varnothing$. Find B in $\mathscr{A}$ such that $(B)\subseteq V$ and $\lambda_{(B_\alpha)}(X\setminus B)=0$. Then $B_{\alpha_0}\cap(X\setminus B)=\varnothing$ for some α_0. Then $B_{\alpha_0}\subseteq B$, so the ideal $(B_{\alpha_0})\subseteq V$. Thus $B_\alpha\to\varnothing$ in G^{cc}.

3.4. COROLLARY. *If G is an* FN *topology on $\mathscr{A}$, then G and G^{cc} have the same null sets.*

Say an FN topology G is *regular* if $G^{cc}=G$. The set of regular FN topologies is a Boolean algebra (although not a sublattice of FN($\mathscr{A}$)) by [1.13, 6]. If G has a complement, then G is regular.

3.5. PROPOSITION. *Let G be an* FN *topology on $\mathscr{A}$. Then the topology $G\vee G^c$ is regular iff G has a complement.*

Proof. Since $(G\vee G^c)^c=G^c\wedge G^{cc}=0$, $(G\vee G^c)^{cc}=D$. So $G\vee G^c$ is regular iff $G\vee G^c=D$ iff G has a complement.

By 3.5 and 2.2 it is easy to construct FN topologies which are not regular. Next we describe some classes of FN topologies which are regular.

3.6. PROPOSITION. *If λ is a* 0–1 *valued submeasure on $\mathscr{A}$, then G_λ is regular.*

Proof. Set $G=G_\lambda$. Then $\Delta_G=\Delta_\lambda$ is G-open. By 3.4, $\Delta_G=\Delta_{G^{cc}}$. If U is a G^{cc}-neighborhood of $\varnothing$, then $\Delta_G\subseteq U$, so U is a G-neighborhood of $\varnothing$. Thus $G^{cc}\subseteq G$. So $G=G^{cc}$.

For the rest of this section we assume that $\mathscr{A}$ is a σ-algebra. Say an FN topology G on $\mathscr{A}$ is *countably additive* if $B_n\to\varnothing$ in G whenever (B_n) is a decreasing sequence in $\mathscr{A}$ and $\bigcap B_n=\varnothing$.

3.7. THEOREM. *If G is countably additive on a σ-algebra $\mathscr{A}$, then G is regular.*

Proof. By 3.3 G^{cc} is countably additive. Hence G^{cc} and G are exhaustive. By 3.1(2), $G^{cc} = G$.

Now we describe G^c in the case where G is countably additive. Say that an ideal Δ in $\mathscr{A}$ *almost supports* an FN topology G if for every G-neighborhood U of $\varnothing$ there is B in Δ such that $(X\backslash B) \subseteq U$.

3.8. PROPOSITION. *Let G be a countably additive* FN *topology and λ a submeasure on $\mathscr{A}$. If $G \wedge G_\lambda = 0$, then Δ_λ almost supports G.*

Proof. Let U be a G-neighborhood of $\varnothing$. Find a closed normal neighborhood V of $\varnothing$ such that $V \subseteq U$. Set $V_0 = V$. Since the map $(A, B) \to A \cup B$ is continuous, by induction we may find a sequence (V_n) of normal neighborhoods of $\varnothing$ such that $V_{n+1} \cup V_{n+1} = \{A \cup B \mid A, B \in V_{n+1}\} \subseteq V_n$ for all $n \geqslant 0$.

For each $n \geqslant 1$ find A_n in $\mathscr{A}$ such that the ideal $(A_n) \subseteq V_n$ and $\lambda(X\backslash A_n) < 1/n$. For $n \geqslant 1$ set $B_n = A_1 \cup \cdots \cup A_n$. Then (B_n) is an increasing sequence and B_n is in $V_1 \cup V_2 \cup \cdots \cup V_n \subseteq V_0 = V$ for each n. Set $A = \bigcup B_n = \bigcup A_n$. Since G is countably additive, $B_n \to A$ in G. Then A is in V. Since V is normal, $(A) \subseteq V \subseteq U$. Since $\lambda(X\backslash A) \leqslant \lambda(X\backslash A_n) < 1/n$ for all n, $\lambda(X\backslash A) = 0$. Then $X\backslash A$ is in Δ_λ and $(A) \subseteq U$, so Δ_λ almost supports G.

3.9. THEOREM. *If G is countably additive on a σ-algebra $\mathscr{A}$, then*

$$G^c = \bigvee\{G_{\lambda(B_\alpha)} \mid (B_\alpha) \text{ is a decreasing net such that } B_\alpha \to \varnothing \text{ in } G\}.$$

Proof. Set $H = \bigvee\{G_{\lambda(B_\alpha)} \mid (B_\alpha)$ is a decreasing net such that $B_\alpha \to \varnothing$ in $G\}$. By 3.2, $H \subseteq G^c$.

Suppose λ is a submeasure such that $G_\lambda \subseteq G^c$. We may assume that $\lambda \leqslant 1$. Let U be a normal G-neighborhood of $\varnothing$. By 3.8 there is A in Δ_λ such that $(X\backslash A) \subseteq U$. Then the decreasing net $(X\backslash A)_{A \in \Delta_\lambda}$ converges to $\varnothing$ in G. Since $\lambda_{(X\backslash A)}(B) = 0$ iff B is in Δ_λ, $\lambda \leqslant \lambda_{(X\backslash A)}$. Then $G_\lambda \subseteq G_{\lambda_{(X\backslash A)}} \subseteq H$. By 1.3(2), $G^c \subseteq H$. Therefore $G^c = H$.

3.10. THEOREM. *Let G be countably additive on a σ-algebra $\mathscr{A}$ and H an* FN *topology on $\mathscr{A}$. Then H is G-order-continuous iff $H \subseteq G$.*

Proof. Suppose H is G-order-continuous. Let (B_α) be a decreasing net in $\mathscr{A}$ such that $B_\alpha \to \varnothing$ in G. Then $B_\alpha \to \varnothing$ in H. By 3.2, $G_{\lambda(B_\alpha)} \wedge H = 0$. By 3.9, $G^c \wedge H = 0$. Then $H \subseteq G^{cc}$. By 3.7, $H \subseteq G$.

The converse is clear.

We translate 3.10 into a theorem about group-valued measures. A finitely additive map ϕ from $\mathscr{A}$ into a commutative topological group gives rise to an FN topology G_ϕ. The collection of all sets $\{A \in \mathscr{A} \mid \phi((A)) \subseteq M\}$ where M is a neighborhood of 0 is a neighborhood base for G_ϕ at $\emptyset$. Let ϕ and ψ be finitely additive from $\mathscr{A}$ into commutative topological groups Y and Z. Say that ϕ is *ψ-continuous* if for every neighborhood (of 0) M in Y there is a neighborhood N in Z such that $\phi(A)$ is in M whenever $\psi((A)) \subseteq N$. Then ϕ is ψ-continuous iff $G_\phi \subseteq G_\psi$. Say that ϕ is *ψ-order-continuous* if $\phi((B_\alpha)) \to \emptyset$ whenever (B_α) is a decreasing net such that $\psi((B_\alpha)) \to \emptyset$.

3.11. THEOREM. *Let ϕ and ψ be maps from a σ-algebra $\mathscr{A}$ into commutative topological groups Y and Z with ψ countably additive. Then ϕ is ψ-order-continuous iff ϕ is ψ-continuous.*

Proof. Suppose ϕ is ψ-order-continuous. Then G_ϕ is G_ψ-order-continuous. Since ψ is countably additive, so is G_ψ. By 3.10, $G_\phi \subseteq G_\psi$. Then ϕ is ψ-continuous.

Again the converse is clear.

Note that 3.11 is a more general version of [Theorem 4.8, 2].

We close with a question. In Section 2 we gave a characterization of the FN topologies which have complements. Is there a nice description of the FN topologies which are regular?

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